Proc. Camb. Phil. Soc. (1965), **61**, 29
Printed in Great Britain

A note on B^*-algebras

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(*Received* 4 *June* 1964)

A normed algebra A is a *pre-B^*-algebra* if its norm satisfies $\|x^*x\| = \|x\|^2$ for all elements $x \in A$; if A is also complete in its norm, then A is a *B^*-algebra* (see (1), page 180). In the study of certain locally convex algebras, the problem arose of expressing the condition that an algebra be a pre-B^*-algebra in terms of its properties as a locally convex algebra, rather than in terms of the norm. A solution to this problem is presented in this note; the application to the theory of locally convex algebras will appear elsewhere.

Let A be a complex normed algebra with an involution $x \to x^*$ ((1), page 178). For simplicity, it will also be assumed that A contains an identity element e; this restriction may easily be removed. The algebra A is said to be *symmetric* if $e + x^*x$ has an inverse for every $x \in A$. Let $\mathfrak{B}$ be the collection of all subsets B of A such that (i) B is absolutely convex, (ii) $B^2 \subset B$, (iii) $B^* = B$, and (iv) B is bounded and closed. The following two theorems will be proved.

THEOREM 1. *If A is a complex normed algebra with identity e and a continuous involution $x \to x^*$, then A is a pre-B^*-algebra (in some norm equivalent to the given one) if and only if*:

(i) *the collection $\mathfrak{B}$ has a greatest member*; *and*

(ii) *for each $x \in A$, there are sequences (u_n), (v_n) in A such that*

$$u_n(e + x^*x) \to e, \quad (e + x^*x)v_n \to e.$$

THEOREM 2. *If A is a complex Banach algebra with identity e and involution $x \to x^*$, then A is a B^*-algebra (in some norm equivalent to the given one) if and only if*:

(i) *the collection $\mathfrak{B}$ has a greatest member*; *and*

(ii) *A is symmetric.*

First, it is remarked that, if A is a pre-B^*-algebra, then it is not difficult to show that the unit ball of A is the greatest member of $\mathfrak{B}$. Also, it is well known that any B^*-algebra is symmetric ((1), page 243). If A is a pre-B^*-algebra, then its completion $\hat{A}$ is a B^*-algebra; thus, in Theorem 1 (ii) both (u_n) and (v_n) may be taken as any sequence in A that tends to $(e + x^*x)^{-1} \in \hat{A}$. Thus one half of each of the above theorems is fairly trivial and we shall merely give the detailed proofs of the non-trivial halves. It is convenient to break the proof into a number of lemmas.

LEMMA 1. *Let A be a complex normed algebra with identity e and involution $x \to x^*$. Let A_1 be a $*$-subalgebra of A that contains e and is such that, for each $x \in A_1$, the element $(e + x^*x)^{-1}$ exists in A. Then*:

(i) *if $h = h^* \in A_1$, $Sp_A(h)$ is real*;

(ii) *if* $x \in A_1$, $Sp_A(x^*x)$ *is real and non-negative.*

(For any $x \in A$, $Sp_A(x)$ denotes the spectrum of x in A.)

The proof of this lemma is very straightforward and we shall omit it. (It is similar to (1) (4·1·7), page 184 and (4·7·6), page 233.)

In Lemmas 2–6, A will be a complex normed algebra with identity e and continuous involution $x \to x^*$ and we shall suppose that A satisfies the conditions (i) and (ii) of Theorem 1. Since the involution is continuous it may be assumed that $\|x\| = \|x^*\|$ so that the unit ball U belongs to $\mathfrak{B}$. Thus, if B is the greatest member of $\mathfrak{B}$,

$$U \subset B \subset \alpha U,$$

for some $\alpha > 0$. It follows that the Minkowski functional of B defines a norm on A that is equivalent to the given one. Thus, without loss of generality, it may be supposed that B is the unit sphere of A; this will be done henceforth. Let $\hat{A}$ be the completion of A; since the map $x \to x^*$, of A into itself, is continuous, it has a unique extension to a map of $\hat{A}$ into itself, which will also be denoted by $x \to x^*$. This map is clearly an involution on $\hat{A}$.

LEMMA 2. *If* $h = h^* \in \hat{A}$ *then* $\|h\| = r_{\hat{A}}(h)$. ($r_{\hat{A}}(h)$ *denotes the spectral radius of* h *in* $\hat{A}$.)

Proof. Suppose first that $h = h^* \in A$. Then clearly $h^2 \in B$ if $h \in B$.

Now let $h^2 \in B$ and choose real $\alpha > 1$ such that $h \in \alpha B$. Then, for $n = 1, 2, \ldots$, $h^{2n} \in B^n \subset B \subset \alpha B$ and $h^{2n+1} \in \alpha BB \subset \alpha B$. Thus if $C = \{e, h, h^2, \ldots\}$, C is bounded and clearly $C^2 \subset C$. It is now simple to show that the closed absolutely convex hull of C belongs to $\mathfrak{B}$ and thus $C \subset B$. In particular, $h \in B$.

Thus $h \in B$ if and only if $h^2 \in B$, so that $\|h^2\| = \|h\|^2$.

Suppose now that $h = h^* \in \hat{A}$. Let (x_n) be some sequence in A such that $x_n \to h$. Then if $h_n = \frac{1}{2}(x_n + x_n^*)$, it follows that $h_n = h_n^* \in A$ and $h_n \to h$. Thus

$$\|h^2\| = \lim_n \|h_n^2\| = \lim_n \|h_n\|^2 = \|h\|^2.$$

The result of the lemma now follows from the formula for spectral radius in a Banach algebra: $r(x) = \lim_n \|x^n\|^{1/n}$.

LEMMA 3. *If* $x \in A$ *then* $(e + x^*x)^{-1}$ *exists in* $\hat{A}$.

Proof. Let $y = e + x^*x$. By hypothesis (ii) of Theorem 1, there is a sequence (u_n) in A such that $u_n y \to e$. Hence there is some integer N such that $\|u_N y - e\| < 1$ and hence such that $u_N y$ is invertible in $\hat{A}$. Thus there is some $u \in \hat{A}$ such that $(uu_N) y = e$; similarly there is some N' and some $v \in \hat{A}$ such that $y(v_{N'} v) = e$. Thus y is invertible in $\hat{A}$.

LEMMA 4. $\hat{A}$ *is a symmetric Banach algebra.*

Proof. Let $x \in \hat{A}$; it must be shown that $e + x^*x$ is invertible in $\hat{A}$.

Choose a sequence (x_n) in A such that $x_n \to x$. Put $y = e + x^*x$, $y_n = e + x_n^* x_n$; then $y_n \in A$, $y_n \to y$. By Lemma 3, y_n^{-1} exists in $\hat{A}$ for each n and so, by Lemma 1 (with $\hat{A}$ for A and A for A_1), $Sp_{\hat{A}}(x_n^* x_n)$ is real and non-negative for each n. Thus

$$Sp_{\hat{A}}(y_n) \subset [1, \infty) \quad \text{and so} \quad Sp_{\hat{A}}(y_n^{-1}) \subset (0, 1]$$

for each n.

https://doi.org/10.1017/S0305004100038597 Published online by Cambridge University Press

Thus, by Lemma 2 (since y_n^{-1} is clearly Hermitian),

$$\|y_n^{-1}\| = r_{\hat{A}}(y_n^{-1}) \leqslant 1 \quad (n = 1, 2, \ldots).$$

Hence
$$\begin{aligned}\|y_n^{-1} - y_m^{-1}\| &= \|y_m^{-1}(y_m - y_n)\, y_n^{-1}\| \\ &\leqslant \|y_m^{-1}\|\, \|y_m - y_n\|\, \|y_n^{-1}\| \\ &\leqslant \|y_m - y_n\|.\end{aligned}$$

Since $y_n \to y$ it follows that (y_n^{-1}) is Cauchy and so has a limit $z \in \hat{A}$. Clearly

$$yz = zy = e.$$

This concludes the proof.

LEMMA 5. *Any maximal commutative *-subalgebra of A is a B^*-algebra.*

Proof. Let C be any maximal commutative *-subalgebra of $\hat{A}$. Since, by Lemma 4, $\hat{A}$ is symmetric, C is symmetric. Also $r_C(x) = r_{\hat{A}}(x) \equiv r(x)$, say $(x \in C)$. If $x = h + ik$ (h, k Hermitian) then $h, k \in C$, $hk = kh$ and $r(x) \leqslant \|x\| \leqslant \|h\| + \|k\| = r(h) + r(k) \leqslant 2r(x)$, the last inequality following since, e.g. $h = \frac{1}{2}(x + x^*)$ (see (1), page 10).

But, restricted to the commutative algebra C, r is a seminorm and in fact, by what has just been shown, it is a norm that is equivalent (on C) to $\|.\|$.

Since C is symmetric, it is well known ((1), pages 189 and 233) that

$$r(x^*x) = r(x)^2 \quad (x \in C),$$

so that C is a B^*-algebra under the norm r, and r is equivalent to the given norm.

LEMMA 6. *$\hat{A}$ is semi-simple.*

Proof. Let R be the radical of $\hat{A}$; for $x \in R$ let $x = h + ik$ (h, k Hermitian). Then $x^* \in R$ ((1), page 179) so $h, k \in R$ and thus $r(h) = r(k) = 0$ ((1), page 56). Thus, by Lemma 2, $h = k = 0$ and so $x = 0$. This proves the lemma.

Proof of Theorem 1. By Lemmas 4 and 6, $\hat{A}$ is a semi-simple symmetric Banach algebra. Hence ((1), pages 236–237) $\hat{A}$ has a faithful *-representation, $x \to T_x$, by an algebra of bounded operators on some Hilbert space H; let $\|.\|_T$ denote the operator norm on $\hat{A}$.

Let $h = h^* \in \hat{A}$ and let C_h be a maximal commutative *-subalgebra of $\hat{A}$ that contains h. By Lemma 5, C_h is a B^*-algebra under the norm r; the restriction to C_h of the representation $x \to T_x$ is thus a *-isomorphism of one B^*-algebra into another (the algebra of bounded operators on H) and this map is therefore an isometry ((1), page 241). Hence, in particular

$$\|h\|_T = r(h) = \|h\|,$$

for any Hermitian $h \in \hat{A}$.

Thus for $x \in \hat{A}$, $x = h + ik$ (h, k Hermitian),

$$\begin{aligned}\|x\| \leqslant \|h\| + \|k\| = \|h\|_T + \|k\|_T \leqslant 2\|x\|_T &\leqslant 2(\|h\|_T + \|k\|_T) \\ &= 2(\|h\| + \|k\|) \\ &\leqslant 4\|x\|,\end{aligned}$$

so that $\|.\|$ and $\|.\|_T$ are equivalent norms on $\hat{A}$. In fact these norms must be identical (on A and hence on $\hat{A}$) since the unit sphere of A with respect to either of them is the set B. Hence $(A; \|.\|)$ is a pre-B^*-algebra and the proof is complete.

Proof of Theorem 2. Let A be a complex Banach algebra that satisfies conditions (i) and (ii) of Theorem 2, the involution in A not being given as continuous. If $\|.\|$ is the given norm on A, a new norm is defined by

$$|x| = \max(\|x\|, \|x^*\|) \quad (x \in A).$$

Then $(A; |.|)$ is a normed algebra (not necessarily complete) and clearly

$$|x| = |x^*|, \quad \|x\| \leqslant |x|, \quad \|x^*\| \leqslant |x| \quad (x \in A).$$

Also if $D \subset A$ and $D = D^*$ then D is $|.|$-bounded if and only if it is $\|.\|$-bounded and so in particular, the collection $\mathfrak{B}$ is the same for the algebra A under either norm. The algebra $(A; |.|)$ thus satisfies all the conditions of Theorem 1 and hence it is a pre-B^*-algebra under a norm equivalent to $|.|$.

Thus $(A; \|.\|)$ is an A^*-algebra ((1), page 181) with an auxiliary norm that is equivalent to $|.|$. Hence the involution on A must be continuous under the given norm ((1), page 187). Thus $\|.\|$ and $|.|$ are equivalent norms so that A is a B^*-algebra under a norm equivalent to the given one. This concludes the proof.

It is simple to find examples that show that both the conditions (i) and (ii) of the theorems are necessary.

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