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The Hodge bundle, the universal \mathcal{O} -section, and the log Chow ring of the moduli space of curves

S. Molcho, R. Pandharipande and J. Schmitt

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ABSTRACT

We bound from below the complexity of the top Chern class λ_g of the Hodge bundle in the Chow ring of the moduli space of curves: no formulas for λ_g in terms of classes of degrees 1 and 2 can exist. As a consequence of the Torelli map, the 0-section over the second Voronoi compactification of the moduli of principally polarized abelian varieties also cannot be expressed in terms of classes of degree 1 and 2. Along the way, we establish new cases of Pixton’s conjecture for tautological relations. In the log Chow ring of the moduli space of curves, however, we prove λ_g lies in the subalgebra generated by logarithmic boundary divisors. The proof is effective and uses Pixton’s double ramification cycle formula together with a foundational study of the tautological ring defined by a normal crossings divisor. The results open the door to the search for simpler formulas for λ_g on the moduli of curves after log blow-ups.

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1. Introduction

1.1 The Hodge bundle

Let $\overline{\mathcal{M}}_g$ be the moduli space of Deligne–Mumford stable curves, and let

$$\pi : \mathcal{C}_g \rightarrow \overline{\mathcal{M}}_g$$

be the universal curve with relative dualizing sheaf ω_π . The rank g Hodge bundle \mathbb{E}_g on $\overline{\mathcal{M}}_g$ is defined by

$$\mathbb{E}_g = \pi_* \omega_\pi.$$

The study of the Chern classes of the Hodge bundle goes back at least to Mumford’s Grothendieck–Riemann–Roch calculation [Mum83] in the 1980s. Starting in the late 1990s, the connection of the Hodge bundle to the deformation theory of the moduli space of stable maps has led to an exploration of Hodge integrals in various contexts (see [AKMV05, ELSV01, FP00a, GP99, LLLZ09, LLZ03, MOOP11, OP04, Pan99]).

The top Chern class¹ of the Hodge bundle

$$\lambda_g = c_g(\mathbb{E}_g) \in \text{CH}^g(\overline{\mathcal{M}}_g)$$

plays a special role for several reasons.

¹ All Chow classes are taken here with \mathbb{Q} -coefficients.

(i) Two *vanishing properties* hold:

$$\lambda_g^2 = 0 \in \text{CH}^{2g}(\overline{\mathcal{M}}_g) \quad \text{and} \quad \lambda_g|_{\Delta_0} = 0 \in \text{CH}^g(\Delta_0),$$

where $\Delta_0 \subset \overline{\mathcal{M}}_g$ is the divisor of curves with a non-separating node. The first vanishing follows from the highest graded part of Mumford’s relation

$$c(\mathbb{E}_g) \cdot c(\mathbb{E}_g^*) = 1,$$

proven in [Mum83, equations (5.4), (5.5)]. The second follows from the existence of a trivial quotient²

$$\mathbb{E}_g \twoheadrightarrow \mathbb{C}$$

determined by the residue at (a branch of) the node (see [FP00b, § 0.4]).

(ii) The class $(-1)^g \lambda_g$ appears in the virtual fundamental class of the moduli of *contracted maps* in the Gromov–Witten theory of target curves. Since the *double ramification cycle* in the degree 0 case is defined via contracted maps, we have

$$\text{DR}_{g,(0,\dots,0)} = (-1)^g \lambda_g \in \text{CH}^g(\overline{\mathcal{M}}_{g,n}),$$

where $\overline{\mathcal{M}}_{g,n}$ is the moduli space of stable pointed curves. See [JPPZ17, §§ 0.5.3 and 3.1].

Another basic consequence is the λ_g -formula [FP03],

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \lambda_g = \binom{2g+n-3}{k_1, \dots, k_n} \cdot \int_{\overline{\mathcal{M}}_{g,1}} \psi_1^{2g-2} \lambda_g,$$

predicted by the Virasoro constraints for degree 0 maps to curves [GP98]. Here

$$\psi_i = c_1(\mathbb{L}_i) \in \text{CH}^1(\overline{\mathcal{M}}_{g,n})$$

is the Chern class of the cotangent line at the i th point. The λ_g -formula plays a central role in the study of the tautological ring $\text{R}^*(\mathcal{M}_{g,n}^{\text{ct}})$ of the moduli space of curves of compact type [Pan12].

(iii) Again as an excess class, $(-1)^g \lambda_g$ appears fundamentally in the local Gromov–Witten theory of surfaces. For example, the Katz–Klemm–Vafa formula [KKV99] proven in [MPT10, PT16] concerns integrals

$$\int_{[\overline{\mathcal{M}}_g(S,\beta)]^{\text{red}}} (-1)^g \lambda_g$$

against the reduced virtual fundamental class of the moduli space of stable maps to $K3$ surfaces. For a recent study of the parallel problem for local log Calabi–Yau surfaces (with integrand $(-1)^g \lambda_g$); see [Bou20].

(iv) The class $(-1)^g \lambda_g$ arises via the pull-back of the universal 0-section of the moduli space of *principally polarized abelian varieties* (PPAVs). Over the moduli space of compact type curves, the connection to PPAVs shows a third vanishing property,

$$\lambda_g|_{\mathcal{M}_g^{\text{ct}}} = 0$$

(see [vdG99]). We will discuss PPAVs further in § 1.2 below.

Our main results here concern the complexity of the class λ_g in the Chow ring. For $\overline{\mathcal{M}}_g$, we bound from below the complexity of formulas for

$$\lambda_g \in \text{CH}^*(\overline{\mathcal{M}}_g).$$

As a consequence of the connection to the moduli of PPAVs, we also bound from below the complexity of formulas for the universal 0-section.

² The quotient is defined on the double cover of Δ_0 obtained by ordering the branches of the node.

The log Chow ring of $(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$ is defined as a colimit over all iterated blow-ups of boundary strata. The usual Chow ring is naturally a subalgebra

$$\text{CH}^*(\overline{\mathcal{M}}_g) \subset \text{logCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g).$$

The main positive result of the paper is the simplicity of λ_g in the log Chow ring. We prove

$$\lambda_g \in \text{divlogCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g),$$

where

$$\text{divlogCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \subset \text{logCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

is the subalgebra generated by logarithmic boundary divisors. While λ_g in Chow is complicated, λ_g in log Chow is as simple as possible! We present several related open questions.

1.2 The 0-section

Let \mathcal{A}_g be the moduli space of PPAVs of dimension g , and let

$$\pi : \mathcal{X}_g \rightarrow \mathcal{A}_g$$

be the universal abelian variety π equipped with a universal 0-section

$$s : \mathcal{A}_g \rightarrow \mathcal{X}_g.$$

The image of the 0-section determines an algebraic cycle class

$$Z_g \in \text{CH}^g(\mathcal{X}_g).$$

The second Voronoi compactification of \mathcal{A}_g has been given a modular interpretation by Alekseev:

$$\mathcal{A}_g \subset \overline{\mathcal{A}}_g^{\text{Alekseev}}.$$

Olsson [Ols12] provided a modular interpretation for the normalization

$$\overline{\mathcal{A}}^{\text{Olsson}} \rightarrow \overline{\mathcal{A}}_g^{\text{Alekseev}}.$$

Our approach here will be equally valid for both $\overline{\mathcal{A}}^{\text{Olsson}}$ and $\overline{\mathcal{A}}_g^{\text{Alekseev}}$. We will simply denote the compactification by

$$\mathcal{A}_g \subset \overline{\mathcal{A}}_g,$$

where $\overline{\mathcal{A}}_g$ stands for either the space of Alekseev or the space of Olsson.

The four important properties³ of the compactification $\overline{\mathcal{A}}_g$ which we will require are as follows.

- The points of $\overline{\mathcal{A}}_g$ parameterize (before normalization) stable semiabelic pairs which are quadruples (G, P, L, θ) where G is a semiabelian variety, P is a projective variety equipped with a G -action, L is an ample line bundle on P , and $\theta \in H^0(P, L)$. The data (G, P, L, θ) satisfy several further conditions (see § 4.2.16 of [Ols12]).
- There is a universal semiabelian variety

$$\overline{\pi} : \overline{\mathcal{X}}_g \rightarrow \overline{\mathcal{A}}_g$$

with a 0-section

$$\overline{s} : \overline{\mathcal{A}}_g \rightarrow \overline{\mathcal{X}}_g$$

corresponding to the semiabelian variety which is the first piece of data of a stable semiabelic pair (the rest of the pair data will not play a role in our study).

³ We follow the notation of [Ols12].

- The usual Torelli map $\tau : \mathcal{M}_g \rightarrow \mathcal{A}_g$ extends canonically,

$$\bar{\tau} : \overline{\mathcal{M}}_g \rightarrow \overline{\mathcal{A}}_g$$

(see [Ale04]).

- The $\bar{\tau}$ -pull-back to $\overline{\mathcal{M}}_g$ of $\overline{\mathcal{X}}_g$ is the universal family

$$\text{Pic}_\epsilon^0 \rightarrow \overline{\mathcal{M}}_g$$

parameterizing line bundles on the fibers of the universal curve

$$\epsilon : \mathcal{C}_g \rightarrow \overline{\mathcal{M}}_g$$

which have degree 0 on every component of any fiber [Ale04].

The image of the 0-section \bar{s} determines an operational Chow class

$$\overline{Z}_g \in \text{CH}_{\text{op}}^g(\overline{\mathcal{X}}_g)$$

since the image is an étale local complete intersection in $\overline{\mathcal{X}}_g$. The class \overline{Z}_g is related to $(-1)^g \lambda_g$ via a pull-back construction. Let

$$t : \overline{\mathcal{M}}_g \rightarrow \text{Pic}_\epsilon^0$$

be the 0-section defined by the trivial line bundle. By the properties of

$$\bar{\pi} : \overline{\mathcal{X}}_g \rightarrow \overline{\mathcal{A}}_g$$

discussed above,

$$\bar{\tau}^* \bar{s}^*(\overline{Z}_g) = t^*(t_*[\overline{\mathcal{M}}_g]).$$

By the standard analysis of the vertical tangent bundle of Pic_ϵ^0 ,

$$t^*(t_*[\overline{\mathcal{M}}_g]) = (-1)^g \lambda_g \in \text{CH}^g(\overline{\mathcal{M}}_g).$$

Indeed, by the excess intersection formula the class $t^*(t_*[\overline{\mathcal{M}}_g])$ equals the top Chern class of the normal bundle of the 0-section of Pic_ϵ^0 . Over $[C] \in \overline{\mathcal{M}}_g$, the fiber of the normal bundle is the first-order deformation space of the trivial line bundle on C . The deformation space is given by

$$H^1(C, \mathcal{O}_C) = H^0(C, \omega_C)^\vee,$$

the fiber of the dual of the Hodge bundle \mathbb{E}_g^\vee with top Chern class $(-1)^g \lambda_g$. We conclude that

$$\bar{\tau}^* \bar{s}^*(\overline{Z}_g) = (-1)^g \lambda_g \in \text{CH}^g(\overline{\mathcal{M}}_g). \tag{1}$$

1.3 Complexity of the 0-section

The study the 0-section over \mathcal{A}_g is related to the double ramification cycle (especially over curves of compact type) (see Hain [Hai13] and Grushevsky-Zakharov [GZ14a]). A central idea there is to use the beautiful formula

$$Z_g = \frac{\Theta^g}{g!} \in \text{CH}^g(\mathcal{X}_g), \tag{2}$$

where $\Theta \in \text{CH}^1(\mathcal{X}_g)$ is the universal symmetric theta divisor trivialized along the 0-section. The proof of (2) in Chow uses the Fourier–Mukai transformation and work of

Deninger and Murre [DM91] (see [BL04, Voi14]). The article [GZ14a] provides a more detailed discussion of the history of (2).

We are interested in the following question: *to what extent is an equation of the form of (2) possible over $\overline{\mathcal{A}}_g$?* A result by Grushevsky and Zakharov along these lines appears in [GZ14b]. As before, let

$$\overline{Z}_g \in \text{CH}_{\text{op}}^g(\overline{\mathcal{X}}_g)$$

be the class of the 0-section \overline{s} . Grushevsky and Zakharov calculate the restriction $\overline{Z}_g|_{\mathcal{U}_g}$ of \overline{Z}_g over a particular open set⁴

$$\mathcal{A}_g \subset \mathcal{U}_g \subset \overline{\mathcal{A}}_g$$

in terms of Θ , a boundary divisor $D \in \text{CH}^1(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$, and a class

$$\Delta \in \text{CH}^2(\overline{\mathcal{X}}_g|_{\mathcal{U}_g}).$$

The result of Grushevsky and Zakharov shows that while the naive extension of (2) does *not* hold over \mathcal{U}_g , the class $\overline{Z}_g|_{\mathcal{U}_g}$ lies in the subalgebra of $\text{CH}^*(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$ generated by classes of degrees 1 and 2. The formula of [GZ14b] is a useful extension of (2).

The divisor classes $\text{CH}_{\text{op}}^1(\overline{\mathcal{X}}_g)$ generate a subalgebra

$$\text{divCH}_{\text{op}}^*(\overline{\mathcal{X}}_g) \subset \text{CH}_{\text{op}}^*(\overline{\mathcal{X}}_g).$$

The first bound from below of the complexity of the class of the 0-section is the following result.

THEOREM 1. *For all $g \geq 3$, we have $\overline{Z}_g \notin \text{divCH}_{\text{op}}^*(\overline{\mathcal{X}}_g)$.*

As a consequence, no divisor formula extending (2) is possible for $\overline{\mathcal{A}}_g$. Though not stated, the analysis of [GZ14b] over \mathcal{U}_g can be used to show that $\overline{Z}_g|_{\mathcal{U}_g}$ is *not* in the subalgebra of $\text{CH}^*(\overline{\mathcal{X}}_g|_{\mathcal{U}_g})$ generated by classes of degree 1. Theorem 1 can therefore also be obtained from [GZ14b].⁵

In fact, we can go further. Let

$$\text{CH}_{\leq k}^*(\overline{\mathcal{X}}_g) \subset \text{CH}_{\text{op}}^*(\overline{\mathcal{X}}_g)$$

be the subalgebra generated by all elements of degree at most k , so that

$$\text{divCH}_{\text{op}}^*(\overline{\mathcal{X}}_g) = \text{CH}_{\leq 1}^*(\overline{\mathcal{X}}_g).$$

THEOREM 2. *For all $g \geq 7$, we have $\overline{Z}_g \notin \text{CH}_{\leq 2}^*(\overline{\mathcal{X}}_g)$.*

By Theorem 2, the Grushevsky–Zakharov formula for $\overline{Z}_g|_{\mathcal{U}_g}$ will require corrections by higher-degree classes when extended over $\overline{\mathcal{A}}_g$. We propose the following conjecture about the complexity of the class \overline{Z}_g .

CONJECTURE A. No extension of (2) over $\overline{\mathcal{A}}_g$ for all g can be written in terms of classes of uniformly bounded degree.

The pull-back relation (1) relates the complexity of the class

$$\lambda_g \in \text{CH}^*(\overline{\mathcal{M}}_g)$$

to the complexity of $\overline{Z}_g \in \text{CH}_{\text{op}}^*(\overline{\mathcal{X}}_g)$. Theorems 1 and 2 will be the immediate consequence of parallel⁶ complexity bounds for λ_g .

⁴ \mathcal{U}_g is the locus determined by semiabelian varieties of torus rank at most 1.

⁵ We thank S. Grushevsky for correspondence about [GZ14b].

⁶ In fact, we will prove in § 2 stronger results in cohomology instead of Chow.

1.4 Complexity of λ_g

The divisor classes $\text{CH}^1(\overline{\mathcal{M}}_g)$ generate a subalgebra

$$\text{divCH}^*(\overline{\mathcal{M}}_g) \subset \text{CH}^*(\overline{\mathcal{M}}_g).$$

The first bound from below of the complexity of λ_g is the following result.

THEOREM 3. *For all $g \geq 3$, we have $\lambda_g \notin \text{divCH}^*(\overline{\mathcal{M}}_g)$.*

Via the pull-back relation (1), Theorem 3 immediately implies Theorem 1. The proof of Theorem 3, presented in §2, starts with explicit calculations in the tautological ring in genera 3 and 4 using the Sage package *admcycles* [DSvZ21]. A boundary restriction argument is then used to inductively control all higher genera.

For the analogue of Theorem 2, let

$$\text{CH}_{\leq k}^*(\overline{\mathcal{M}}_g) \subset \text{CH}^*(\overline{\mathcal{M}}_g)$$

be the subalgebra generated by all elements of degree at most k . A similar strategy (with a much more complicated initial calculation in genus 5) yields the following result which implies Theorem 2.

THEOREM 4. *For all $g \geq 7$, we have $\lambda_g \notin \text{CH}_{\leq 2}^*(\overline{\mathcal{M}}_g)$.*

The proofs of Theorems 3 and 4 require new cases of Pixton's conjecture about the ideal of relations in the tautological ring

$$\text{R}^*(\overline{\mathcal{M}}_{g,n}) \subset \text{CH}^*(\overline{\mathcal{M}}_{g,n}).$$

PROPOSITION 5. *Pixton's relations generate all relations among tautological classes in $\text{R}^4(\overline{\mathcal{M}}_{4,1})$ and $\text{R}^5(\overline{\mathcal{M}}_{5,1})$.*

While the above arguments become harder to pursue in general for $\text{CH}_{\leq k}^*(\overline{\mathcal{M}}_g)$, we expect the following assertion to hold.

CONJECTURE B. For fixed k , $\lambda_g \in \text{CH}_{\leq k}^*(\overline{\mathcal{M}}_g)$ holds only for finitely many g .

Of course, Conjecture B implies Conjecture A.

1.5 Log Chow

Theorems 1–4 about the classes \overline{Z}_g and λ_g are in a sense negative results since formula types are excluded. Our main positive result about λ_g concerns the larger log Chow ring

$$\text{CH}^*(\overline{\mathcal{M}}_g) \subset \text{logCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g).$$

The log Chow ring and the subalgebra

$$\text{divlogCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

generated by logarithmic boundary divisors are defined carefully in §3. Our perspective, using limits over log blow-ups, requires the least background in log geometry. A more intrinsic approach to the definitions can be found in [Bar18].

THEOREM 6. *For all $g \geq 2$, we have $\lambda_g \in \text{divlogCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$.*

Our proof of Theorem 6 is constructive: we start with Pixton's formula for the double ramification cycle for constant maps [JPPZ17] and show that each term lies in $\text{divlogCH}^*(\overline{\mathcal{M}}_g)$. In principle, it is possible to obtain bounds for the necessary log blow-ups from the proof, but these will certainly not be optimal. Finding a minimal (or efficient) sequence of log-blows of $(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$ after which λ_g lies in the subalgebra of logarithmic boundary divisors is an interesting question.

A crucial part of the proof of Theorem 6 is the study in § 5 of the logarithmic tautological ring,

$$R^*(X, D) \subset CH^*(X),$$

defined by a normal crossings divisor $D \subset X$ in a non-singular variety X . Tautological classes are defined here using the Chern roots of the normal bundle of logarithmic strata $S \subset X$. The precise definitions are given in § 5.1.

We prove three main structural results about logarithmic tautological classes.

- (i) $R^*(X, D) \subset \text{divlogCH}^*(X, D)$.
- (ii) pull-backs of tautological classes under log blow-ups are tautological.
- (iii) push-forwards of tautological classes under log blow-ups are tautological.

Our first proof of (i) is presented in § 5.2 via an explicit analysis of *explosions*: sequences of blow-ups associated to logarithmic strata of X . A second approach to (i)–(iii), via the geometry of the Artin fan of (X, D) , is given in § 5.5. The Artin fan perspective, advocated by D. Ranganathan,⁷ is theoretically more flexible.

After Pixton’s formula for the double ramification cycle for constant maps is shown to lie in $R^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$, property (i) implies Theorem 6. Since Pixton’s formula and the proof of (i) are both effective, it is possible in principle to compute divisor expressions for λ_g . The result reveals the essential simplicity of λ_g and opens the door to the search for a simpler formula in divisors.

The proof of Theorem 6 yields a refined result: only logarithmic boundary divisors over

$$\Delta_0 \subset \overline{\mathcal{M}}_g$$

are needed to generate λ_g . The parallel result is also true for pointed curves:

$$\lambda_g \in \text{divlogCH}^*(\overline{\mathcal{M}}_{g,n}, \Delta_0)$$

for $2g - 2 + n > 0$.

We have seen that $(-1)^g \lambda_g$ is a special case of the double ramification cycle. The general double ramification cycle

$$DR_{g,A} \in CH^g(\overline{\mathcal{M}}_{g,n})$$

is defined with respect to a vector of integers $A = (a_1, \dots, a_n)$ satisfying

$$\sum_{i=1}^n a_i = 0.$$

In [HPS19, Appendix A], the double ramification cycle was lifted to log Chow,⁸

$$\widetilde{DR}_{g,A} \in \text{logCH}^g(\overline{\mathcal{M}}_{g,n}). \tag{3}$$

Motivated by Theorem 6, we conjecture⁹ a uniform divisorial property of the lifted double ramification cycle (3).

CONJECTURE C. For all g and A , we have $\widetilde{DR}_{g,A} \in \text{divlogCH}^*(\overline{\mathcal{M}}_{g,n})$ where

$$\text{divlogCH}^*(\overline{\mathcal{M}}_{g,n}) \subset \text{logCH}^*(\overline{\mathcal{M}}_{g,n})$$

is the subalgebra generated by logarithmic boundary divisors together with the cotangent line classes ψ_1, \dots, ψ_n .

⁷ See Ranganathan’s April 2020 lecture ‘Gromov–Witten theory and logarithmic intersections’ at the Algebraic Geometry and Moduli Zoominar at ETH Zürich. A foundational development will appear in [MR21].

⁸ The paper [HPS19] is primarily formulated in the language of the related bChow ring, which we discuss below and treat in detail in § 7.

⁹ In developments after the paper was completed, Conjecture C was proven; see § 6.6 for a discussion.

Finally, we return to the Θ -formula (2) for Z_g . Is an extension of the Θ -formula possible over $\overline{\mathcal{M}}_g$ in $\log\text{CH}^*(\overline{\mathcal{M}}_g)$? More specifically, can we find

$$T \in \log\text{CH}^1(\overline{\mathcal{M}}_g)$$

which satisfies the following two properties?

- (i) The restriction of T over the moduli of curves $\mathcal{M}_g^{\text{ct}}$ of compact type is 0.
- (ii) $(-1)^g \lambda_g = \frac{T^g}{g!} \in \log\text{CH}^g(\overline{\mathcal{M}}_g)$.

Property (i) is imposed since

$$\Theta|_{Z_g} = 0 \in \text{CH}^1(Z_g)$$

by the trivialization condition for Θ . Unfortunately, the answer is *no* even for genus 2.

PROPOSITION 7. *There does not exist a class $T \in \log\text{CH}^1(\overline{\mathcal{M}}_2)$ satisfying the restriction property (i) and*

$$(-1)^2 \lambda_2 = \frac{T^2}{2!} \in \log\text{CH}^2(\overline{\mathcal{M}}_2).$$

The Θ -formula for $(-1)^g \lambda_g$ can *not* be extended in a straightforward way in $\text{CH}^g(\overline{\mathcal{M}}_g)$ or $\log\text{CH}^g(\overline{\mathcal{M}}_g)$. However,

$$\lambda_g \in \log\text{CH}^g(\overline{\mathcal{M}}_g)$$

is a degree g polynomial in the logarithmic boundary divisors over $\Delta_0 \subset \overline{\mathcal{M}}_g$.

Question D. Find a polynomial formula in logarithmic boundary divisors for λ_g in log Chow (without using Pixton’s formula).

The larger bChow ring of $\overline{\mathcal{M}}_g$ is defined as a limit over *all* blow-ups:

$$\text{CH}^*(\overline{\mathcal{M}}_g) \subset \log\text{CH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \subset \text{bCH}^*(\overline{\mathcal{M}}_g).$$

The bChow ring is by far the largest of the three Chow constructions. In §7, we show that the main questions of the paper become trivial in bChow. In fact, for every non-singular variety X , we have

$$\text{divbCH}^*(X) = \text{bCH}^*(X).$$

The logarithmic geometry of $\overline{\mathcal{M}}_g$ is therefore the natural place to study Question D for λ_g .

2. λ_g in the Chow ring

2.1 Proof of Theorem 3

Recall that the tautological rings $(R^*(\overline{\mathcal{M}}_{g,n}))_{g,n}$ are defined as the smallest system of \mathbb{Q} -subalgebras with unit of the Chow rings $(\text{CH}^*(\overline{\mathcal{M}}_{g,n}))_{g,n}$ closed under push-forwards by gluing and forgetful maps (see [FP00b, Pan18] for more details). The tautological subring $\text{RH}^*(\overline{\mathcal{M}}_{g,n})$ is defined as the image of the cycle map

$$R^*(\overline{\mathcal{M}}_{g,n}) \twoheadrightarrow \text{RH}^*(\overline{\mathcal{M}}_{g,n}) \subset \text{H}^{2*}(\overline{\mathcal{M}}_{g,n}).$$

We will use the complex degree grading for RH^* and the real degree grading (as usual) for H^* . Let

$$\text{divRH}^*(\overline{\mathcal{M}}_{g,n}) \subset \text{RH}^*(\overline{\mathcal{M}}_{g,n}) \quad \text{and} \quad \text{divH}^*(\overline{\mathcal{M}}_{g,n}) \subset \text{H}^{2*}(\overline{\mathcal{M}}_{g,n})$$

be the subrings generated respectively by $\text{RH}^1(\overline{\mathcal{M}}_{g,n})$ and $\text{H}^2(\overline{\mathcal{M}}_{g,n})$. Since

$$\text{RH}^1(\overline{\mathcal{M}}_{g,n}) = \text{H}^2(\overline{\mathcal{M}}_{g,n}),$$

by [AC98, Theorem 2.2] we have

$$\operatorname{divRH}^*(\overline{\mathcal{M}}_{g,n}) = \operatorname{divH}^{2*}(\overline{\mathcal{M}}_{g,n}). \tag{4}$$

We will use the complex degree grading for both divRH^* and divH^* . Since

$$\operatorname{CH}^1(\overline{\mathcal{M}}_{g,n}) \cong \operatorname{H}^2(\overline{\mathcal{M}}_{g,n})$$

via the cycle class map, we obtain a surjection

$$\operatorname{divCH}^*(\overline{\mathcal{M}}_{g,n}) \twoheadrightarrow \operatorname{divH}^*(\overline{\mathcal{M}}_{g,n}) \subset \operatorname{H}^{2*}(\overline{\mathcal{M}}_{g,n}).$$

The following stronger result implies Theorem 3.

THEOREM 3/COHOMOLOGY. *For all $g \geq 3$, we have $\lambda_g \notin \operatorname{divH}^*(\overline{\mathcal{M}}_g)$.*

Proof. For $g = 3$, we have complete control of the tautological rings in Chow and cohomology since the intersection pairing to $\operatorname{R}_0(\overline{\mathcal{M}}_g) \cong \mathbb{Q}$ is non-degenerate for tautological classes (see [Fab90]). In particular,

$$\operatorname{R}^*(\overline{\mathcal{M}}_3) \cong \operatorname{RH}^*(\overline{\mathcal{M}}_3).$$

In degree 3,

$$\operatorname{divRH}^3(\overline{\mathcal{M}}_3) \subset \operatorname{RH}^3(\overline{\mathcal{M}}_3)$$

is a nine-dimensional subspace of a 10-dimensional space. Explicit calculations with the Sage program *admcycles* [DSvZ21] show that $\lambda_3 \notin \operatorname{divRH}^3(\overline{\mathcal{M}}_3)$. We conclude that $\lambda_3 \notin \operatorname{divH}^*(\overline{\mathcal{M}}_3)$ by (4).

Adding one marked point, we can consider the case of $\overline{\mathcal{M}}_{3,1}$. Again it is known that all (even) cohomology classes on $\overline{\mathcal{M}}_{3,1}$ are tautological (see [SvZ20, §5.1]). Thus, again by Poincaré duality, the intersection pairing on $\operatorname{RH}^*(\overline{\mathcal{M}}_{3,1})$ is perfect and hence we can completely identify these groups in terms of generators and relations. One finds that

$$\operatorname{divRH}^3(\overline{\mathcal{M}}_{3,1}) \subset \operatorname{RH}^3(\overline{\mathcal{M}}_{3,1})$$

is a 28-dimensional subspace of a 29-dimensional space. But remarkably, a calculation by *admcycles* shows

$$\lambda_3 \in \operatorname{divRH}^3(\overline{\mathcal{M}}_{3,1}).$$

The containment appears miraculous. Is there a geometric explanation?

The tautological ring $\operatorname{RH}^*(\overline{\mathcal{M}}_{4,1})$ is also completely under control in codimension 4:

$$\operatorname{divRH}^4(\overline{\mathcal{M}}_{4,1}) \subset \operatorname{RH}^4(\overline{\mathcal{M}}_{4,1})$$

is a 103-dimensional subspace of a 191-dimensional space. An *admcycles* calculation shows that

$$\lambda_4 \notin \operatorname{divRH}^4(\overline{\mathcal{M}}_{4,1}). \tag{5}$$

Result (5) implies $\lambda_4 \notin \operatorname{divRH}^4(\overline{\mathcal{M}}_4)$ by a pull-back argument and

$$\lambda_4 \notin \operatorname{divH}^*(\overline{\mathcal{M}}_4)$$

since divisor classes are tautological.

For $g \geq 5$, a boundary restriction argument is pursued. Suppose, for contradiction, that

$$\lambda_g \in \text{divH}^g(\overline{\mathcal{M}}_g). \tag{6}$$

Then, by pull-back, we have

$$\lambda_g \in \text{divH}^g(\overline{\mathcal{M}}_{g,1}). \tag{7}$$

Consider the standard boundary inclusion

$$\delta : \overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2} \rightarrow \overline{\mathcal{M}}_{g,1}.$$

As usual, we have

$$\delta^*(\lambda_g) = \lambda_{g-1} \otimes \lambda_1. \tag{8}$$

Then (7) implies

$$\lambda_{g-1} \otimes \lambda_1 \in \text{divH}^g(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2}). \tag{9}$$

Since $H^1(\overline{\mathcal{M}}_{g-1,1})$ and $H^1(\overline{\mathcal{M}}_{1,2})$ both vanish,

$$\text{divH}^*(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2}) = \text{divH}^*(\overline{\mathcal{M}}_{g-1,1}) \otimes \text{divH}^*(\overline{\mathcal{M}}_{1,2}).$$

We can therefore write $\text{divH}^g(\overline{\mathcal{M}}_{g-1,1} \times \overline{\mathcal{M}}_{1,2})$ as

$$\begin{aligned} & \text{divH}^g(\overline{\mathcal{M}}_{g-1,1}) \otimes \text{divH}^0(\overline{\mathcal{M}}_{1,2}) \\ \oplus & \text{divH}^{g-1}(\overline{\mathcal{M}}_{g-1,1}) \otimes \text{divH}^1(\overline{\mathcal{M}}_{1,2}) \\ \oplus & \text{divH}^{g-2}(\overline{\mathcal{M}}_{g-1,1}) \otimes \text{divH}^2(\overline{\mathcal{M}}_{1,2}). \end{aligned} \tag{10}$$

Since by (8) the degree of $\delta^*(\lambda_g)$ splits as $(g - 1) + 1$ on the two factors, we conclude that

$$\begin{aligned} \lambda_{g-1} \otimes \lambda_1 & \in \text{divH}^{g-1}(\overline{\mathcal{M}}_{g-1,1}) \otimes \text{divH}^1(\overline{\mathcal{M}}_{1,2}) \\ \implies \lambda_{g-1} & \in \text{divH}^{g-1}(\overline{\mathcal{M}}_{g-1,1}), \end{aligned}$$

using that $\lambda_1 \neq 0 \in \text{divH}^1(\overline{\mathcal{M}}_{1,2})$. By descending induction, we contradict (5). Therefore (7) and hence also (6) must be false. \square

2.2 With marked points

The proof of Theorem 3 in cohomology shows that

$$\lambda_g \notin \text{divH}^g(\overline{\mathcal{M}}_{g,1}) \tag{11}$$

for $g \geq 4$. By using (11) as a starting point, we can study

$$\lambda_g \in \text{divH}^g(\overline{\mathcal{M}}_{g,n})$$

for $g \geq 4$ and $n \geq 2$ using the boundary restrictions

$$\widehat{\delta} : \overline{\mathcal{M}}_{g,n-1} \times \overline{\mathcal{M}}_{0,3} \rightarrow \overline{\mathcal{M}}_{g,n}.$$

The argument used in the proof then easily yields the following statement with markings.

THEOREM 3/MARKINGS. *For all $g \geq 4$ and $n \geq 0$, we have*

$$\lambda_g \notin \text{divH}^*(\overline{\mathcal{M}}_{g,n}).$$

2.3 Proof of Theorem 4

Define the subalgebra of tautological classes

$$\text{RH}_{\leq k}^*(\overline{\mathcal{M}}_{g,n}) \subset \text{RH}^*(\overline{\mathcal{M}}_{g,n})$$

generated by classes of complex degrees less than or equal to k . Since all divisors are tautological,

$$\text{divRH}^*(\overline{\mathcal{M}}_{g,n}) = \text{RH}_{\leq 1}^*(\overline{\mathcal{M}}_{g,n}).$$

The arguments in §§ 2.1 and 2.2 naturally generalize to address the following question: *when does*

$$\lambda_{g-r} \in \text{RH}_{\leq k}^{g-r}(\overline{\mathcal{M}}_{g,n})$$

hold?

A crucial case of the question (from the point of view of boundary restriction arguments) is for $n = 1$. Let $\mathbf{Q}_g(r, k)$ be the statement

$$\lambda_{g-r} \notin \text{RH}_{\leq k}^{g-r}(\overline{\mathcal{M}}_{g,1}),$$

which may be true or false.

For example, $\mathbf{Q}_g(r, g - r)$ is false essentially by definition. In fact,

$$\mathbf{Q}_g(s, g - r) \text{ is false for all } s \geq r$$

for the same reason. In fact, depending on the parity of $g - r$, it is also false for s slightly below r :

$$\mathbf{Q}_g(r - 1, g - r) \text{ is false whenever } g - r \text{ is odd.}$$

To see this, note that the even Chern character $\text{ch}_{g-(r-1)}(\mathbb{E}_g)$ vanishes by [Mum83, Corollary (5.3)]. Expressing it in terms of Chern classes $\lambda_i = c_i(\mathbb{E}_g)$ using Newton's identities, we have

$$0 = \text{ch}_{g-(r-1)}(\mathbb{E}_g) = \frac{(-1)^{g-r+1}}{(g-r+1)!} \lambda_{g-r+1} + (\text{polynomial in } \lambda_1, \dots, \lambda_{g-r}).$$

This proves that λ_{g-r+1} can be written in terms of tautological classes of degrees $1, \dots, g - r$, showing $\mathbf{Q}_g(r - 1, g - r)$ to be false.

The boundary arguments used in §§ 2.1 and 2.2 yield the following two results.

PROPOSITION 8. *If $\mathbf{Q}_g(r, k)$ is true, then $\mathbf{Q}_{g+1}(r, k)$ and $\mathbf{Q}_{g+1}(r + 1, k)$ are true.*

PROPOSITION 9. *If $\mathbf{Q}_g(r, k)$ is true, then*

$$\lambda_{g-r} \notin \text{RH}_{\leq k}^{g-r}(\overline{\mathcal{M}}_{g,n})$$

for all $n \geq 0$.

Since the $k = 1$ case has already been analyzed, we now consider $k = 2$. The first relevant *admcycles* calculation is

$$\lambda_3 \notin \text{RH}_{\leq 2}^3(\overline{\mathcal{M}}_{4,1}),$$

so $\mathbf{Q}_4(1, 2)$ is true. The corresponding subspace here is of dimension 91 inside a 93-dimensional space. As a consequence of Propositions 8 and 9, we obtain the following result.

PROPOSITION 10. *For all $g \geq 4$ and $n \geq 0$, we have*

$$\lambda_{g-1} \notin \text{RH}_{\leq 2}^{g-1}(\overline{\mathcal{M}}_{g,n}).$$

A much more complicated *admcycles* calculation shows that

$$\lambda_5 \notin \text{RH}_{\leq 2}^5(\overline{\mathcal{M}}_{5,1}),$$

so $\text{Q}_5(0, 2)$ is true. The corresponding subspace here is of dimension 1314 inside a 1371-dimensional space. As a consequence of Propositions 8 and 9, we find that

$$\lambda_g \notin \text{RH}_{\leq 2}^g(\overline{\mathcal{M}}_{g,n}) \tag{12}$$

for all $g \geq 5$ and $n \geq 0$. For $g \geq 7$, the equality

$$\text{RH}^2(\overline{\mathcal{M}}_g) = \text{H}^4(\overline{\mathcal{M}}_g)$$

is shown by combining results of Edidin [Edi92] and Boldsen [Bol12]. We provide a summary of the argument in Appendix A. For $g \geq 7$, the cycle map

$$\text{CH}_{\leq 2}^*(\overline{\mathcal{M}}_g) \rightarrow \text{H}^{2*}(\overline{\mathcal{M}}_g)$$

therefore factors through $\text{RH}_{\leq 2}^*(\overline{\mathcal{M}}_g)$. Then the non-containment (12) completes the proof of Theorem 4.

2.4 Cases of Pixton’s conjecture (Proposition 5)

For the proofs of Theorem 3 and 4, dimensions and bases of the following graded parts of tautological rings are required:

$$\begin{aligned} \text{RH}^4(\overline{\mathcal{M}}_{4,1}), \quad \dim_{\mathbb{Q}} &= 191, \\ \text{RH}^5(\overline{\mathcal{M}}_{5,1}), \quad \dim_{\mathbb{Q}} &= 1314. \end{aligned}$$

These cases can be analyzed (via *admcycles*) since the dual pairings are found to have kernels exactly spanned by Pixton’s relations. A discussion of the *admcycles* calculation is presented in Appendix B.

Pixton has conjectured that his relations always provide all tautological relations. Dual pairings are known to be insufficient to prove Pixton’s conjecture in all cases; see [Pan18, PPZ15] for a more complete discussion.

3. The log Chow ring

3.1 Definitions

Let (X, D) be a non-singular variety¹⁰ X with a normal crossings divisor

$$D = D_1 \cup \dots \cup D_\ell \subset X$$

with ℓ irreducible components. The divisor $D \subset X$ is called the *logarithmic boundary*. An *open stratum*

$$S \subset X$$

is an irreducible quasi-projective subvariety satisfying two properties.

- (i) S is étale locally the transverse intersections of the branches of the D_i which meet S .
- (ii) S is maximal with respect to (i).

The set $U = X \setminus D$ is an open stratum. Every open stratum is non-singular. A *closed stratum* is the closure of an open stratum.

¹⁰ For a non-singular Deligne–Mumford stack X and a normal crossings divisor $D \subset X$, the definitions are the same.

If all D_i are non-singular and all intersections

$$D_{i_1} \cap \cdots \cap D_{i_k}$$

are irreducible and non-empty, then there are exactly 2^ℓ open strata.

Our main interest will be in the case $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ where the normal crossings divisors have self-intersections. The open strata defined above for $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ are the same as the usual open strata of the moduli space of stable curves.

An open stratum $S \subset X$ is *simple* if the closure

$$\overline{S} \subset X$$

is non-singular. A *simple blow-up* of (X, D) is a blow-up of X along the closure $\overline{S} \subset X$ of a simple stratum. Let

$$\tilde{X} \rightarrow X \tag{13}$$

be a simple blow-up along \overline{S} . Let

$$\tilde{D} = \tilde{D}_1 \cup \cdots \cup \tilde{D}_\ell \cup E \subset \tilde{X}$$

be the union of the strict transforms \tilde{D}_i of D_i along with the exceptional divisor E of the blow-up (13). Then (\tilde{X}, \tilde{D}) is also a non-singular variety with a normal crossings divisor. An *iterated blow-up*

$$(\hat{X}, \hat{D}) \rightarrow (X, D)$$

is a finite sequence of simple blow-ups of varieties with normal crossings divisors.¹¹

The log Chow group of (X, D) is defined as a colimit over all iterated blow-ups,

$$\log\text{CH}^*(X, D) = \varinjlim_{Y \in \log\text{B}(X, D)} \text{CH}^*(Y).$$

Here, $\log\text{B}(X, D)$ is the category of iterated blow-ups of (X, D) : objects in $\log\text{B}(X, D)$ are iterated blow-ups of (X, D) and morphisms in $\log\text{B}(X, D)$ are iterated blow-ups.

Since (X, D) is the trivial iterated blow-up of itself, there is canonical algebra homomorphism

$$\text{CH}^*(X) \rightarrow \log\text{CH}^*(X, D)$$

which is injective (since an inverse map of \mathbb{Q} -vectors spaces is obtained by proper push-forward). We therefore view $\text{CH}^*(X)$ as a subalgebra of $\log\text{CH}^*(X, D)$. Every Chow class on X canonically determines a log Chow class for (X, D) .

3.2 Calculation in genus 2

We will prove Proposition 7: *there does not exist a class $T \in \log\text{CH}^1(\overline{\mathcal{M}}_2)$ satisfying*

$$T|_{\mathcal{M}_2^{\text{st}}} = 0 \quad \text{and} \quad \lambda_2 = \frac{T^2}{2!} \in \log\text{CH}^2(\overline{\mathcal{M}}_2).$$

Proof. Denote by $\pi_* : \log\text{CH}^*(\overline{\mathcal{M}}_2) \rightarrow \text{CH}^*(\overline{\mathcal{M}}_2)$ the push-forward from log Chow to ordinary Chow. We will prove a stronger claim: *there does not exist a class $T \in \log\text{CH}^1(\overline{\mathcal{M}}_2)$ satisfying*

$$T|_{\mathcal{M}_2^{\text{st}}} = 0 \quad \text{and} \quad \pi_* \left(\lambda_2 - \frac{T^2}{2!} \right) = 0 \in \text{CH}^2(\overline{\mathcal{M}}_2). \tag{14}$$

¹¹ An iterated blow-up is a special type of log blow-up. Since we are taking a limit, we do not have to consider all log blow-ups.

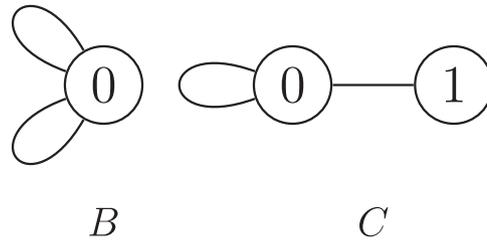


FIGURE 1. The stable graphs associated to the codimension 2 boundary strata B and C contained in U_2 .

Denote by $U_2 \subseteq \overline{\mathcal{M}}_2$ the open subset obtained by removing all closed strata of codimension at least 3. By the excision exact sequence of Chow groups, we have

$$\text{CH}^2(U_2) \cong \text{CH}^2(\overline{\mathcal{M}}_2)$$

and thus we can verify the stronger claim by working over U_2 .

The open set U_2 has open strata of codimension 1 and 2. Since blow-ups along codimension 1 strata do not change U_2 , the only simple blow-ups

$$U'_2 \rightarrow U_2$$

are along codimension 2 open strata (all of which are special in U_2). Since the codimension 2 open strata of U_2 do not intersect (or self-intersect), we obtain a \mathbb{P}^1 -bundle as an exceptional divisor which contains 0- and ∞ -sections¹² which are codimension 2 strata of U'_2 . The iterated blow-ups

$$\widehat{U}_2 \rightarrow U_2$$

are then simply towers of blow-ups of these codimension 2 toric strata in successive exceptional divisors.

Assume $\mathbb{T} \in \text{logCH}^1(U_2)$ satisfies the conditions (14). Since \mathbb{T} restricts to zero over the compact type locus, \mathbb{T} can be represented as

$$\mathbb{T} \in \text{CH}^1(\widehat{U}_2)$$

on an iterated blow-up

$$\widehat{U}_2 \rightarrow U_2$$

with all blow-up centers living over strata in the complement of the compact type locus.

There are a single codimension 1 stratum $\Delta_0 \subset U_2$ and two codimension 2 strata $B, C \subset U_2$ contained in the complement of the compact type locus (see Figure 1).

Denote by E_B^1, \dots, E_B^ℓ and E_C^1, \dots, E_C^m the exceptional divisors of blow-ups with centers lying over B and C . Then \mathbb{T} has a representation¹³

$$\mathbb{T} = a \cdot [\Delta_0] + \sum_{i=1}^{\ell} b_i [E_B^i] + \sum_{j=1}^m c_j [E_C^j].$$

After taking the square and pushing forward, we claim that

$$\pi_*(\mathbb{T}^2) = x \cdot [\Delta_0]^2 + y \cdot [B] + z \cdot [C], \tag{15}$$

¹² Depending upon monodromy, there are either two distinct sections or one double section, that is, a closed subset whose map to the base is finite of degree 2.

¹³ Here, $[\Delta_0]$ is defined via pull-back (not strict transformation).

with $x, y, z \in \mathbb{Q}$ satisfying

$$x = a^2 \geq 0 \quad \text{and} \quad z \leq 0.$$

The claim follows from the following observations.

- In T^2 , all mixed terms $[\Delta_0] \cdot [E_B^i]$ and $[\Delta_0] \cdot [E_C^j]$ vanish after push-forward to U_2 , since

$$\pi_*([\Delta_0] \cdot [E_B^i]) = [\Delta_0] \cdot \pi_*[E_B^i] = [\Delta_0] \cdot 0 = 0.$$

- Similarly, since $B \cap C = \emptyset$ in U_2 (as we have removed the codimension 3 stratum of $\overline{\mathcal{M}}_2$), we have $[E_B^i] \cdot [E_C^j] = 0$.
- Denote by $\mathbf{M} \in \text{Mat}_{\mathbb{Q}, m \times m}$ the matrix defined by

$$\pi_*([E_C^{j_1}] \cdot [E_C^{j_2}]) = \mathbf{M}_{j_1, j_2}[C].$$

A basic fact is that \mathbf{M} is negative definite (see [Mum61, § 1]). Therefore, for $\mathbf{b} = (b_i)_{i=1}^\ell$, we have

$$\pi_*\left(\sum_{j=1}^m b_j [E_C^j]\right)^2 = \underbrace{(\mathbf{b}^\top \mathbf{M} \mathbf{b})}_{=z \leq 0} [C].$$

- The push-forward

$$\pi_*\left(\sum_{i=1}^\ell b_i [E_B^i]\right)^2$$

is supported on B and thus is a multiple $y \cdot [B]$ of the fundamental class of B .

After substituting (15) in the second condition of (14), we conclude the existence of $x, y, z \in \mathbb{Q}$ with $x \geq 0$ and $z \leq 0$ satisfying

$$x \cdot [\Delta_0]^2 + y \cdot [B] + z \cdot [C] = 2\lambda_2 \in \text{CH}^2(U_2). \tag{16}$$

Using *admcycles* (see Appendix B.3), we can explicitly identify all classes in (16) in

$$\text{CH}^2(U_2) \cong \mathbb{Q}^2.$$

The corresponding affine linear equation has the solution space

$$x = z - \frac{1}{120}, \quad y = -\frac{5}{24} \cdot z + \frac{11}{2880}.$$

But for $z \leq 0$, we have

$$z - \frac{1}{120} < 0,$$

which contradicts the assumption that $x \geq 0$. Therefore, there cannot exist a class

$$T \in \text{logCH}^1(U_2)$$

satisfying conditions (14). □

4. Relationship with logarithmic geometry

4.1 Overview

The definitions of § 3 are natural from the perspective of logarithmic geometry. The choice of the divisor D on X can be seen as the choice of a log structure on X . We briefly recall the relevant definitions and constructions of logarithmic geometry.

4.2 Definitions

A log structure on a scheme X is a sheaf of monoids M_X on the étale site of X together with a homomorphism¹⁴

$$\exp : M_X \rightarrow \mathcal{O}_X$$

which induces an isomorphism $\exp^{-1}(\mathcal{O}_X^*) \cong \mathcal{O}_X^*$ on units.

- Morphisms of log schemes $(X, M_X) \rightarrow (Y, M_Y)$ are morphisms of schemes

$$f : X \rightarrow Y$$

together with homomorphisms of sheaves of monoids $f^{-1}M_Y \rightarrow M_X$ which are compatible with the structure map $f^{-1}\mathcal{O}_Y \rightarrow \mathcal{O}_X$ in the obvious sense.

- Log structures can be pulled back. Given a morphism of schemes

$$f : X \rightarrow Y$$

and a log structure M_Y on Y , there is an induced log structure f^*M_Y on X , generated by $f^{-1}M_Y$ and the units \mathcal{O}_X^* .

The basics of log schemes can be found in Kato’s original article on the subject [Kat89].

The category of log schemes is, in practice, too large for geometric study. It is therefore common to work in smaller categories by requiring additional properties to hold. For our purposes, we will work only with in the category of fine and saturated log schemes, usually termed *f.s. log schemes*. The prototype of such a log scheme is

$$A_P = \text{Spec}(k[P]),$$

the spectrum of the algebra generated by a *fine and saturated monoid* P : a finitely generated monoid P which injects into its Grothendieck group P^{gp} and which is saturated there,

$$nx \in P \text{ for } n \in \mathbb{N}, \quad x \in P^{\text{gp}} \implies x \in P.$$

The sheaf M_{A_P} here is the subsheaf of \mathcal{O}_{A_P} generated by P and the units of \mathcal{O}_{A_P} .

All of the log schemes which arise for us will be comparable to A_P on the level of log structures. More precisely, we require our log schemes X to admit the following local charts: for each $x \in X$, there must be an étale neighborhood

$$i : U \rightarrow X,$$

an f.s. monoid P , and a map $g : U \rightarrow A_P$ such that

$$i^*M_X = g^*M_{A_P}.$$

Since we are always working with f.s. log schemes, the chart P at x can in fact always be chosen to be isomorphic to the characteristic monoid¹⁵

$$\overline{M}_{X,\bar{x}} = M_{X,\bar{x}}/\mathcal{O}_{X,\bar{x}}^*$$

at x .

4.3 Normal crossings pairs

Let us now return to the situation of interest for this paper: a pair (X, D) of a non-singular scheme (or Deligne–Mumford stack) with a normal crossings divisor $D \subset X$. The pair (X, D)

¹⁴ \mathcal{O}_X here is sheaf of monoids under multiplication. In particular, the map \exp transfers the addition in M_X to the usual multiplication of functions in \mathcal{O}_X , motivating its name.

¹⁵ Since we are working with étale sheaves, the stalk is computed in the étale topology; \bar{x} denotes the étale stalk.

determines a sheaf M_X on the étale site of X by setting

$$M_X(p : U \rightarrow X) = \{f \in \mathcal{O}_U : f \text{ is a unit on } p^{-1}(X - D)\}$$

for each étale map $p : U \rightarrow X$. The sheaf of units \mathcal{O}_X^* is a subsheaf of M_X . We write

$$\overline{M}_X = M_X / \mathcal{O}_X^*$$

for the *characteristic monoid* of X . Normal crossings pairs (X, D) , with the log structure described above, are precisely the log schemes which are log smooth over the base field $\text{Spec } k$ with trivial log structure.

When the irreducible components of D do not have self-intersections, the log structure M_X of (X, D) can be defined on the Zariski topology of X . The result is a technically simpler theory. The pair (X, D) is then called a *toroidal embedding (without self-intersection)* in [KKMS73]. However, for a general pair (X, D) , M_X can only be defined on the étale site of X . The general étale case differs from the Zariski case in two key aspects: the irreducible components of D can self-intersect, and the characteristic monoid \overline{M}_X , while locally constant on a stratum, can globally acquire monodromy.

The characteristic monoid \overline{M}_X is a constructible sheaf on X . The connected components of the loci on which \overline{M}_X is locally constant define a stratification of X , which is precisely the stratification of §3.1. Indeed, for a geometric point $x \in X$,

$$\overline{M}_{X,\bar{x}} = \mathbb{N}^r$$

where r is the number of branches (in the étale topology) of D that contain x .

A combinatorial space can be built from the information contained in \overline{M}_X . There are two basic approaches. The first, which is more geometric and more evidently combinatorial, is to build the *cone complex* $C(X, D)$ of (X, D) . We briefly outline the construction (details can be found in [CCUW20, ACMUW16]).

We begin with the case where M_X is defined Zariski locally on X (when the irreducible components of D do not have self-intersections). Then $C(X, D)$ is a rational polyhedral cone complex (see [KKMS73]).

- For each point $x \in X$, the characteristic monoid $\overline{M}_{X,\bar{x}}$ determines a rational polyhedral cone

$$\sigma_{X,x} = \text{Hom}_{\text{Monoids}}(\overline{M}_{X,\bar{x}}, \mathbb{R}_{\geq 0})$$

together with an integral structure

$$N_{X,x} = \text{Hom}(\overline{M}_{X,\bar{x}}^{\text{gp}}, \mathbb{Z}).$$

- When x belongs to a stratum $S \subset X$ and y belongs to the closure $\overline{S} \subset X$, there are canonical inclusions

$$\sigma_{X,x} \subset \sigma_{X,y}, \quad N_{X,x} \subset N_{X,y}.$$

- We glue the cones $\sigma_{X,x}$ together with their integral structures to form the complex

$$C(X, D) = \varinjlim_{x \in X} (\sigma_{X,x}, \sigma_{X,x} \cap N_{X,x}).$$

- More effectively, instead of working with all points $x \in X$, we can take the finite set $\{x_S\}$ of the generic points of the strata of (X, D) . Then

$$C(X, D) = \varinjlim_{x_S} (\sigma_{X,x_S}, \sigma_{X,x_S} \cap N_{X,x_S}).$$

In other words, $C(X, D)$ is the dual intersection complex of (X, D) .

When M_X is defined only on the étale site, we build the cone complex $C(X, D)$ by descent.

- We find an étale (but not necessarily proper), strict ($f^*M_X = M_Y$) cover $f : Y \rightarrow X$ which is *as fine as possible* (called atomic or small in the literature): the log structure on Y is defined on the Zariski site of Y , and each connected component of Y has a unique closed stratum. Taking a further such cover V of the fiber product $Y \times_X Y$ if necessary, we find a groupoid presentation

$$V \rightrightarrows Y \rightarrow X.$$

- We define

$$C(X, D) = \varinjlim [C(V) \rightrightarrows C(Y)]$$

in the category of stacks (with respect to the topology generated by face inclusions) over cone complexes. The construction is carried out in detail in [CCUW20], where it is also shown that it is independent of the choice of groupoid presentation.

Moreover, $C(X, D)$ is a complex of cones, but no longer a rational polyhedral cone complex. For each point $x \in X$, there is a canonical map

$$\sigma_{X,x} \rightarrow C(X, D),$$

but the map may no longer be injective. As the étale local branches of the divisor D may be connected globally on X , the faces of the cones $\sigma_{X,x}$ may be glued to each other in $C(X, D)$, and they may naturally acquire automorphisms coming from the monodromy of the branches of D .

4.4 Artin fans

An equivalent combinatorial space is the Artin fan \mathcal{A}_X of (X, D) . The Artin fan is defined by gluing, instead of the dual cones $\sigma_{X,x}$ of $\overline{M}_{X,\bar{x}}$, the quotient stacks

$$\mathcal{A}_{\overline{M}_{X,\bar{x}}} = [\mathrm{Spec}(k[\overline{M}_{X,\bar{x}}]) / \mathrm{Spec}(k[\overline{M}_{X,\bar{x}}^{\mathrm{gp}}])].$$

The gluing is exactly the same as for $C(X, D)$ as explained above. When M_X is defined on the Zariski site of X ,

$$\mathcal{A}_X = \varinjlim_{x \in X} \mathcal{A}_{\overline{M}_{X,\bar{x}}} = \varinjlim_{x_S} \mathcal{A}_{\overline{M}_{X,\bar{x}_S}},$$

and when M_X is defined only on the étale site of X ,

$$\mathcal{A}_X = \varinjlim [\mathcal{A}_V \rightrightarrows \mathcal{A}_Y],$$

for an atomic presentation $\varinjlim [V \rightrightarrows Y] = X$ as before.

The Artin fan \mathcal{A}_X captures exactly the same combinatorial information as the cone complex $C(X, D)$, but is geometrically less intuitive. Nevertheless, the Artin fan has the advantage of coming with a *smooth* morphism of stacks

$$\alpha : X \rightarrow \mathcal{A}_X.$$

4.5 Logarithmic modifications

The cone complex $C(X, D)$ encodes an important operation: *logarithmic modification* of X . Logarithmic modifications correspond to subdivisions of $C(X, D)$. A subdivision of $C(X, D)$ is, by definition, a compatible subdivision of all the cones $\sigma_{X,x}$ compatible with the gluing relations.

Each cone in the subdivision $\sigma'_{X,x} \rightarrow \sigma_{X,x}$ determines dually a map $\overline{M}_{X,\overline{x}} \rightarrow \overline{M}'_{X,\overline{x}}$, and so a map

$$[\mathrm{Spec}(k[\overline{M}'_{X,\overline{x}}])/\mathrm{Spec}(k[\overline{M}^{\mathrm{gp}}_{X,\overline{x}}])] \rightarrow [\mathrm{Spec}(k[\overline{M}_{X,\overline{x}}])/\mathrm{Spec}(k[\overline{M}^{\mathrm{gp}}_{X,\overline{x}}])].$$

The compatibility of the subdivisions with respect to the gluing relations in $C(X, D)$ implies that these maps glue to a *proper* and *birational* representable map

$$\mathcal{A}'_X \rightarrow \mathcal{A}_X.$$

Then we define

$$X' = X \times_{\mathcal{A}_X} \mathcal{A}'_X \rightarrow X$$

which is proper, birational, and representable over X . Moreover, X' has an induced log structure, and there is a map

$$\mathcal{A}'_X \rightarrow \mathcal{A}_{X'}$$

which is proper, Deligne–Mumford type, étale, and bijective.

The map $\mathcal{A}'_X \rightarrow \mathcal{A}_{X'}$ – called the relative Artin fan of X' over X in the literature – is not necessarily representable, as the various monodromy groups of the strata of \mathcal{A}_X may act non-faithfully on the strata of \mathcal{A}'_X , whereas the monodromy groups of the strata of X' act faithfully on $\mathcal{A}_{X'}$ by definition. In this way the strata of \mathcal{A}'_X become trivial gerbes over the strata of $\mathcal{A}_{X'}$. In a sense, $\mathcal{A}_{X'}$ can be considered as a relative coarse moduli space for \mathcal{A}'_X .¹⁶

Geometrically, subdivisions come in three levels of generality as follows.

- General subdivisions simply produce proper birational maps $X' \rightarrow X$, which are isomorphisms over $X - D$. Such maps are called *logarithmic modifications*
- Log blow-ups are a special kind of subdivision. They are the subdivisions of $C(X, D)$ into the domains of linearity of a piecewise linear function on $C(X, D)$, and they correspond to a sheaf of monomial ideals,

$$I \subset M_X.$$

The map $X' \rightarrow X$ is then projective and is the normalization of the blow-up of X along the sheaf of ideals $\exp(I) \subset \mathcal{O}_X$.

- Star subdivisions along simple strata S correspond to the most basic logarithmic modifications. The strata of X are, by construction, in bijection with the cones of $C(X, D)$. We obtain a subdivision by subdividing σ_{X,x_S} along its barycenter (see [CLS11, Definition 3.3.13]). A simple blow-up along \overline{S} corresponds precisely to the star subdivision of the cone σ_{X,x_S} . Further applications of the star subdivision operation are discussed in § 5.3.

Although star subdivisions are the simplest and most basic subdivisions, we need not consider more general subdivisions for our purposes. We are only concerned with statements that are valid over some arbitrarily fine subdivision, and the star subdivisions along simple strata are cofinal in this setting: for each subdivision

$$C(X, D)' \rightarrow C(X, D),$$

there is a further subdivision $C(X, D)'' \rightarrow C(X, D)'$ such that the composition $C(X, D)'' \rightarrow C(X, D)$ is the composition of star subdivisions along simple strata (see [Oda88, Chapter 1.7]). So the reader can restrict attention to simple blow-ups without any loss of generality.

¹⁶ In fact, this can be made precise: $\mathcal{A}_{X'}$ is the relative coarse moduli space of \mathcal{A}'_X with respect to the map $\mathcal{A}'_X \rightarrow \mathbf{Log}$ to the stack parameterizing log structures.

We define a category $\text{logM}(X, D)$ whose objects are log modifications

$$X' \rightarrow X$$

obtained via subdivisions of $C(X, D)$. There is a unique morphism $X'' \rightarrow X'$ if and only if X'' is a log modification of X' . Following [Bar18], we then define

$$\text{logCH}^*(X, D) = \varinjlim_{X' \in \text{logM}(X)} \text{CH}^*(X').$$

As simple blow-ups are cofinal among log modifications, we have, equivalently,

$$\text{logCH}^*(X, D) = \varinjlim_{X' \in \text{logB}(X, D)} \text{CH}^*(X')$$

as defined in §3.1.

5. The divisor subalgebra of log Chow

5.1 Definitions

Let (X, D) be a non-singular variety X with a normal crossings divisor

$$D = D_1 \cup \dots \cup D_\ell \subset X$$

with ℓ irreducible components. Let

$$\text{divlogCH}^*(X, D) \subset \text{logCH}^*(X, D)$$

be the subalgebra generated by the classes of all the components of the associated normal crossings divisors of all iterated blow-ups of X .

Let $S \subset X$ be an open stratum of codimension s , let $\bar{S} \subset X$ be the closure, and let

$$\epsilon : \tilde{S} \rightarrow X$$

be the normalization of \bar{S} equipped with a canonical map ϵ to X . The normalization \tilde{S} is non-singular and separates the branches of the self-intersections of \bar{S} . The map ϵ is an immersion locally on the source and therefore has a well-defined normal bundle

$$\mathbf{N}_\epsilon = \epsilon^*T_X/T_{\tilde{S}}$$

of rank s .

An open stratum $S \subset X$ of codimension s is étale locally cut out by s branches of the full divisor D . These s branches are partitioned by monodromy orbits over S . Each monodromy orbit determines a summand of \mathbf{N}_ϵ . We obtain a canonical splitting of \mathbf{N}_ϵ corresponding to monodromy orbits

$$\mathbf{N}_\epsilon = \bigoplus_{\gamma \in \text{Orb}(S)} \mathbf{N}_\epsilon^\gamma, \quad \text{rank}(\mathbf{N}_\epsilon^\gamma) = |\gamma|,$$

where $\text{Orb}(S)$ is the set of monodromy orbits of the branches of D cutting out S , and $|\gamma|$ is the number of branches in the orbit γ . For polynomials P_γ in the Chern classes of $\mathbf{N}_\epsilon^\gamma$, we define

$$[S, \{P_\gamma\}_{\gamma \in \text{Orb}(S)}] = \epsilon_* \left(\prod_{\gamma \in \text{Orb}(S)} P_\gamma(\mathbf{N}_\epsilon^\gamma) \right) \in \text{CH}^*(X). \tag{17}$$

We define *normally decorated classes* by the following more general construction. Let G be the monodromy group of the s branches of D which cut out S . Over \tilde{S} , there is a principal G -bundle

$$\mu : \tilde{P} \rightarrow \tilde{S}$$

over which the s branches determine s line bundles

$$N_1, \dots, N_s. \tag{18}$$

The G -action on \tilde{P} permutes the line bundles (18) via the original monodromy representation. Let P_G be any G -invariant polynomial in the Chern classes $c_1(N_i)$. Since $P_G(c_1(N_1), \dots, c_1(N_s))$ is G -invariant,

$$P_G(c_1(N_1), \dots, c_1(N_s)) \in \text{CH}^*(\tilde{S}).$$

We define a *normally decorated strata class* by

$$[S, P_G] = \epsilon_*(P_G(c_1(N_1), \dots, c_1(N_s))) \in \text{CH}^*(X).$$

Construction (17) is a special case of a normally decorated strata class.

A fundamental result about the log Chow ring of (X, D) is the following inclusion.

THEOREM 11. *Let (X, D) be a non-singular variety with a normal crossings divisor. Let $S \subset X$ be an open stratum. Every normally decorated class associated to S lies in $\text{divlogCH}^*(X, D)$.*

We give two proofs of Theorem 11. In §5.2 we give a very concrete iterated blow-up of X and an explicit computation expressing the normally decorated class as a sum of products of divisors. On the other hand, in Corollary 16 we give a more conceptual explanation based on the study of the Chow group of the Artin fan of the pair (X, D) .

5.2 Proof of Theorem 11

Theorem 11 is almost trivial if every irreducible component D_i of D is non-singular. The complexity of the argument occurs only if there are irreducible components with self-intersections.

Proof. Let $S \subset X$ be an open stratum of codimension s . The first case to consider is when S is simple. Then the closure

$$\bar{S} \subset X$$

is non-singular and no normalization is needed,

$$\epsilon : \bar{S} \rightarrow X.$$

Let G be the monodromy of the s branches of D which cut out S . We must prove

$$[S, P_G] = \epsilon_*(P_G(c_1(N_1), \dots, c_1(N_s))) \in \text{divlogCH}^*(X)$$

for every G -invariant polynomial P_G .

We argue by induction on the degree of P_G . The base case is when P_G is of degree 0. We can take $P_G = 1$, and we must prove

$$[S, 1] = \epsilon_*[S] \in \text{divlogCH}^*(X, D). \tag{19}$$

Our argument requires a blow-up construction which we term an explosion.

The *explosion* of (X, D) along a simple stratum S ,

$$e : E_S(X, D) \rightarrow X, \tag{20}$$

is defined by a sequence of blow-ups of X . To describe the blow-ups locally¹⁷ near a point $p \in S$, let

$$B_1, \dots, B_s$$

be the branches of D cutting out S near p .

- At the zeroth stage, we blow up S , the intersection of all s branches B_1, \dots, B_s .

Consider next the strict transform of the intersection of $s - 1$ branches. For each choice of $s - 1$ branches, the strict transform of the intersection is non-singular of codimension $s - 1$ over an open set of $p \in X$. Moreover, the strict transforms of the intersections of different sets of $s - 1$ branches are disjoint over an open set of $p \in X$.

- At the first stage, we blow up all s of these strict transforms of intersections of $s - 1$ branches.

Then the strict transforms of the intersections of $s - 2$ branches among B_1, \dots, B_s are non-singular of codimension $s - 2$ and disjoint over an open set of $p \in X$.

- At the second stage, we blow up all $\binom{s}{2}$ of these strict transforms of intersections of $s - 2$ branches.

We proceed in the above pattern until we have completed $s - 1$ stages.

- At the j th stage, we blow up all $\binom{s}{j}$ strict transforms of intersections of $s - j$ branches.

The explosion (20) is the result after stage $j = s - 1$.¹⁸ Since the above blow-ups are defined symmetrically with respect to the branches B_i , the definition is well defined globally on X .

Near S , all the prescribed blow-ups are of simple loci, but non-simplicity may occur away from S . In order for the explosion to be a sequence of simple blow-ups, some extra blow-ups may be required far from S . Since we will only be interested in the geometry near S , the blow-ups related to non-simplicity away from S are not important for our argument (and are not included in our notation).

A local study shows the following properties of the explosion

$$e : E_S(X, D) \rightarrow X,$$

near S .

- (i) The inverse image $e^{-1}(S) \subset E_S(X, D)$ is a non-singular irreducible subvariety which we denote by $E_S(S)$ and call the *exceptional divisor* of the explosion. We denote the inclusion by

$$\iota : E_S(S) \rightarrow E_S(X, D).$$

- (ii) Let N_S be the rank s normal bundle of S in X . The fibers of the projective normal bundle

$$P(N_S) \rightarrow S \tag{21}$$

have a canonical (unordered) set of s coordinate hyperplanes determined by the s local branches of D cutting out S . In the fibers of (21), these relative hyperplanes determine s coordinate points, $\binom{s}{2}$ coordinate lines, $\binom{s}{3}$ coordinate planes, and so on.

¹⁷ Throughout the proof of Theorem 11, the terms ‘local’, ‘near’, and ‘open’ refer to the Euclidean topology since we must separate branches.

¹⁸ At stage $j = s - 1$, we are blowing up divisors, so no change occurs in the space. Still, to uniformize later notation, we include this $j = s - 1$ stage and declare the divisorial center of this trivial blow-up to be its exceptional divisor.

(iii) The restriction of the explosion morphism to the exceptional divisor

$$e_S : E_S(S) \rightarrow S$$

is obtained from $P(N_S) \rightarrow S$ by first blowing up the coordinate points, and then blowing up the strict transforms of the coordinate lines, and so on. For

$$1 \leq j \leq s - 1,$$

the j th stage of the construction of the explosion restricts to the blow-up of the strict transform of the $(j - 1)$ -dimensional coordinate linear spaces of the fibers of (21).

(iv) On $E_S(S)$, we have a distinguished set of divisors

$$E_0, E_1, \dots, E_{s-1} \in \text{CH}^1(E_S(S)).$$

Here, E_0 is the pull-back to $E_S(S)$ of

$$\mathcal{O}_{P(N_S)}(-1) \rightarrow P(N_S)$$

determined by the zeroth stage of the construction of the explosion. Then $E_j \in \text{CH}^1(E_S(S))$ is the pull-back to $E_S(S)$ of the exceptional divisor obtained from the blow-up of the strict transform of the $(j - 1)$ -dimensional coordinate linear spaces in the fibers of (21).

(v) Every class of the form

$$[E_S(S)] \cdot F(E_0, \dots, E_{s-1}) \in \text{CH}^*(E_S(X, D)),$$

where F is a polynomial, lies in the divisor ring of log Chow,

$$[E_S(S)] \cdot F(E_0, \dots, E_{s-1}) \in \text{divlogCH}^*(X, D).$$

The claim follows from the geometric construction of the explosion. To start, $E_S(S)$ is a component of the associated normal crossings divisor of $E_S(X, D)$. For each $0 \leq j \leq s - 1$, E_j comes from the pull-back of a divisor stratum of the blow-up at the j th stage.

To the explosion geometry, we can apply Fulton's excess intersection formula. We start with the zeroth stage:

$$e_0 : X_0 \rightarrow X$$

is the blow-up along S , and

$$e_0^*[S] = [P(N_S)] \cdot c_{s-1} \left(\frac{N_S}{\mathcal{O}_{P(N_S)}(-1)} \right).$$

When we pull back $e_0^*[S]$ all the way to $E_S(X, D)$, we obtain¹⁹

$$e^*[S] = [E_S(S)] \cdot c_{s-1} \left(\frac{N_S}{\mathcal{O}_{P(N_S)}(-1)} \right).$$

By definition, we have

$$c(\mathcal{O}_{P(N_S)}(-1)) = 1 + E_0.$$

By property (v) above for the explosion geometry, to prove

$$\epsilon_*[S] \in \text{divlogCH}^*(X, D), \tag{22}$$

we need only show that

$$c_k(N_S) = F_k(E_0, \dots, E_{s-1}) \in \text{CH}^k(E_S(S)) \tag{23}$$

for polynomials F_k , $1 \leq k \leq s - 1$.

¹⁹ We have omitted the pull-backs in the notation inside the argument of c_{s-1} .

Claim (23) is established directly by the following basic formula of the explosion geometry. For $0 \leq j \leq s - 1$, let

$$L_j = \sum_{i=0}^j E_i.$$

Let σ_k be the k th elementary symmetric polynomial. Then we claim that

$$c_k(N_S) = \sigma_k(L_0, \dots, L_{s-1}) \in \text{CH}^k(E_S(S)). \tag{24}$$

Once we prove (24), this immediately shows (23) and thus, as explained above, establishes (22). We remind ourselves that (22) represents the base case $P_G = 1$ of our inductive proof that $[S, P_G] \in \text{divlogCH}^*(X)$.

Let $T = (\mathbb{C}^*)^s$ and let $t_i : T \rightarrow \mathbb{C}^*$ be the projection to the i th factor, which we interpret as the weight of the standard representation of this i th factor. To show formula (24), we consider the universal T -equivariant model where $S \subset X$ is

$$\mathbf{0} \in \mathbb{C}^s$$

and the logarithmic boundary $H \subset \mathbb{C}^s$ is the union of the s coordinate hyperplanes. Then the T -action on

$$e_0 : E_0(\mathbb{C}^s, H) \rightarrow \mathbf{0}$$

has $s!$ isolated T -fixed points naturally indexed by elements of the symmetric group Σ_s . The weights of the divisors

$$L_0, \dots, L_{s-1}$$

with their canonical T -equivariant lifts at the T -fixed point $\gamma \in \Sigma_s$ are

$$t_{\gamma(1)}, t_{\gamma(2)}, t_{\gamma(3)}, \dots, t_{\gamma(s)}$$

respectively. Formula (24) then follows immediately for the T -equivariant model. The general case of (24) is a formal consequence.

We now will establish the induction step. Let $S \subset X$ be a simple stratum of codimension s with monodromy group²⁰ G of the branches of D cutting out S . We must prove

$$[S, P_G] = \epsilon_*(P_G(c_1(N_1), \dots, c_1(N_s))) \in \text{divlogCH}^*(X, D)$$

for every G -invariant polynomial P_G . By induction, we assume the truth of the statement for polynomials of lower degree.

Let P_G be a G -equivariant polynomial in $c_1(N_1), \dots, c_1(N_s)$ of degree $d > 0$. We will prove a stronger property for the induction argument:

$$\epsilon_*(P_G(c_1(N_1), \dots, c_1(N_s))) \in \text{divlogCH}^*(X, D)$$

can be expressed as a linear combination of terms of the form

$$\widehat{D}_1 \widehat{D}_2 \cdots \widehat{D}_d$$

where the \widehat{D}_i are components of the logarithmic boundary of an iterated blow-up of the explosion $E_S(X, D)$ and \widehat{D}_1 lies over

$$E_S(S) \subset E_S(X, D).$$

Our proof of the base of the induction establishes the stronger property.

²⁰ The geometry involved in the proof of the base case of the induction was fully symmetric with respect to the branches, so the group G did not play a role.

We can assume P_G is the summation²¹ M_G of the G -orbit of a degree d monomial M ,

$$M_G = \frac{1}{|\text{Stab}(M)|} \sum_{g \in G} g(M).$$

We will study the geometry of the exceptional divisor of the explosion

$$e_S : E_S(S) \rightarrow S$$

locally over an analytic open set $U_p \subset S$ of $p \in S$.

Over small enough U_p , we can separate all the branches B_1, \dots, B_s of D which cut out S , and we can write

$$M = c_1(N_1)^{m_1} \dots c_1(N_s)^{m_s} = B_1^{m_1} \dots B_s^{m_s}. \tag{25}$$

Over U_p , we can separate all the exceptional divisors of all the blow-ups in the construction of

$$E_S(S) \rightarrow \mathbb{P}(\mathbb{N}_S)$$

explained in (iii) above. There are $2^s - 2$ such exceptional divisors in bijective correspondence to all the proper non-zero coordinate linear subspaces of the fiber $\mathbb{N}_S|_p$ of \mathbb{N}_S at p . We denote these $2^s - 2$ exceptional divisors by E_Λ , where

$$\Lambda \subset \mathbb{N}_S|_p$$

is a proper coordinate linear space. As before, we denote the pull-back of $\mathcal{O}_{\mathbb{P}(\mathbb{N}_S)}(-1)$ to $E_S(S)$ by E_0 .

Via the pull-back formula for B_i , we have

$$e^*(N_i) = E_0 + \sum_{\Lambda \subset H_i} E_\Lambda \in \text{CH}^1(e^{-1}(U_p)), \tag{26}$$

where $H_i \subset \mathbb{N}_S|_p$ is the hyperplane associated to B_i . We now substitute formula (26) into (25) to find that

$$M \in \mathbb{Q}[E_0, \{E_\Lambda\}_\Lambda].$$

Of course, M has degree d in the divisors E_0 and $\{E_\Lambda\}_\Lambda$.

Let M^E be a monomial of degree d in the divisors

$$E_0 \text{ and } \{E_\Lambda\}_\Lambda. \tag{27}$$

The monodromy group G acts²² canonically on the set (27) leaving E_0 fixed. Let

$$M_G^E = \frac{1}{|\text{Stab}(M^E)|} \sum_{g \in G} g(M^E)$$

be the summation over the G -orbit of M^E . Since M_G^E is G -invariant, M_G^E is a well-defined class

$$M_G^E \in \text{CH}^d(E_S(S)).$$

To prove the stronger induction step, we need only prove²³ that

$$\iota_* \left(M_G^E \cdot c_{s-1} \left(\frac{\mathbb{N}_S}{\mathcal{O}_{\mathbb{P}(\mathbb{N}_S)}(-1)} \right) \right) \in \text{divlogCH}^*(X, D) \tag{28}$$

²¹ The stabilizer factor occurs to correct for overcounting.

²² The G -action on $\{E_\Lambda\}_\Lambda$ preserves the dimension of Λ . Moreover, for a group element $g \in G$, if $g(E_\Lambda) \neq E_\Lambda$, then

$$g(E_\Lambda) \cap E_\Lambda = \emptyset.$$

²³ Recall that ι is the inclusion $\iota : E_S(S) \rightarrow E_S(X, D)$.

can be expressed as a linear combination of terms of the form

$$\widehat{D}_1 \widehat{D}_2 \cdots \widehat{D}_d,$$

where the \widehat{D}_i are components of the logarithmic boundary of an iterated blow-up of the explosion $E_S(X, D)$ and \widehat{D}_1 lies over $E_S(S)$. To see why the claim for (28) is enough, we write

$$\begin{aligned} e^*[S, M_G] &= \sum_{M_G^E} e^*[S] \cdot M_G^E \\ &= \sum_{M_G^E} [E_S(S)] \cdot c_{s-1} \left(\frac{N_S}{\mathcal{O}_{\mathbb{P}(N_S)}(-1)} \right) \cdot M_G^E \\ &= \sum_{M_G^E} \iota_* \left(M_G^E \cdot c_{s-1} \left(\frac{N_S}{\mathcal{O}_{\mathbb{P}(N_S)}(-1)} \right) \right). \end{aligned}$$

The first equality is written with the understanding that $e^*[S]$ is supported on $E_S(S)$.

To study M_G^E , we take a geometric approach. If M^E is just E_0^d , then (28) is already of the claimed form by our analysis in the base case. Otherwise, M^E has at least one factor E_Λ . Since $\{E_\Lambda\}_\Lambda$ is a set of simple normal crossings divisors on $E_S(S)$, we claim that we can write M^E (if non-zero) as

$$M^E = E_{\Lambda_1} \cdots E_{\Lambda_t} \cdot \widetilde{M}^E,$$

where $E_{\Lambda_1}, \dots, E_{\Lambda_t}$ are distinct divisors with a non-empty transverse intersection

$$I_{U_p} = E_{\Lambda_1} \cap \cdots \cap E_{\Lambda_t} \text{ over } U_p.$$

Moreover, we can assume every divisor of the monomial \widetilde{M}^E contains I_{U_p} . Indeed, we construct inductively for $i = 1, 2, \dots$ a representation

$$M^E = E_{\Lambda_1} \cdots E_{\Lambda_i} \cdot \widetilde{M}_i^E$$

such that the E_{Λ_j} are distinct and have non-zero transverse intersection. For $i = 1$ this is just our assumption that M^E has some factor $E_\Lambda =: E_{\Lambda_1}$. On the other hand, given the representation above for some i , if all factors $E_{\Lambda'}$ of \widetilde{M}_i^E contain $E_{\Lambda_1} \cap \cdots \cap E_{\Lambda_i}$, we are done, setting $t = i$. If there is an $E_{\Lambda'}$ not satisfying this, we set $E_{\Lambda_{i+1}} = E_{\Lambda'}$. If the intersection $E_{\Lambda_1} \cap \cdots \cap E_{\Lambda_{i+1}}$ was empty, then $M^E = 0$, giving a contradiction. Thus, the intersection is non-empty, and transverse by the fact that the E_Λ are a normal crossings divisor. We continue inductively, and this construction concludes after at most d steps.

When the monodromy invariant M_G^E is considered, we obtain a non-singular subvariety of $E_S(S)$ of codimension t ,

$$V \subset E_S(S),$$

which is a simple stratum of $E_S(X, D)$,

$$\epsilon^V : V \rightarrow E_S(X, D).$$

Over U_p , the subvariety V restricts to the union²⁴ of the distinct G -translates of I_{U_p} . The crucial geometric observation is

$$\iota_*(M_G^E) = \epsilon_*^V(\widetilde{P}) \in \text{CH}^*(E_S(X, D)),$$

where \widetilde{P} is defined by \widetilde{M}^E and is of degree at most $d - 1$.

²⁴ The distinct G -translates of I_{U_p} are disjoint (see footnote 22).

We can apply the strong induction property: the class

$$\epsilon_*^V(\tilde{P}) \in \text{divlogCH}^*(X, D)$$

can be expressed as a linear combination of terms of the form

$$\widehat{D}_1 \widehat{D}_2 \cdots \widehat{D}_d$$

where the \widehat{D}_i are components of the logarithmic boundary of an iterated blow-up of the explosion of V in $E_S(X, D)$ and \widehat{D}_1 lies over

$$E_V(V) \subset E_S(X, D).$$

Then the claim

$$\iota_* \left(M_G^E \cdot c_{s-1} \left(\frac{N_S}{\mathcal{O}_{\mathbb{P}(N_S)}(-1)} \right) \right) \in \text{divlogCH}^*(X, D) \tag{29}$$

holds by the analysis of

$$c_{s-1} \left(\frac{N_S}{\mathcal{O}_{\mathbb{P}(N_S)}(-1)} \right)$$

on $E_S(S)$ in the base case of the induction. Since each monomial

$$\widehat{D}_1 \widehat{D}_2 \cdots \widehat{D}_d$$

of $\epsilon_*^V(\tilde{P})$ lies over $E_V(V)$, which in turn lies over $E_S(S)$, the analysis of the base case yields the desired result (29).

The induction argument is complete, so we have proven Theorem 11 in the case where S is a simple stratum of (X, D) . The general case follows by repeated application of the result for a simple stratum.

Let $S \subset X$ be a stratum with a singular closure

$$\overline{S} \subset X.$$

The first step is to blow up simple strata in \overline{S} ,

$$\widehat{X} \rightarrow X,$$

until the strict transform of \overline{S} ,

$$\widehat{S} \subset \widehat{X},$$

is non-singular. Since S is simple stratum of the blow-up \widehat{X} , we can apply Theorem 11 to $S \subset \widehat{X}$.

Via the blow-down map, we have

$$\widehat{S} \rightarrow \overline{S}.$$

There are two discrepancies to handle before deducing Theorem 11 for normally decorated classes associated to $S \subset X$ from the result for normally decorated classes associated to $S \subset \widehat{X}$.

- (i) The fundamental class $[\widehat{S}] \in \text{CH}^*(\widehat{X})$ is not the pull-back of $[\overline{S}] \in \text{CH}^*(X)$.
- (ii) The normal directions of $\widehat{S} \subset \widehat{X}$ differ from the pull-backs of the normal directions of $\overline{S} \subset X$.

However, both discrepancies are corrected by applying the simple stratum result to the lower-dimensional strata occurring in $\widehat{S} \setminus S$. □

5.3 Explosion geometry and barycentric subdivision

The explosion operation $E(X, D)$ along a simple stratum $S \subset X$, which appeared in the proof § 5.2, is an essentially combinatorial operation that has a natural interpretation in terms of the geometry of the cone complex $C(X, D)$.

Consider first a cone σ of dimension n in a lattice N , and let A_σ be the associated toric variety. Let \mathcal{A}_σ be the associated Artin fan, which is simply the stack quotient of A_σ by the corresponding dense torus T_σ . The logarithmic stratification of A_σ is precisely the stratification defined by the orbits of T_σ , and there is a bijective dimension-reversing correspondence between faces of σ and strata. We write $\sigma(k)$ for the k -dimensional faces of σ and thus the codimension k strata of A_σ .

For each face τ of σ , the barycenter b_τ of τ is the sum

$$b_\tau = \sum_{v_i \in \tau \cap \sigma(1)} v_i$$

of the primitive vectors along the extremal rays of τ . For any flag

$$\tau_0 \subset \tau_1 \subset \dots \subset \tau_k$$

of faces of σ , the barycenters $b_{\tau_0}, \dots, b_{\tau_k}$ span a cone. The set of all such cones, for all flags in σ , forms a subdivision of σ , which we call the *barycentric subdivision* $\tilde{\sigma}$ of σ .

Alternatively, we can build the barycentric subdivision inductively: at step 1, we start with the star subdivision over the barycenter of faces in $\sigma(n)$ (where σ has dimension n), then take the star subdivision over faces in $\sigma(n - 1)$, and so on, terminating after $n - 1$ steps with $\sigma(2)$, after which the operation no longer has any effect. We thus produce a sequence of $n - 1$ subdivisions

$$\tilde{\sigma} = \sigma_{n-1} \rightarrow \sigma_{n-1} \cdots \rightarrow \sigma_1 \rightarrow \sigma_0 = \sigma$$

When $\sigma = \mathbb{R}_{\geq 0}^n$, which is our main case of interest, the barycentric subdivision has $n!$ maximal cones.

The barycentric subdivision of σ produces a log modification

$$\tilde{A}_\sigma \rightarrow A_\sigma,$$

which is in fact a log blow-up. More precisely, we have constructed the subdivision $\tilde{A}_\sigma \rightarrow A_\sigma$ as a sequence

$$\tilde{A}_\sigma = A_{n-1} \rightarrow A_{n-2} \rightarrow \dots \rightarrow A_1 \rightarrow A_0 = A_\sigma$$

and the map $A_k \rightarrow A_{k-1}$ is determined by the subdivision $\sigma_k \rightarrow \sigma_{k-1}$, which is the subdivision corresponding to the domains of linearity of a piecewise linear function (see [KKMS73] for the construction). In the case of interest,

$$\sigma = \mathbb{R}_{\geq 0}^n,$$

the map $A_1 \rightarrow A_0$ is the blow-up of \mathbb{A}^n at the origin, $A_2 \rightarrow A_1$ is the blow-up along the strict transforms of the coordinate lines, and in general $A_k \rightarrow A_{k-1}$ is the blow-up along the strict transforms of the dimension $k - 1$ hyperplanes of \mathbb{A}^n in A_{k-1} . Thus, the barycentric subdivision of \mathbb{A}^n is precisely the explosion of \mathbb{A}^n along the origin.

The barycentric subdivision construction is clearly equivariant and therefore descends to the Artin fan \mathcal{A}_σ of A_σ . Furthermore, the subdivision is the same on isomorphic faces of σ and invariant with respect to automorphisms of σ . Consequently, given any cone complex C , the barycentric subdivisions of individual cones glue to a global subdivision of C , and that is true even if faces of C are identified or if there is monodromy in C . Thus, for a normal crossings pair

(X, D) , we can define the barycentric subdivision $\tilde{C}(X, D)$ of the cone complex $C(X, D)$, and equivalently, a log blow-up

$$\tilde{\mathcal{A}}_X \rightarrow \mathcal{A}_X$$

of the Artin fan. We also obtain *globally* a log blow-up

$$(\tilde{X}, \tilde{D}) = X \times_{\mathcal{A}_X} \tilde{\mathcal{A}}_X \rightarrow (X, D)$$

with Artin fan $\mathcal{A}_{\tilde{X}} = \tilde{\mathcal{A}}_X$.

The explosion of § 5.2 can only be defined locally around a simple stratum S . A quasi-projective stratum S (not necessarily simple) of a normal crossings pair (X, D) corresponds to a cone σ of $C(X, D)$. More precisely, the quasi-projective stratum S corresponds to the interior of σ , and the whole of σ corresponds to a canonical open set U in X that contains S as its minimal stratum: the open set U consists of all quasi-projective strata whose closure contains S . The explosion $\mathbf{E}_S(U, D|_U)$ is well defined.

The cone σ has a cover by $\mathbb{R}_{\geq 0}^n$, and, more precisely, by a quotient of $\mathbb{R}_{\geq 0}^n$ obtained by potentially identifying faces and taking a quotient by a group G . The group G is precisely the monodromy group of the divisors D that cut out S considered in § 5, and the interior σ° of σ is in fact the stack quotient $[\mathbb{R}_{> 0}^n/G]$. Similarly, the Artin fan \mathcal{U} of U has an analogous étale cover by the groupoid quotient of $[\mathbb{A}^n/\mathbb{G}_m^n \times G]$, with S corresponding to the minimal stratum

$$B(\mathbb{G}_m^n \times G) \subset \mathcal{U}.$$

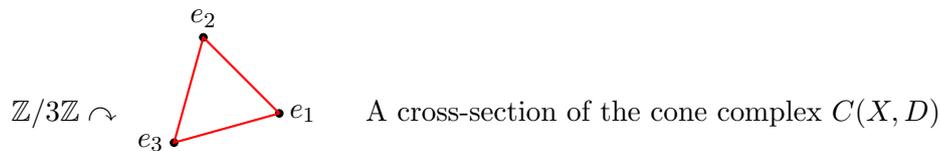
The cover is not representable, but is representable over S . From the discussion of the barycentric subdivision of \mathbb{A}^n , we see that $\mathbf{E}_S(U, D|_U)$ is precisely the barycentric subdivision $\tilde{X} \rightarrow X$ restricted to U . We may thus view the barycentric subdivision as globalizing the explosion geometry.

If the stratum S is simple, the explosion of § 5.2 is defined over a neighborhood of \bar{S} . However, the extension no longer coincides with the barycentric subdivision. The barycentric subdivision performs additional blow-ups, first blowing up all minimal strata in the closure of S (and also strata around \bar{S} whose closure does not necessarily meet S).

We illustrate the concepts discussed above through an example. Let (X, D) be a log scheme whose cone complex is the cone over an equilateral triangle, with all edges identified and with monodromy $\mathbb{Z}/3\mathbb{Z}$. For example, we can construct (X, D) by taking

$$X \rightarrow B$$

to be a family with fiber \mathbb{A}^3 over a non-singular base B satisfying $\pi_1(B) = \mathbb{Z}$, so that the generator of $\pi_1(B)$ cyclically permutes the coordinate hyperplanes of \mathbb{A}^3 . The divisor $D \subset X$ is then the union of these coordinate hyperplanes over B .



The log scheme (X, D) has four strata: the open set $X - D$, corresponding to the empty face of the triangle (or, equivalently, the vertex of the cone over the triangle); the interior of the divisor D corresponding to the vertex

$$e_1 = e_2 = e_3;$$

the locus which is étale locally the intersection of exactly two irreducible components of D corresponding to edge

$$\overline{e_1e_2} = \overline{e_1e_3} = \overline{e_2e_3};$$

and the triple point singularity of D corresponding to the whole triangle. We name the strata Q, R, S, T , respectively. While T is simple, S is not, since

$$\overline{S} = S \cup T$$

is not normal. The strata are taken bijectively to points of the Artin fan via the map

$$\alpha : X \rightarrow \mathcal{A}_X$$

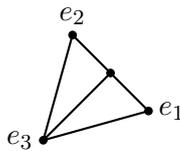
We depict the Artin fan as four points, each isomorphic to $B\mathbb{G}_m^k \times G$ as indicated, with points drawn increasingly bigger to describe the topology (the closure contains all smaller points).

$$\begin{array}{ccccccc}
 B\mathbb{G}_m^3 \times \mathbb{Z}/3\mathbb{Z} = \alpha(T) & & B\mathbb{G}_m = \alpha(R) & & & & \text{Artin fan } \mathcal{A}_X \\
 \bullet & & \bullet & & \bullet & & \\
 & & B\mathbb{G}_m^2 = \alpha(S) & & \text{Spec}\mathbb{C} = \alpha(Q) & &
 \end{array}$$

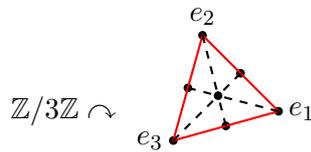
Consider the explosion of the quasi-projective stratum S depicted by the open line segment $\overline{e_1e_2}$. The open set U over which the explosion is defined is $Q \cup R \cup S$. The explosion of S is the barycentric subdivision of $\overline{e_1e_2}$:



However, the above explosion does not extend away from U . The blow-up of \overline{S} , over an étale cover of X is depicted as



But the blow-up does not descend to X as it does not respect the face identifications/automorphisms of $C(X, D)$. The barycentric subdivision is depicted as



The corresponding log blow-up restricts to the explosion over U . Over X , the log blow-up is not the blow-up of \overline{S} , but the explosion of T .

5.4 Tautological classes

Let (X, D) be a non-singular variety with a normal crossings divisor. We define the *logarithmic tautological ring*

$$R^*(X, D) \subset CH^*(X)$$

to be the \mathbb{Q} -linear subspace spanned by all normally decorated strata classes (which is easily seen to be closed under the intersection product). Theorem 11 can then be written as

$$R^*(X, D) \subset \text{divlogCH}^*(X, D).$$

The logarithmic tautological ring of (X, D) depends strongly on the divisor D . For example, if X is irreducible and $D = \emptyset$, then there is only one stratum and

$$R^*(X, \emptyset) = \mathbb{Q}.$$

For the moduli space of curves, the inclusion

$$R^*(\overline{\mathcal{M}}_g, \Delta_0) \subset R^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

is proper for $g \geq 2$. Furthermore, the inclusion

$$R^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g) \subset R^*(\overline{\mathcal{M}}_g)$$

in the standard tautological ring²⁵ is proper for $g \geq 3$ since $R^*(\overline{\mathcal{M}}_g)$ contains κ and ψ classes which do not appear in the logarithmic constructions.

Let (X, D) be a non-singular variety with a normal crossings divisor. Let

$$\pi : \tilde{X} \rightarrow X$$

be a simple blow-up of (X, D) . Let $\tilde{D} \subset \tilde{X}$ be the associated normal crossings divisor. We will prove the following two basic properties of logarithmic tautological rings.

THEOREM 12. *The pull-back*

$$\pi^* : R^*(X, D) \rightarrow \text{CH}^*(\tilde{X})$$

has image in $R^*(\tilde{X}, \tilde{D})$.

THEOREM 13. *The push-forward*

$$\pi_* : R^*(\tilde{X}, \tilde{D}) \rightarrow \text{CH}^*(X)$$

has image in $R^*(X, D)$.

By Theorems 12 and 13, we can simply write

$$\pi^* : R^*(X, D) \rightarrow R^*(\tilde{X}, \tilde{D}), \quad \pi_* : R^*(\tilde{X}, \tilde{D}) \rightarrow R^*(X, D).$$

Theorems 12 and 13 will be proven in §5.6 via the geometry of the Artin fan. As a consequence, we will present a more conceptual (but less constructive) proof of Theorem 11.

5.5 The Chow ring of the Artin fan

Let (X, D) be a non-singular variety with a normal crossings divisor. We relate here the normally decorated strata classes of (X, D) to Chow classes on the Artin fan \mathcal{A}_X of (X, D) . Here, since \mathcal{A}_X is a smooth, finite type algebraic stack stratified by quotient stacks, it has well-defined Chow groups $\text{CH}^*(\mathcal{A}_X)$ with an intersection product as defined in [Kre99]. Note that for our proof below it will not be necessary to recall the precise definition from [Kre99], since we only use some properties and examples of these Chow groups (such as the existence of an excision sequence) that we recall when needed. Also, we stress again that all Chow groups below are with \mathbb{Q} -coefficients.

As we explain in §4.4, there is a smooth morphism to the Artin fan,

$$\alpha : X \rightarrow \mathcal{A}_X.$$

²⁵ $R^*(\overline{\mathcal{M}}_g)$ is definitely not equal to $R^*(\overline{\mathcal{M}}_g, \emptyset)$!

THEOREM 14. *There is a canonical isomorphism*

$$\mathrm{CH}^*(\mathcal{A}_X) \cong \mathrm{PP}^*(C(X, D))$$

between the Chow ring of \mathcal{A}_X and the algebra of piecewise polynomial functions on the cone complex $C(X, D)$.

Proof. By construction, the Artin fan \mathcal{A}_X has a presentation as a colimit

$$\mathcal{A}_X = \varinjlim_{x \in \mathcal{S}} \mathcal{A}_x,$$

where \mathcal{S} is a finite diagram, each map \mathcal{A}_x is a stack of the form $[\mathbb{A}^n/\mathbb{G}_m^n]$, and all maps in the diagram are étale. First, we note that for the individual stacks $\mathcal{A}_x = [\mathbb{A}^n/\mathbb{G}_m^n]$ we have

$$\mathrm{CH}^*([\mathbb{A}^n/\mathbb{G}_m^n]) \cong \mathrm{CH}^*([\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m^n]) \cong \mathbb{Q}[x_1, \dots, x_n]. \tag{30}$$

The first equality is because

$$[\mathbb{A}^n/\mathbb{G}_m^n] \rightarrow [\mathrm{Spec}(\mathbb{C})/\mathbb{G}_m^n]$$

is a vector bundle and induces an isomorphism of Chow groups by [Kre99, Theorem 2.1.12 (vi)]. The second equality is because the equivariant Chow ring of a product of tori is a polynomial algebra [EG98, §3.2], which can be identified with polynomials on the cone $\sigma_{X,x}$ associated to \mathcal{A}_x (appearing in the colimit presentation of $C(X, D)$).

For the entire Artin fan \mathcal{A}_X , we claim

$$\mathrm{CH}^* \mathcal{A}_X = \varinjlim_{x \in \mathcal{S}} \mathrm{CH}^* \mathcal{A}_x. \tag{31}$$

If we can show equality (31), then Theorem 14 will follow since the result holds for each term on the right-hand side by (30). Piecewise polynomial functions on $C(X, D)$ are defined by the corresponding limit presentation.

All the stacks appearing in (31) are very special: they are non-singular and have a stratification with strata isomorphic to

$$B(\mathbb{G}_m^n \rtimes G)$$

where G is a finite group. For the argument below, it will be more convenient to index Chow groups by the dimension of the cycles (instead of the codimension) and prove²⁶

$$\mathrm{CH}_*(\mathcal{A}_X) = \varinjlim_{x \in \mathcal{S}} \mathrm{CH}_*(\mathcal{A}_x). \tag{32}$$

Let \mathcal{C} denote the full 2-subcategory of the 2-category of algebraic stacks with $\mathrm{Ob}(\mathcal{C})$ given by algebraic stacks \mathcal{A} with a stratification by stacks of the form $B(\mathbb{G}_m^n \rtimes G)$, with G a finite group. Similarly, let \mathcal{C}° be the full 2-subcategory of \mathcal{C} with objects given by stacks of the form $B\mathbb{G}_m^n$. We start with a stack²⁷ $\mathcal{A}_X \in \mathcal{C}$ with a colimit presentation

$$\mathcal{A}_X = \varinjlim_{x \in \mathcal{S}} \mathcal{A}_x = \mathcal{A}_X,$$

where $\mathcal{A}_x \in \mathcal{C}^\circ$ and all maps in the diagram are étale. We will prove (32) by induction on the number of strata of \mathcal{A}_X .

²⁶ A similar formula and computation for the Chow groups of the stack of expanded pairs appears in [Oes19].

²⁷ The case of interest is the Artin fan \mathcal{A}_X of X , but in the argument we allow \mathcal{A}_X to be arbitrary in \mathcal{C} in order to run the induction.

Assume first that there is a unique stratum,

$$\mathcal{A}_X = B(\mathbb{G}_m^n \rtimes G),$$

and all maps in the diagram \mathcal{S} are isomorphisms. Then the groupoid

$$\varinjlim_{x \in \mathcal{S}} \mathcal{A}_x$$

is equivalent to the quotient $B\mathbb{G}_m^n/G$, and the statement is equivalent to

$$\mathrm{CH}_*(B(\mathbb{G}_m^n \rtimes G)) = \mathrm{CH}_*(B\mathbb{G}_m^n)^G,$$

which is true (see [BS21, Lemma 2.20]). In general, we pick an open stratum $U \in \mathcal{A}_X$ with preimage $U_x \in \mathcal{A}_x$. Then, by [Kre99, Proposition 4.2.1], we have an exact sequence

$$\mathrm{CH}(U, 1) \longrightarrow \mathrm{CH}(Z) \longrightarrow \mathrm{CH}(\mathcal{A}_X) \longrightarrow \mathrm{CH}(U) \longrightarrow 0$$

with $Z = \mathcal{A}_X - U$. Since U is of the form $U = B(\mathbb{G}_m^n \rtimes G)$, we can use [BS21, Proposition 2.14, Remark 2.21] to see that

$$\mathrm{CH}(U, 1) = \mathrm{CH}(U) \otimes_{\mathbb{Q}} \mathrm{CH}(\mathrm{Spec}(\mathbb{C}), 1).$$

Then, by [BS21, Remark 2.18], the connecting homomorphism $\mathrm{CH}(U, 1) \rightarrow \mathrm{CH}(Z)$ vanishes. So we obtain an exact sequence

$$0 \longrightarrow \mathrm{CH}(Z) \longrightarrow \mathrm{CH}(\mathcal{A}_X) \longrightarrow \mathrm{CH}(U) \longrightarrow 0,$$

and the same sequence holds with \mathcal{A}_X replaced by \mathcal{A}_x , U by U_x , and Z by $Z_x = \mathcal{A}_x - U_x$. As projective limits are left exact, we obtain the following diagram.

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{CH}(Z) & \longrightarrow & \mathrm{CH}(\mathcal{A}_X) & \longrightarrow & \mathrm{CH}(U) \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \varprojlim_{x \in \mathcal{S}} \mathrm{CH}(Z_x) & \longrightarrow & \varprojlim_{x \in \mathcal{S}} \mathrm{CH}(\mathcal{A}_x) & \longrightarrow & \varprojlim_{x \in \mathcal{S}} \mathrm{CH}(U_x) \end{array}$$

By induction, the left and right vertical arrows are isomorphisms. But the bottom row is exact as well: the composed map

$$\mathrm{CH}(\mathcal{A}_X) \rightarrow \mathrm{CH}(U) \cong \varprojlim_{x \in \mathcal{S}} \mathrm{CH}(U_x)$$

is surjective and factors through $\varprojlim_{x \in \mathcal{S}} \mathrm{CH}(\mathcal{A}_x)$. Thus, the map

$$\mathrm{CH}(\mathcal{A}_X) \rightarrow \varprojlim_{x \in \mathcal{S}} \mathrm{CH}(\mathcal{A}_x)$$

is an isomorphism as well. □

Theorem 14 has clear precursors in the toric context by Payne [Pay06] and Brion [Bri94]. In the logarithmic context, we were directly motivated by ideas of Ranganathan. A development of the theory for general log schemes can be found in [MR21].

THEOREM 15. *The logarithmic tautological ring*

$$\mathbf{R}^*(X, D) \subset \mathrm{CH}^*(X)$$

coincides with the image $\alpha^ \mathrm{CH}^*(\mathcal{A}_X) \subset \mathrm{CH}^*(X)$.*

Proof. Fix a stratum $S \subset X$ with closure $\bar{S} \subset X$, and normalization

$$\epsilon : \tilde{S} \rightarrow \bar{S} \subset X.$$

Consider the cone complex $C(X, D)$ and the Artin fan \mathcal{A}_X of (X, D) with

$$\alpha : X \rightarrow \mathcal{A}_X.$$

Let \tilde{P} be the total space of the principal G -bundle over the normalization \tilde{S} defined by the branches of D in §5.1,

$$\mu : \tilde{P} \rightarrow \tilde{S}, \quad \mu_X = \epsilon \circ \mu : \tilde{P} \rightarrow X.$$

We observe that all the relevant geometry is pulled back from the Artin fan \mathcal{A}_X : the stratum S corresponds to the stratum

$$\alpha(S) = \mathcal{S} \subset \mathcal{A}_X$$

with closure $\bar{\mathcal{S}} = \alpha(\bar{S})$. Let $\tilde{\mathcal{S}}$ be the normalization of $\bar{\mathcal{S}}$, and let

$$\mu : \tilde{\mathcal{P}} \rightarrow \tilde{\mathcal{S}}, \quad \mu_{\mathcal{A}} = \tilde{\mathcal{P}} \rightarrow \mathcal{A}_X$$

be the total space of the principal G -bundle over $\tilde{\mathcal{S}}$. Then

$$S = \mathcal{S} \times_{\mathcal{A}_X} X, \quad \bar{S} = \bar{\mathcal{S}} \times_{\mathcal{A}_X} X, \quad \tilde{S} = \tilde{\mathcal{S}} \times_{\mathcal{A}_X} X, \quad \tilde{P} = \tilde{\mathcal{P}} \times_{\mathcal{A}_X} X.$$

Furthermore, since the map α is smooth, we find that $N_{\tilde{S}/X}$ is the pull-back of $N_{\tilde{\mathcal{S}}/\mathcal{A}_X}$, and the splitting of $N_{\tilde{S}/X}$ on \tilde{P} into line bundles is pulled back from the splitting of $N_{\tilde{\mathcal{S}}/\mathcal{A}_X}$ on $\tilde{\mathcal{P}}$. In other words, we have the following Cartesian diagram.

$$\begin{array}{ccc} \tilde{P} & \xrightarrow{\alpha_P} & \tilde{\mathcal{P}} \\ \downarrow \mu_X & & \downarrow \mu_{\mathcal{A}} \\ X & \xrightarrow{\alpha} & \mathcal{A}_X \end{array}$$

Normally decorated strata classes on \bar{S} have the form $\mu_{X*} \alpha_P^*(\gamma)$ for $\gamma \in \text{CH}^*(\tilde{\mathcal{P}})$. As α, α_P are smooth, $\mu_{X*} \alpha_P^* = \alpha^* \mu_{\mathcal{A}*}$. Therefore,

$$R(X, D) \subset \alpha^* \text{CH}^*(\mathcal{A}_X).$$

In fact, the argument shows more precisely that

$$R(X, D) = \alpha^* R(\mathcal{A}_X, \mathcal{D})$$

for $\mathcal{D} = \alpha(D)$ the corresponding divisor in \mathcal{A}_X . In other words, the logarithmic tautological ring of (X, D) , which is generated by the Chern roots of the normal bundles on the various monodromy torsors of the strata of (X, D) , is the pull-back of the logarithmic tautological ring of \mathcal{A}_X , generated by the analogous constructions over the strata of \mathcal{A}_X . Thus, it suffices to show that the normally decorated strata classes of \mathcal{A}_X generate the Chow ring of \mathcal{A}_X . We may thus reduce to proving the theorem for \mathcal{A}_X .

So let γ be a class in $\text{CH}^*(\mathcal{A}_X)$. We must show that

$$\gamma \in R(\mathcal{A}_X, \mathcal{D}).$$

We may assume that γ is supported on $\bar{\mathcal{S}}$ for some stratum $\mathcal{S} \subset \mathcal{A}_X$. Suppose, by induction, we have shown that every such class supported on a stratum $\bar{\mathcal{S}'}$ with

$$\dim \mathcal{S}' < \dim \mathcal{S}$$

is in $R(\mathcal{A}_X, \mathcal{D})$. Suppose further that we can find a class $\delta \in R(\mathcal{A}_X, \mathcal{D})$ such that γ equals δ on \mathcal{S} . Then

$$\gamma - \delta \in \text{CH}^*(\mathcal{A}_X)$$

is supported on lower-dimensional strata and therefore lies in $R(\mathcal{A}_X, \mathcal{D})$, so that we have $\gamma \in R(\mathcal{A}_X, \mathcal{D})$ as well. Thus, the induction hypothesis ensures that, for a given dimension $\dim \mathcal{S}$, we can remove strata \mathcal{S}' with $\dim \mathcal{S}' < \dim \mathcal{S}$, and thus it suffices to prove the statement with the additional assumption that \mathcal{S} is closed in \mathcal{A}_X . Note that this reduction also suffices to handle the base of the induction: the minimal dimensional strata of \mathcal{A}_X are automatically closed.

Suppose then that γ is a class supported on \mathcal{S} , and \mathcal{S} is closed of codimension n in \mathcal{A}_X . Then $\mathcal{S} \cong B(\mathbb{G}_m^n \rtimes G)$, \mathcal{A}_X is a quotient of $[\mathbb{A}^n/\mathbb{G}_m^n]$ by an étale equivalence relation in a neighborhood of \mathcal{S} , and the monodromy torsor $\mu : \tilde{\mathcal{P}} \rightarrow \mathcal{S}$ is isomorphic to $B\mathbb{G}_m^n$.

We can use this to describe the normal bundle $N_{\mathcal{S}/\mathcal{A}_X}$ on \mathcal{S} : the data of this vector bundle on \mathcal{S} are equivalent to specifying the bundle $\mu^*N_{\mathcal{S}/\mathcal{A}_X}$ on $\tilde{\mathcal{P}}$ together with a G -action. It is given by

$$\mu^*N_{\mathcal{S}/\mathcal{A}_X} = \bigoplus_{i=1}^n N_i := \bigoplus_{i=1}^n \mathcal{O}(\mathcal{D}_i)|_{\tilde{\mathcal{P}}}$$

where \mathcal{D}_i is the i th hyperplane divisor in $[\mathbb{A}^n/\mathbb{G}_m^n]$. The monodromy group G acts by permuting the hyperplanes \mathcal{D}_i cutting out \mathcal{S} , and this action lifts to a corresponding action permuting the direct summands N_i above. In particular, while the pull-back $\mu^*N_{\mathcal{S}/\mathcal{A}_X}$ is a direct sum, the individual direct summands are in general not invariant under the G -action, and thus $N_{\mathcal{S}/\mathcal{A}_X}$ is not actually split on \mathcal{S} .

Still, on $\tilde{\mathcal{P}}$ we have that the classes $x_i := c_1(N_i)$ form a generating set for the algebra

$$\text{CH}(\tilde{\mathcal{P}}) \cong \mathbb{Q}[x_1, \dots, x_n].$$

On the other hand, the map μ gives an isomorphism

$$\mu^* : \text{CH}(\mathcal{S}) \cong \text{CH}(\tilde{\mathcal{P}})^G$$

with inverse $(1/|G|)\mu_*$, since we are working with rational Chow groups. Thus, γ is the image of $(1/|G|)\mu^*\gamma$ under μ_* , which is a G -invariant polynomial in the x_i . This shows that γ is a normally decorated strata class, completing the proof. \square

Theorem 15 immediately implies that $R^*(X, D) \subset \text{CH}^*(X)$ is closed under the intersection product (a claim which was left to the reader in §5.4). On the other hand, it is not immediate to see which piecewise polynomial corresponds to which normally decorated strata class. The precise correspondence between piecewise polynomials and normally decorated strata classes has now been established in [HMPPS22, §6].

5.6 Proofs of Theorems 12 and 13

Fix a normal crossings pair (X, D) with Artin fan \mathcal{A}_X and map

$$\alpha : X \rightarrow \mathcal{A}_X.$$

Consider an arbitrary smooth log modification

$$f : \tilde{X} \rightarrow X,$$

necessarily of the form (\tilde{X}, \tilde{D}) , with an associated map

$$\tilde{\alpha} : \tilde{X} \rightarrow \mathcal{A}_{\tilde{X}}.$$

By definition, the log modification $\tilde{X} \rightarrow X$ is pulled back to X from a log modification $\tilde{\mathcal{A}}_X \rightarrow \mathcal{A}_X$ of Artin fans, and we have a diagram

$$\begin{array}{ccccc} \tilde{X} & \xrightarrow{\beta} & \tilde{\mathcal{A}}_X & \xrightarrow{c} & \mathcal{A}_{\tilde{X}} \\ \downarrow f & & \downarrow g & & \\ X & \xrightarrow{\alpha} & \mathcal{A}_X & & \end{array}$$

with the square being Cartesian, the map c proper, Deligne–Mumford type, étale and bijective, and $\tilde{\alpha} = c \circ \beta$. By Theorem 15,

$$R^*(X, D) = \alpha^*CH^*(\mathcal{A}_X) \quad \text{and} \quad R^*(\tilde{X}, \tilde{D}) = \tilde{\alpha}^*CH^*(\mathcal{A}_{\tilde{X}}).$$

Since the map c is proper, Deligne–Mumford type, étale and bijective, it induces an isomorphism

$$c^* : CH^*(\mathcal{A}_{\tilde{X}}) \rightarrow CH^*(\tilde{\mathcal{A}}_X)$$

between rational Chow groups, and thus we also have

$$R^*(\tilde{X}, \tilde{D}) = \beta^*CH^*(\tilde{\mathcal{A}}_X)$$

As $f_*\beta^*(\tilde{\delta}) = \alpha^*g_*(\tilde{\delta})$, we have

$$f_*R^*(\tilde{X}, \tilde{D}) = R^*(X, D),$$

where we conclude equality instead of inclusion since g_* is surjective.²⁸ Similarly, since $f^*\alpha^*(\delta) = \beta^*g^*(\delta)$, we have $f^*R^*(X, D) \subset R^*(\tilde{X}, \tilde{D})$.

Combining Theorem 15 with the techniques used in the proof above also provides a second proof of Theorem 11 based on the study of the Artin fan. The crucial observation is as follows. Suppose (X, D) is a normal crossings pair with D ‘as simple as possible’: D is normal crossings in the Zariski topology, and the non-empty intersections of the branches of D are connected. Equivalently, this means that $C(X, D)$ is the cone over an abstract simplicial complex, that is, can be piecewise linearly embedded into a vector space. Then the ring of piecewise polynomials on $C(X, D)$ has a global description in terms of the Stanley–Reisner ring,

$$PP(C(X, D)) = \mathbb{Q}[x_r]/N,$$

where the variables x_r range over the rays of $C(X, D)$, and N is the ideal of non-faces, that is, generated by monomials $x_{i_1} \cdots x_{i_k}$ ranging over the collections i_1, \dots, i_k of rays which do not form a cone in $C(X, D)$. A fortiori, this presentation implies that $CH(\mathcal{A}_X)$ is generated by divisors.

While the piecewise polynomials of a general (X, D) do not admit this simple description, the observation is relevant in our context because any sufficiently fine log blow-up of (X, D) has this form. For example, the double barycentric subdivision (\hat{X}, \hat{D}) of (X, D) always has this form. Applying barycentric subdivision once on an arbitrary $C(X, D)$ produces a cone complex with no self-intersection (and thus no monodromy), but where two cones possibly share the same set of rays (i.e. the intersection of a set of branches of the divisor is disconnected). Applying barycentric subdivision a second time separates such cones, ensuring that each cone is uniquely characterized by its set of rays, and thus produces a cone complex $C(\hat{X}, \hat{D})$ which is the cone over a simplicial complex.

²⁸ For an interpretation of the push-forward g_* in terms of piecewise polynomials, we refer the reader to [Bri94, §2.3] where the toric setting is studied. These ideas are used in the calculations of [HMPPS22].

COROLLARY 16. We have $R^*(X, D) \subset \text{divlogCH}^*(X, D)$.

Proof. Let $(\widehat{X}, \widehat{D})$ be the log blow-up corresponding to the double barycentric subdivision,

$$\begin{array}{ccc} \widehat{X} & \xrightarrow{\widehat{\alpha}} & \widehat{\mathcal{A}}_X \\ \downarrow f & & \downarrow g \\ X & \xrightarrow{\alpha} & \mathcal{A}_X. \end{array}$$

As above, $\widehat{\mathcal{A}}_X$ is the relative Artin fan of $\widehat{X} \rightarrow X$, and the Artin fan $\mathcal{A}_{\widehat{X}}$ has the same rational Chow ring as $\widehat{\mathcal{A}}_X$. Let $\gamma \in R^*(X, D)$. By Theorem 15, $\gamma \in \alpha^* \text{CH}^*(\mathcal{A}_X)$ and therefore

$$f^*(\gamma) \in \widehat{\alpha}^* \text{CH}^*(\mathcal{A}_{\widehat{X}}).$$

Since $\text{CH}^*(\mathcal{A}_{\widehat{X}})$ is generated by divisors, we have $f^*(\gamma) \in \text{divCH}^*(\widehat{X})$. □

The proof of Theorem 15 immediately yields a finer statement: $R^*(X, D)$ lies in the subalgebra generated by logarithmic divisors of the log blow-up associated to the second barycentric subdivision of the Artin fan of (X, D) ; in fact, the subalgebra generated by logarithmic divisors of the log blow-up associated to any log blow-up $(\widetilde{X}, \widetilde{D})$ with $C(\widetilde{X}, \widetilde{D})$ the cone over a simplicial complex. The double barycentric subdivision of any normal crossings pair (X, D) is always a canonical such choice, but, for any given example, a much more efficient choice $(\widetilde{X}, \widetilde{D})$ may be available.

6. Pixton’s formula for $\lambda_g \in \text{CH}^*(\overline{\mathcal{M}}_g)$

6.1 Strata

Pixton’s formula for the double ramification cycle $\text{DR}_{g,A} \in \text{CH}^g(\overline{\mathcal{M}}_{g,n})$ is expressed as a sum over strata of $(\overline{\mathcal{M}}_{g,n}, \partial\overline{\mathcal{M}}_{g,n})$ indexed by the set $\mathbf{G}_{g,n}$ of stable graphs. We present here Pixton’s formula with an emphasis on the special case

$$\text{DR}_{g,\emptyset} = (-1)^g \lambda_g \in \text{CH}^g(\overline{\mathcal{M}}_g).$$

We refer the reader to [JPPZ17, Pan99] for a more detailed discussion about double ramification cycles, stable graphs, Pixton’s formula, and the relation to classical Abel–Jacobi theory.

6.2 Weightings

Let $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ satisfy $\sum_{i=1}^n a_i = 0$. Let

$$\Gamma \in \mathbf{G}_{g,n}$$

be a stable graph²⁹ of genus g with n legs. A *weighting* of Γ is a function on the set of half-edges,

$$w : \text{H}(\Gamma) \rightarrow \mathbb{Z},$$

which satisfies the following three properties:

- (i) for all $h_i \in \text{L}(\Gamma)$, corresponding to the marking $i \in \{1, \dots, n\}$,

$$w(h_i) = a_i;$$

- (ii) for all $e \in \text{E}(\Gamma)$, corresponding to two half-edges $h, h' \in \text{H}(\Gamma)$;

$$w(h) + w(h') = 0,$$

²⁹ Here and in Pixton’s formula in § 6.3, we follow the notation of [JPPZ17, §§ 0.3 and 0.4]. The factors of 2 are treated equivalently but slightly differently in [BHPSS20, JPPZ20].

(iii) for all $v \in V(\Gamma)$,

$$\sum_{v(h)=v} w(h) = 0,$$

where the sum is taken over all $n(v)$ half-edges incident to v .

In the case $A = \emptyset$, the set of half-edges $H(\Gamma)$ has no legs ($n = 0$).

Let r be a positive integer. A *weighting mod r* of Γ is a function,

$$w : H(\Gamma) \rightarrow \{0, \dots, r - 1\},$$

which satisfies exactly properties (i)–(iii) above, but with the equalities replaced, in each case, by the condition of *congruence mod r* . The set $W_{\Gamma,r}$ of such weightings w is finite, with cardinality $r^{h^1(\Gamma)}$.

6.3 Formula for double ramification cycles

Let $A = (a_1, \dots, a_n) \in \mathbb{Z}^n$ satisfy $\sum_{i=1}^n a_i = 0$. Let r be a positive integer. We denote by

$$P_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$$

the degree d component of the tautological class

$$\sum_{\Gamma \in \mathcal{G}_{g,n}} \sum_{w \in W_{\Gamma,r}} \frac{1}{|\text{Aut}(\Gamma)|} \frac{1}{r^{h^1(\Gamma)}} \xi_{\Gamma*} \left[\prod_{i=1}^n \exp(a_i^2 \psi_{h_i}) \cdot \prod_{e=(h,h') \in E(\Gamma)} \frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}} \right] \tag{33}$$

in $R^*(\overline{\mathcal{M}}_{g,n})$.

The following fundamental polynomiality property of $P_g^{d,r}(A)$ has been proven by Pixton (see [JPPZ17, Appendix]).

PROPOSITION 17 (Pixton). *For fixed g, A , and d , the class*

$$P_g^{d,r}(A) \in R^d(\overline{\mathcal{M}}_{g,n})$$

is polynomial in r (for all sufficiently large r).

We denote by $P_g^d(A)$ the value at $r = 0$ of the polynomial associated to $P_g^{d,r}(A)$ by Proposition 17. In other words, $P_g^d(A)$ is the *constant* term of the associated polynomial in r . Pixton’s formula for double ramification cycles is

$$DR_{g,A} = 2^{-g} P_g^d(A) \in CH^g(\overline{\mathcal{M}}_{g,n}).$$

6.4 Examples in the $A = \emptyset$ case

For the reader’s convenience, we present in Figure 2 the first few examples³⁰ of Pixton’s formula for λ_g obtained by calculating $(-1)^g DR_{g,\emptyset}$.

Each labeled graph Γ describes a moduli space $\overline{\mathcal{M}}_{\Gamma}$ (a product of moduli spaces associated with the vertices of Γ), a tautological class $\alpha \in R^*(\overline{\mathcal{M}}_{\Gamma})$, and a natural map

$$\xi_{\Gamma} : \overline{\mathcal{M}}_{\Gamma} \rightarrow \overline{\mathcal{M}}_g.$$

Our convention in the formulas below is that the graph Γ represents the cycle class $(\xi_{\Gamma})_* \alpha$. For instance, assume the graph carries no ψ -classes and the class α equals 1. Since the map ξ_{Γ} is of

³⁰ The graphics are by F. Janda.

Genus 1.

$$\lambda_1 = \frac{1}{24} \text{diagram}$$

Genus 2.

$$\lambda_2 = \frac{1}{240} \text{diagram} + \frac{1}{1152} \text{diagram}$$

Genus 3.

$$\lambda_3 = \frac{1}{2016} \text{diagram} + \frac{1}{2016} \text{diagram} - \frac{1}{672} \text{diagram} + \frac{1}{5760} \text{diagram} - \frac{13}{30240} \text{diagram} - \frac{1}{5760} \text{diagram} + \frac{1}{82944} \text{diagram}$$

Genus 4.

$$\lambda_4 = \frac{1}{11520} \text{diagram} + \frac{1}{3840} \text{diagram} - \frac{1}{2880} \text{diagram} - \frac{1}{3840} \text{diagram} - \frac{1}{1440} \text{diagram} - \frac{1}{1920} \text{diagram} - \frac{1}{2880} \text{diagram} - \frac{1}{3840} \text{diagram} + \frac{1}{48384} \text{diagram} + \frac{1}{48384} \text{diagram} + \frac{1}{115200} \text{diagram} + \frac{1}{960} \text{diagram} - \frac{23}{100800} \text{diagram} - \frac{1}{57600} \text{diagram} - \frac{1}{16128} \text{diagram} - \frac{1}{16128} \text{diagram} - \frac{1}{57600} \text{diagram} - \frac{1}{16128} \text{diagram} - \frac{1}{16128} \text{diagram} - \frac{23}{100800} \text{diagram} + \frac{23}{100800} \text{diagram} + \frac{23}{50400} \text{diagram} + \frac{1}{16128} \text{diagram} + \frac{1}{115200} \text{diagram} + \frac{1}{276480} \text{diagram} - \frac{13}{725760} \text{diagram} - \frac{1}{138240} \text{diagram} - \frac{43}{1612800} \text{diagram} - \frac{13}{725760} \text{diagram} - \frac{1}{276480} \text{diagram} + \frac{1}{7962624} \text{diagram}$$

FIGURE 2. Formulas for the classes λ_g for $g = 1, \dots, 4$.

degree $|\text{Aut}(\Gamma)|$ onto its image, the cycle class represented by Γ is then $|\text{Aut}(\Gamma)|$ times the class of the image of ξ_Γ .

6.5 Proof of Theorem 6

We analyze Pixton’s formula in the $A = \emptyset$ case,

$$\lambda_g = (-1)^g \text{DR}_{g,\emptyset} \in \text{CH}^g(\overline{\mathcal{M}}_g).$$

Since $A = \emptyset$, the sum (33) is over stable graphs $\Gamma \in \mathbf{G}_g$ corresponding to strata of $(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$.

- By the definition of a *weighting mod r*, the weights

$$w(h), \quad w(h')$$

on the two halves of every separating edge e of Γ must both be 0. The factor in Pixton’s formula for e ,

$$\frac{1 - \exp(-w(h)w(h')(\psi_h + \psi_{h'}))}{\psi_h + \psi_{h'}}$$

then vanishes and kills the contribution of Γ to $\mathbb{P}_g^g(\emptyset)$. Therefore, non-vanishing terms in the sum (33) must correspond to graphs with no separating edges.

- Since $A = \emptyset$, the term

$$\prod_{i=1}^n \exp(a_i^2 \psi_{h_i})$$

drops out of (33).

- The classes which do appear in (33) are the normal bundle terms $\psi_h + \psi_{h'}$ at each edge of Γ .

Since the formula (33) respects the automorphisms of the stable graph Γ , we obtain the following result.

PROPOSITION 18. *The class $\lambda_g \in \text{CH}^g(\overline{\mathcal{M}}_g)$ is a sum of normally decorated classes associated to strata of $(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$ corresponding to stable graphs $\Gamma \in \mathbf{G}_g$ with no separating edges.*

Theorem 6 is then an immediate consequence of Proposition 18 and Theorem 11. Proposition 18 reflects a very special property of λ_g obtained from Pixton’s formula. Since every edge of every stable graph $\Gamma \in \mathbf{G}_g$ which appears in Pixton’s formula for λ_g is non-separating, we actually have

$$\lambda_g \in \mathbf{R}^*(\overline{\mathcal{M}}_g, \Delta_0).$$

Theorem 11 then implies a refinement of Theorem 6,

$$\lambda_g \in \text{divlogCH}^*(\overline{\mathcal{M}}_g, \Delta_0).$$

By applying Pixton’s formula for the double ramification cycle

$$\text{DR}_{g,(0,\dots,0)} = (-1)^g \lambda_g \in \text{CH}^g(\overline{\mathcal{M}}_{g,n}),$$

an identical argument yields

$$\lambda_g \in \text{divlogCH}^*(\overline{\mathcal{M}}_{g,n}, \Delta_0)$$

for $2g - 2 + n > 0$.

6.6 More general double ramification cycles

Let $A = (a_1, \dots, a_n)$ be a vector of integers satisfying $\sum_{i=1}^n a_i = 0$. Pixton’s formula for the double ramification cycle

$$\text{DR}_{g,A} \in \mathbf{R}^*(\overline{\mathcal{M}}_{g,n}),$$

together with Theorem 11, yields the following result (the proof of which is exactly the same as the proof of Theorem 6).

THEOREM 19. *We have $\text{DR}_{g,A} \in \text{divlogCH}^*(\overline{\mathcal{M}}_{g,n})$ where*

$$\text{divlogCH}^*(\overline{\mathcal{M}}_{g,n}) \subset \text{logCH}^*(\overline{\mathcal{M}}_{g,n})$$

is the subalgebra generated by logarithmic boundary divisors together with the cotangent line classes ψ_1, \dots, ψ_n .

Theorem 19 provides half of the proof of Conjecture C concerning the lifted double ramification cycle $\widetilde{\text{DR}}_{g,A}$. There are now three proofs of the other half of the conjecture via three

different approaches. The first two are by Abel–Jacobi theory in [HS22] and by controlling the difference between $\mathrm{DR}_{g,A}$ and $\widetilde{\mathrm{DR}}_{g,A}$ in an appropriate blow-up of $\overline{\mathcal{M}}_{g,n}$ in [MR21]. The third, presented in [HMPPS22], proves the conjecture directly by giving a formula for (a representative of) $\widetilde{\mathrm{DR}}_{g,A}$ in terms of ψ -classes and piecewise polynomials.

The special case $A = (0, \dots, 0)$ related to the class λ_g is simpler since no cotangent line classes appear at the markings in Pixton’s formula. Moreover, there is no change in the lift for $A = (0, \dots, 0)$:

$$\mathrm{DR}_{g,(0,\dots,0)} = \widetilde{\mathrm{DR}}_{g,(0,\dots,0)} \in \underline{\mathrm{divlogCH}}^*(\overline{\mathcal{M}}_{g,n}).$$

The ω^k -twisted double ramification cycle [Hol19] is also governed by Pixton’s formula [BHPSS20],

$$\mathrm{DR}_{g,A}^k \in \mathbf{R}^*(\overline{\mathcal{M}}_{g,n}), \quad \sum_{i=1}^n a_i = k(2g - 2).$$

The analogue of Theorem 19 can be proven for the ω^k -twisted double ramification cycle, but the divisor subalgebra of $\log\mathrm{CH}^*(\overline{\mathcal{M}}_{g,n})$ must include κ_1 together with the cotangent line classes ψ_i and the logarithmic boundary divisors. Conjecture C can then also be promoted to a statement for the lifted ω^k -twisted double ramification cycle (again including κ_1 in the subalgebra).

6.7 Pixton’s generalized boundary strata classes

In [Pix18], Pixton defined a subalgebra of the tautological ring $\mathbf{R}^*(\overline{\mathcal{M}}_{g,n})$ spanned by *generalized boundary strata classes*: tautological classes $[\Gamma]$ associated to prestable graphs Γ of genus g with n legs.

If Γ is a semistable graph (every genus 0 vertex is incident to at least two legs or half-edges), then Pixton’s definition takes a simple form. Let Γ' be the stabilization of Γ . The class $[\Gamma]$ is defined as a push-forward under the gluing map $\xi_{\Gamma'}$ of products of classes ψ_1, \dots, ψ_n and classes $\psi_h + \psi_{h'}$ for half-edges (h, h') forming an edge of Γ' . The analysis of § 6.5 then implies

$$[\Gamma] \in \underline{\mathrm{divlogCH}}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

in the semistable case.

Pixton’s boundary class for more general unstable graphs has κ classes and will likely not lie in any version of $\underline{\mathrm{divlogCH}}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$.

7. The bChow ring

Let X be a non-singular variety. Given the additional data of a normal crossings divisor $D \subset X$, we defined the log Chow ring of the pair (X, D) . This is a variant of a much larger ring, the *bChow ring* of X . We define

$$\mathrm{bCH}^*(X) = \varinjlim_{Y \in \mathbf{B}(X)} \mathrm{CH}^*(Y),$$

where $\mathbf{B}(X)$ is the category of non-singular blow-ups of X : objects in $\mathbf{B}(X)$ are proper birational maps

$$Y \rightarrow X$$

with Y non-singular, and morphisms in $\mathbf{B}(X)$ are proper birational maps over X . For a longer introduction to the bChow ring, see [HPS19]. Some of the ideas involved go back to papers of Shokurov [Sho96, Sho03]. See also Aluffi [Alu05] for similar constructions.

Let $[Z \rightarrow X]$ and $[Y \rightarrow X]$ be objects of $\mathbf{B}(X)$. If $Z \rightarrow X$ factors as

$$Z \rightarrow Y \rightarrow X,$$

then there is a unique morphism from $[Z \rightarrow X]$ to $[Y \rightarrow X]$ in $\mathbf{B}(X)$, and we call $Z \rightarrow X$ a *refinement* of $Y \rightarrow X$. The transition maps in the above colimit are given by pull-backs

$$f^* : \mathrm{CH}^*(Y) \rightarrow \mathrm{CH}^*(Z)$$

for refinements $Z \xrightarrow{f} Y \rightarrow X$.

Unlike $\log\mathrm{CH}^*(X)$, the bChow ring does *not* depend upon the choice of a normal crossings divisor $D \subset X$. However, given such a choice there is always a tower of natural inclusions

$$\mathrm{CH}^*(X) \subset \log\mathrm{CH}^*(X) \subset \mathrm{bCH}^*(X).$$

Since the centers of the blow-up are so restricted in the definition of $\log\mathrm{CH}^*(X)$, we view $\mathrm{CH}^*(X)$ and $\log\mathrm{CH}^*(X)$ as relatively close in size. On the other hand, $\mathrm{bCH}^*(X)$ is very much larger.

Let $\mathrm{divbCH}^*(X)$ be the subalgebra of $\mathrm{bCH}^*(X)$ generated by divisors. More precisely,

$$\mathrm{divbCH}^*(X) = \varinjlim_{Y \in \mathbf{B}(X)} \mathrm{divCH}^*(Y).$$

While the proof of the claim

$$\lambda_g \in \mathrm{divlogCH}^*(\overline{\mathcal{M}}_g, \partial\overline{\mathcal{M}}_g)$$

depended upon special properties of λ_g , the parallel bChow statement

$$\lambda_g \in \mathrm{divbCH}^*(\overline{\mathcal{M}}_g)$$

immediately follows from a general result.

THEOREM 20. *For every non-singular quasi-projective variety³¹ X , bChow is generated by divisor classes,*

$$\mathrm{divbCH}^*(X) = \mathrm{bCH}^*(X).$$

Proof. Let $\alpha \in \mathrm{CH}^*(Y)$ for an object $[Y \rightarrow X]$ in $\mathbf{B}(X)$. We will find a refinement $Z \rightarrow Y$ for which

$$f^*a \in \mathrm{divCH}(Z).$$

Since Y is non-singular and quasi-projective, the Chern classes of vector bundles generate $\mathrm{CH}^*(Y)$. We can assume $\alpha = c_i(E)$ for a vector bundle E on Y . By [Hir64, Corollary 2], there is a blow-up

$$g : W \rightarrow Y$$

where W is non-singular and g^*E contains a subline bundle L ,

$$0 \rightarrow L \rightarrow g^*E \rightarrow g^*E/L \rightarrow 0.$$

Applying the same argument to the quotient bundle g^*E/L , we find inductively a non-singular blow-up

$$f : Z \rightarrow Y$$

for which f^*E has a filtration with line bundles as quotients. Therefore,

$$f^*c_i(E) = c_i(f^*E)$$

is in $\mathrm{divCH}^*(Z)$. □

³¹ The statement holds verbatim for non-singular Deligne–Mumford stacks which admit finite resolutions of sheaves by vector bundles.

The quasi-projective hypothesis is used only for vector bundle resolutions. In fact, the hypothesis is not necessary. Theorem 20 can be proven locally near any cycle

$$S \subset X$$

by successive blow-ups along non-singular centers to resolve S and appropriately modify the Chern classes of the normal bundle of S . We leave the details for the interested reader.

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Appendix A. The fourth cohomology group of $\overline{\mathcal{M}}_g$

In the proof of Theorem 4, we require the equality³²

$$H^4(\overline{\mathcal{M}}_g) = \text{RH}^2(\overline{\mathcal{M}}_g). \tag{A.1}$$

for sufficiently large g . In other words, the fourth cohomology group of $\overline{\mathcal{M}}_g$ is spanned by tautological classes for sufficiently high g .

Equality (A.1) was first proven by Edidin [Edi92] for $g \geq 12$. Edidin bounded the Betti number $h^4(\overline{\mathcal{M}}_g)$ from above and then showed by intersection calculations that the span of the tautological classes³³ in codimension 2 achieves the required rank. Edidin used the interior result

$$H^4(\mathcal{M}_g) = \text{RH}^2(\mathcal{M}_g) \tag{A.2}$$

proven by Harer [Har85] for $g \geq 12$. The interior statement (A.2) was later proven for $g \geq 9$ by Ivanov [Iva93] and strengthened further to $g \geq 7$ by Boldsen [Bol12] which improved Edidin’s bound.

THEOREM A.1 [Edi92, Iva93, Bol12]. *We have $H^4(\overline{\mathcal{M}}_g) = \text{RH}^2(\overline{\mathcal{M}}_g)$ for $g \geq 7$.*

³² We use, as before, the complex grading on RH^* .

³³ Edidin does not use the language of tautological classes as we now do, but all of his generators are in fact tautological: they are given by the classes κ_2, κ_1^2 , push-forwards of λ - and ψ -classes under boundary divisor gluing maps, and fundamental classes of strata of codimension 2.

Appendix B. Computations in *admcycles*

B.1 Verification of Pixton's conjecture

In [Pix12], Pixton proposed a set of relations between tautological classes on the moduli spaces $\overline{\mathcal{M}}_{g,n}$ of stable curves. These were proven to hold in cohomology [PPZ15] and in Chow [Jan17]. Furthermore, Pixton conjectured that his relations span the *complete* set of relations among tautological classes. The relations were implemented by Pixton in the mathematical software SageMath [Sag20] and later incorporated in the SageMath package *admcycles*. Assuming Pixton's conjecture, the software computes a basis of the \mathbb{Q} -vector spaces $R^d(\overline{\mathcal{M}}_{g,n})$ and express tautological classes in the basis.

In Proposition 5, we state that Pixton's conjecture holds for the spaces

$$R^4(\overline{\mathcal{M}}_{4,1}) \quad \text{and} \quad R^5(\overline{\mathcal{M}}_{5,1}).$$

Assuming the conjecture, *admcycles* computes the rank of these two spaces to be 191 and 1371, respectively. If the conjecture were false, the rank of one (or both) of the groups would have to be strictly smaller. However, using *admcycles*, we verify that the ranks of the intersection pairings

$$R^4(\overline{\mathcal{M}}_{4,1}) \otimes R^6(\overline{\mathcal{M}}_{4,1}) \rightarrow \mathbb{Q} \quad \text{and} \quad R^5(\overline{\mathcal{M}}_{5,1}) \otimes R^8(\overline{\mathcal{M}}_{5,1}) \rightarrow \mathbb{Q}$$

are bounded from below by 191 and 1371, respectively. The rank bounds are obtained by taking generating sets of $R^4(\overline{\mathcal{M}}_{4,1})$ and $R^5(\overline{\mathcal{M}}_{5,1})$ and computing the matrix of pairings with generators in $R^6(\overline{\mathcal{M}}_{4,1})$ and $R^8(\overline{\mathcal{M}}_{5,1})$, respectively. For the rank bounds of pairing, we do *not* assume anything about the relations between the above generators, though we are allowed to use the known relations [PPZ15] to reduce the size of the generating sets.

The computations were performed on a server of the Max Planck Institute for Mathematics in Bonn,³⁴ taking two days in the case of $\overline{\mathcal{M}}_{4,1}$ and 31 days for $\overline{\mathcal{M}}_{5,1}$. Without substantial improvements to the algorithm, it is thus unlikely that Pixton's conjecture can be verified in this way for significantly larger g , n , and d . We warmly thank the Max Planck Institute for providing the computer infrastructure for our computations.

B.2 Computations in proofs of Theorems 3 and 4

Once we have verified Pixton's conjecture (as above)³⁵ for $RH^d(\overline{\mathcal{M}}_{g,n})$, we can explicitly check whether

$$\lambda_d \in RH_{\leq k}^d(\overline{\mathcal{M}}_{g,n}).$$

Several such checks used in the proofs of Theorems 3 and 4 were made using *admcycles*.

We provide below an example of the computation showing that the class λ_3 is not contained in the space

$$\text{div}RH^3(\overline{\mathcal{M}}_3) \subset RH^3(\overline{\mathcal{M}}_3),$$

which is a nine-dimensional subspace of a 10-dimensional space. We first create the list `divcl` of divisor classes on $\overline{\mathcal{M}}_3$, compute the set of triple products of such classes, and then take the span `divR` of the vectors representing them in a basis of $RH^3(\overline{\mathcal{M}}_3)$. We verify that `divR` is 9-dimensional inside the 10-dimensional ambient space $RH^3(\overline{\mathcal{M}}_3)$. Finally, we compute the class λ_3 and verify that the associated vector `Lv` is not contained in `divR`.

³⁴ The program ran on a single thread of the available CPU (Intel Xeon Prozessor E5-2667 v2), taking about 60 GB of RAM due to the large amounts of intermediate data to store (the list of Pixton's relations, sets of tautological generators, etc.).

³⁵ Our verification method also then shows that $R^d(\overline{\mathcal{M}}_{g,n}) = RH^d(\overline{\mathcal{M}}_{g,n})$.

```
sage: from admcycles import *
sage: divcl = tautgens(3,0,1)
sage: divp = [a*b*c for a in divcl for b in divcl for c in divcl]
sage: divR = span(u.toTautbasis() for u in divp)
sage: (divR.rank(), divR.degree())
(9, 10)
sage: L = lambdaclass(3,3,0)
sage: Lv = L.toTautbasis()
sage: Lv in divR
False
```

B.3 Proof of Proposition 7

We record below the computation in *admcycles* used in the proof of Proposition 7. We create the classes $\lambda_2, [\Delta_0], [B]$, and $[C]$ and represent the class defined by

$$2\lambda_2 - x \cdot [\Delta_0]^2 - y \cdot [B] - z \cdot [C]$$

in the vector diff with respect to a basis of $\text{CH}^2(\overline{\mathcal{M}}_2) = \mathbb{R}^2(\overline{\mathcal{M}}_2)$. We then solve the equation $\text{diff}=0$ to find the formula for x and y in terms of the variable z used in the proof.

We remark that in the definition of the class `Delta0` we need to divide by 2 since this is the degree of the gluing morphism parameterizing the boundary divisor Δ_0 .

```
sage: from admcycles import *
sage: lambda2 = lambdaclass(2,2,0)
sage: Delta0 = 1/2 * irrdiv(2,0)
sage: gammaB = StableGraph([0], [[1,2,3,4]], [(1,2), (3,4)])
sage: B = gammaB.boundary_pushforward()
sage: gammaC = StableGraph([0,1], [[1,2,3], [4]], [(1,2), (3,4)])
sage: C = gammaC.boundary_pushforward()
sage: x, y, z = var('x, y, z')
sage: diff = (2*lambda2 - x*Delta0^2 - y*B - z*C).toTautbasis()
sage: diff
(476*x + 1824*y - 96*z - 3, -144*x - 576*y + 24*z + 1)
sage: solve([diff[i]==0 for i in (0,1)], x,y,z)
[[x == r1 - 1/120, y == -5/24*r1 + 11/2880, z == r1]]
```

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