Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2021.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

105.A (Stan Dolan)

The three face angles at one vertex of a tetrahedron are right angles. Find, with proof, all such tetrahedra for which each face has integer area and the insphere has unit diameter.

105.B (Michael Fox)

In the unsymmetrical plane quadrilateral *ABCD*, which can be convex, concave or crossed, the edges *AB* and *CD* are not perpendicular. Prove that the nine-point circles of triangles *BCD*, *ACD*, *ABD* and *ABC* meet at a unique point *N*. Prove also that if some or all of the four vertices are replaced by their reflections in *N*, then each new nine-point circle passes through *N* and has the same radius as the circle it replaces.

105.C (Dorin Marghidanu)

For $0 < x < \frac{1}{2}\pi$, prove that

$$\left(\sin^2 x\right)^{\sin^2 x}\left(\cos^2 x\right)^{\cos^2 x} + \left(\sin^2 x\right)^{\cos^2 x}\left(\cos^2 x\right)^{\sin^2 x} \leqslant 1.$$

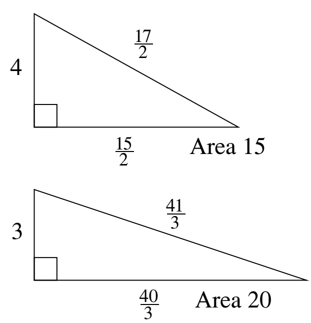
105.D (Isaac Sofair)

Prove that $\displaystyle\int_0^{\pi/3} \tan^{-1}\sqrt{\sec\theta + 1}\,d\theta = \frac{5\pi^2}{48}$.

Solutions and comments on **104.E**, **104.F**, **104.G**, **104.H** (July 2020).

104.E (Stan Dolan)

Consider the products obtained by multiplying pairs of numbers from a Pythagorean triple. For the (3, 4, 5) triple, the two largest products of 15 and 20 are both the areas of the right-angled triangles with rational sides shown in the Figure.



Is this result true for all Pythagorean triples?

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Answer: Yes, the result true for all Pythagorean triples.

Most solvers employed the standard parametrisation of Pythagorean triples but Steve Abbott, Prithwijit De, M. G. Elliott, James Mundie and Krish Nigam avoided this by the following argument.

In the Pythagorean triple $a^2 + b^2 = c^2$, the two largest products are ac and bc. Then $b^2 + \left(\frac{2ac}{b}\right)^2 = \frac{(c^2 - a^2)^2 + 4a^2c^2}{b^2} = \left(\frac{a^2 + c^2}{b}\right)^2$, and the area of the right-angled triangle with rational sides b, $\frac{2ac}{b}$, $\frac{a^2 + c^2}{b}$ is ac.

Similarly, switching a and b shows that the area of the right-angled triangle with rational sides a, $\frac{2bc}{a}$, $\frac{b^2 + c^2}{a}$ is bc.

James Mundie embedded the problem in the framework of elliptic curves, finding other pairs of rational Pythagorean triples than those given above and conjecturing that there are, in fact, infinitely many such pairs.

Correct solutions were received from: S. Abbott, M. V. Channakeshava, N. Curwen, P. De, M. G. Elliott, G. Howlett, P. F. Johnson, P. King, J. A. Mundie, K. Nigam, J. Siehler, C. Starr, A. Tee and the proposer Stan Dolan.

104.F (Martin Lukarevski)

Let m be an arbitrary integer and let $k = 3m + 1$. Find all integer solutions x and y of the equation

$$(x + y - k)(x + y + k) = 1 + xy.$$

Answer: There are no integer solutions.

Most solvers found a succinct solution by working mod 3 and we summarise the most common approaches below.

The equation $(x + y - k)(x + y + k) = 1 + xy$ may be rewritten as

$$x^2 + xy + y^2 = k^2 + 1. \qquad (*)$$

Checking all possible values of x, y (mod 3) shows that $x^2 + xy + y^2$ only takes the values 0, 1 (mod 3) whereas $k^2 + 1 \equiv 2 \pmod 3$. Hence there are no solutions.

Alternatively, $(*)$ may be rewritten as

$$(x - y)^2 + 3xy = k^2 + 1.$$

But squares are congruent to 0, 1 (mod 3) whereas $k^2 + 1 \equiv 2 \pmod 3$.

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The same argument works on rewriting (∗) as

$$(x - y)^2 + 3(x + y)^2 = 4\left(k^2 + 1\right).$$

The same methods show that (∗) also has no solutions when $k \equiv 2 \pmod 3$.

Peter Johnson investigated the case $k \equiv 0 \pmod 3$ where (∗) may be written as $x^2 + xy + y^2 = 9K^2 + 1$ and showed that the complete solution set is based on $(x, y, K) = \left(p_n^2,\ 3q_n^2,\ p_nq_n\right)$ where $p_n + q_n\sqrt{3} = \left(2 + \sqrt{3}\right)^n$, together with $(1, -1,\ 0)$ and allowed automorphisms of the solution set.

Correct solutions were received from: S. Abbott, M. V. Channakeshava, N. Curwen, S. Dolan, M. G. Elliott, A. P. Harrison, G. Hasanzade, G. Howlett, P. F. Johnson, K. Nigam, J. Siehler, C. Starr, L. Y. J. Wang, L. Wimmer and the proposer Martin Lukarevski.

104.G (Michael Fox)

A line r passes through the midpoints of the direct common tangents d and d' of two given circles u and v. A circle w with centre W touches u and v, the contacts being either both external or both internal. Prove that lines drawn through W perpendicular to d and d' meet r in points J and K that lie on the circle w.

Is there a corresponding property if w, still touching both u and v, surrounds u but excludes v?

The solutions received were all carefully argued and employed methods ranging from coordinate geometry and conics to trigonometry and inversion. The succinct solution below was submitted by Stan Dolan.

By definition, the midpoint of a common tangent to u and v (either direct or transverse) has the same power with respect to each circle. The line r is therefore the radical axis.

As in Figure 1, let the circles u, v, w have radii u, v, w, respectively. Let the centres U, V, W of u, v, w be distances a, b, c, respectively from the radical axis and let W be distance y from the line of centres of u and v.

Note that common direct tangents to u and v are at angles $\pm \sin^{-1}\left(\frac{u - v}{a + b}\right)$ to their line of centres. These angles are $\pm \sin^{-1}\left(\frac{u + v}{a + b}\right)$ for common transverse tangents.

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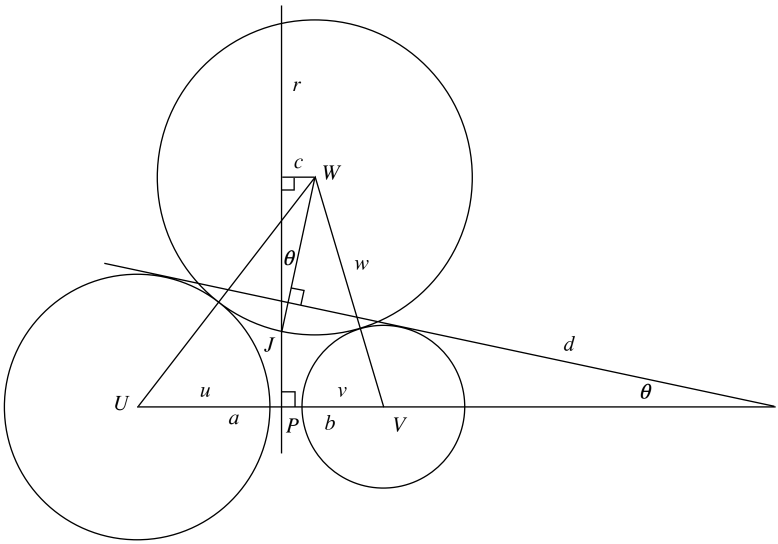

FIGURE 1

(i) *If both contacts are external*

Then

$$a^2 - u^2 = b^2 - v^2 \tag{1}$$

(since the point P has the same power with respect to circles u and v).

Also, from Figure 2,

$$(w + u)^2 = (a + c)^2 + y^2 \tag{2}$$

and

$$(w + v)^2 = (b - c)^2 + y^2. \tag{3}$$

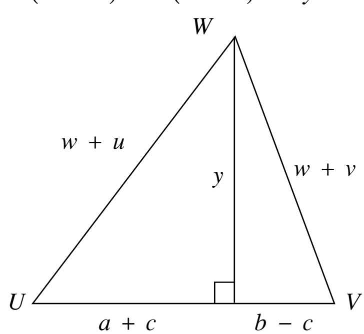

FIGURE 2

Then (1) + (2) − (3) gives $c = \dfrac{w(u - v)}{a + b}$.

Since $\sin\theta = \dfrac{u - v}{a + b}$, we have $c = w\sin\theta$ so J lies on w (see Figure 1). The same argument shows that K also lies on w.

(ii) *If both contacts are internal*

The same equations hold as in (i), but with u and v replaced by $-u$ and $-v$. This does not affect $\pm\sin^{-1}\left(\dfrac{u - v}{a + b}\right)$.

(iii) *If the contact is internal and one external*

The same equations hold with the sign of one of u or v changed. The property therefore still holds, but with transverse tangents.

Correct solutions were received from: S. Dolan, M. G. Elliott, A. P. Harrison, G. Howlett, P. F. Johnson and the proposer Michael Fox.

104.H (Abdurrahim Yilmaz)

Two particles P_1 and P_2 are moving in the xy-plane. P_1 is moving up the y-axis with speed v; at time $t = 0$ it is at the origin. P_2 is always moving straight towards P_1 with speed kv, $k > 1$; at time $t = 0$ it is at the point $A(a, 0)$ with $a > 0$. The particles stop when P_2 meets P_1.

(a) Find the value of $a - k$ if the difference in the distances travelled by P_1 and P_2 is numerically equal to k.

(b) Find the only pair of integers (k, a) such that the length of the path of P_2 is numerically equal to the area of the region bounded by the path and the axes.

Answer: (a) $a - k = 1$; (b) $k = 2, a = 10$.

The first stage in the solution is to derive the equation of the pursuit curve followed by P_2 and solvers used a standard approach to this.

At time t, the positions of the particles are $P_1(0, vt)$ and $P_2(x, y)$ where $\dfrac{dy}{dx} = \dfrac{y - vt}{x}$.

Differentiating $x\dfrac{dy}{dx} = y - vt$ and writing s for arc-length gives

$$x\frac{d^2y}{dx^2} = -v\frac{dt}{dx} = -v\frac{dt}{ds}\frac{ds}{dx} = \frac{1}{k}\sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

Separating variables and integrating yields

$$\sinh^{-1}\left(\frac{dy}{dx}\right) = \frac{1}{k}(\ln x - \ln a) = \ln\left(\frac{x}{a}\right)^{1/k}$$

and hence

$$\frac{dy}{dx} = \frac{1}{2}\left[\left(\frac{x}{a}\right)^{1/k} - \left(\frac{x}{a}\right)^{-1/k}\right].$$

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A final integration gives the pursuit curve equation for P_2,

$$y = \frac{k}{2}\left[\frac{a^{-1/k}x^{1+1/k}}{k+1} - \frac{a^{1/k}x^{1-1/k}}{k-1}\right] + \frac{ak}{k^2-1}.$$

When the particles meet and stop, $x = 0$ and $y = \dfrac{ak}{k^2-1}$. This is the distance travelled by P_1; P_2 has thus travelled a distance $\dfrac{ak^2}{k^2-1}$.

For (a), we require $\dfrac{ak^2}{k^2-1} - \dfrac{ak}{k^2-1} = k$ which simplifies to $\dfrac{ak}{k+1} = k$ and $a - k = 1$.

Integrating the pursuit curve equation for P_2 shows that the area beneath it is

$$\int_0^a y\,dx = \frac{1}{2}k^2a^2\left[\frac{1}{(2k+1)(k+1)} - \frac{1}{(2k-1)(k-1)}\right] + \frac{a^2k}{k^2-1} = \frac{a^2k}{4k^2-1}.$$

For (b), we require $\dfrac{a^2k}{4k^2-1} = \dfrac{ak^2}{k^2-1}$ from which

$$a = \frac{k\left(4k^2-1\right)}{k^2-1} = 4k + \frac{3k}{k^2-1}.$$

Since k and $k^2 - 1$ are coprime, $k^2 - 1$ divides 3 so that $k = 2$ and $a = 10$.

Correct solutions were received from: N. Curwen, S. Dolan, M. G. Elliott, A. P. Harrison, G. Howlett, P. F. Johnson, J. A. Mundie, C. Starr and the proposer Abdurrahim Yilmaz.

N. J. L.

Robin Chapman

Readers will have been saddened at the news of Robin Chapman's death in October 2020. He spent most of his career in the Mathematics Department of the University of Exeter and his main research interests were in number theory, algebra, and combinatorics. He was also a prolific problem solver: his name and elegant solutions appeared in those journals with a problems section, including the *Gazette* and the *American Mathematical Monthly* where he frequently succeeded in solving most of the problems set. He also made other contributions to the *Gazette* as a long-standing referee, author of articles and book reviewer.

Graham Hoare

As this issue of the *Gazette* was going to press, we learned the sad news that Graham Hoare died in January 2021. He edited Problem Corner from 1983 to 2003 and made numerous other contributions to the work of the Mathematical Association. A full appreciation will be given in the July 2021 *Gazette*.

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