Proceedings of the Edinburgh Mathematical Society (2021) **64**, 1028–1036
doi:10.1017/S0013091521000766

A WEIGHT-HOMOGENOUS CONDITION TO THE REAL JACOBIAN CONJECTURE IN $\mathbb{R}^2$

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(Received 9 April 2020; first published online 19 November 2021)

Abstract It is known that a polynomial local diffeomorphism $(f,\, g) : \mathbb{R}^2 \to \mathbb{R}^2$ is a global diffeomorphism provided the higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ do not have real linear factors in common. Here, we give a weight-homogeneous framework of this result. Our approach uses qualitative theory of differential equations. In our reasoning, we obtain a result on polynomial Hamiltonian vector fields in the plane, generalization of a known fact.

Keywords: Real Jacobian conjecture; global injectivity; weight-homogeneous polynomials

2020 *Mathematics subject classification:* Primary: 14R15;
Secondary: 35F05; 35A30

1. Introduction

Let $F = (f,\, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map whose Jacobian determinant satisfies

$$\det DF(x, y) \neq 0 \tag{1}$$

for all $(x,\, y) \in \mathbb{R}^2$. The map F is locally a diffeomorphism but, after the family of counterexamples found by Pinchuk [15], we know that F is not necessarily globally injective. Pinchuk's counterexamples disprove the *real Jacobian conjecture*, i.e. the claim that polynomial maps satisfying (1) are injective.

A natural problem is then to look for additional conditions that guarantee the real Jacobian conjecture. For instance, if the Jacobian determinant of F is a *constant* different from zero, then its injectivity is unknown up to now, and this problem is part of the famous *Jacobian conjecture*, which is unsolved until these days, see [9].

Conditions on the degree of F were established in [1,3,13]. Conditions on the spectrum of DF, also valid for non-polynomial maps, can be found in [7,10]. The aim of this paper is to provide different conditions to the validity of the real Jacobian conjecture. Our

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https://doi.org/10.1017/S0013091521000766 Published online by Cambridge University Press

main result is Theorem 1, which turns out to be a generalization of the main result of [4]. Theorem 1 is also related to the work [6], as explained below. In order to enunciate the theorem, we need some preliminary concepts.

Let s_1 and s_2 be positive integers and set $s = (s_1, s_2)$. We say that a polynomial function $f : \mathbb{R}^2 \to \mathbb{R}$ is *s-weight-homogeneous* if there is a non-negative integer d such that

$$f(\alpha^{s_1}x, \alpha^{s_2}y) = \alpha^d f(x, y)$$

for all $\alpha \in \mathbb{R}$, $\alpha > 0$, and for all $(x, y) \in \mathbb{R}^2$. In this case, we call d the *weight-degree* of f and s the *weight-exponent* of f. When $s = (1, 1)$ we simply say that f is *homogeneous* of *degree* d. Given a weight-exponent s and a polynomial $f : \mathbb{R}^2 \to \mathbb{R}$, we can uniquely write $f = f_0 + f_1 + \cdots + f_r$ where f_i is a s-weight-homogeneous polynomial of weight degree i. In this case, when $f_r \neq 0$, we say that f_r is the *higher s-weight-homogeneous part of* f and we also say that r is the weight degree of f. It is straightforward to check the validity of the following *Euler formula* for a s-weight-homogeneous polynomial f with weight degree d:

$$df(x, y) = s_1 x f_x(x, y) + s_2 y f_y(x, y). \tag{2}$$

Here f_x (respectively f_y) is the partial derivative of f with respect to x (respectively y). It is also clear in this case that f_x (respectively f_y) is s-weight-homogeneous with weight degree $d - s_1$ (respectively $d - s_2$). Finally, if $p(x, y) = (ax + by)^k q(x, y)$, with p and q polynomial functions, k a positive integer and $a, b \in \mathbb{R}$, then we say that $ax + by$ is a *real linear factor* of f. We observe that a s-weight-homogeneous polynomial p, with $s_1 \neq s_2$, can have a real linear factor only in case $a = 0$ or $b = 0$. Now we can formulate our main result.

Theorem 1. *Let $F = (f, g) : \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map satisfying (1) and such that there is $z \in \mathbb{R}^2$ with $F(z) = (0, 0)$.*

(a) If either the higher homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ do not have real linear factors in common, or

(b) if the higher homogeneous term of $f^2 + g^2$ does not have a factor $(ax + by)^2$, with $ab \neq 0$, and there is a weight-exponent s such that the higher s-weight-homogeneous terms of the polynomials $ff_x + gg_x$ and $ff_y + gg_y$ do not have real linear factors in common,

then F is injective.

The main result of [4] is only statement (a) of Theorem 1, with $z = (0, 0)$. The following is an example where the injectivity follows from Theorem 1 but not from [4]. Let $F(x, y) = (x + y + x^2, y + x^2)$. We have $\det DF = 1$ and

$$ff_x + gg_x = x + y + 3x^2 + 4xy + 4x^3, \quad ff_y + gg_y = x + 2y + 2x^2.$$

The higher homogeneous terms of these polynomials are $4x^3$ and $2x^2$, respectively, and so the assumptions of [4] are not satisfied. Now the higher homogeneous term of $f^2 + g^2$ is $2x^4$ and, with weight exponent $s = (1, 2)$, the higher s-weight-homogeneous terms of

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the above polynomials are $4x(y + x^2)$ and $2(y + x^2)$, respectively, that do not have real linear factors in common. So F is injective by Theorem 1.

We point out that the assumptions on Theorem 1 are not necessary for the global injectivity of a polynomial local diffeomorphism, as can be seen by the polynomial diffeomorphism $G(x, y) = (x + (x - y)^2, y + (x - y)^2)$. Here $\det DG = 1$, the higher homogeneous part of $f^2 + g^2$ is $2(x - y)^4$ and the higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ are $4(x - y)^3$ and $-4(x - y)^3$, respectively, and so we can not use Theorem 1.

It is important to mention here that a standard fact in algebraic geometry is that if a polynomial map (f, g) satisfying (1) has no points at infinity in $\mathbb{RP}^2$, i.e., the higher homogeneous term of $f^2 + g^2$ has no real linear factors, then (f, g) is a proper map, and so it is a diffeomorphism, according to [16]. A generalization of this to the quasi-homogenous frame is the bidimensional counterpart of [6, Theorem A]: polynomial maps $F = (f, g)$ satisfying (1) and such that the higher s-weight-homogeneous parts of f and g have $(0, 0)$ as an isolated common zero, are injective. We observe that in the first example F above, the higher s-weight-homogenous parts of f and g are (y, y), $(y + x^2, y + x^2)$ or (x^2, x^2), depending whether $2s_1 < s_2$, $2s_1 = s_2$ or $2s_1 > s_2$, respectively. None of them have $(0, 0)$ as an isolated common zero, and hence this example (satisfying the hypotheses of Theorem 1) does not satisfy the assumptions of [6, Theorem A]. On the other hand, we do not know if our Theorem 1 implies this result of [6], although in case the s-weight degree of f and g is equal, for some weight s, it does, as proven in Lemma 6.

We emphasize that our proofs rely on qualitative theory of differential equations and uses a characterization of injectivity of F via centres of a suitable Hamiltonian vector field associated to F. In our reasoning, we prove a result on polynomial Hamiltonian vector fields in the plane, Proposition 4, which is a generalization of a result of [5] that we think is interesting on its own.

In § 2, we summarize this and other results needed to the proof of Theorem 1, which is performed in § 3.

After the completion of this work, we took knowledge of the paper [14], a partly expository paper with very nice connections between global injectivity and dynamics. The main result of [14] is that a polynomial map (f, g) satisfying (1) is globally injective provided the complexification of the algebraic curve $f = 0$ has *one place at infinity* (meaning that the curve $f = 0$ is irreducible and the pre-image of the *desingularization map* of the curve intersected with the infinity line in $\mathbb{CP}^2$ has only one point, see the precise definition in [14]). This result is different from our Theorem 1 as the polynomial local diffeomorphism $(f, g)(x, y) = (x + x^3, y + y^3)$ satisfies the assumption (a) of Theorem 1 but f and g are not irreducible, and so cannot have one place at infinity.

2. Preliminary results and a new condition for degenerate hyperbolic sectors at infinity

We begin this section by explaining the characterization of injectivity of polynomial maps mentioned in the introduction section. By a *centre* of a vector field $\mathcal{V}$, we mean as usually an equilibrium point v of $\mathcal{V}$ having a neighbourhood U such that $U \setminus \{v\}$ is filled with non-constant periodic orbits of $\mathcal{V}$. The *period annulus* of the centre is the maximum neighbourhood of v with this property. We say that a centre is *global* if its period annulus is the whole plane.

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In what follows we assume that $F = (f, g)$ is a polynomial map satisfying (1). Let the function $H : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$H(x, y) = \frac{f(x, y)^2 + g(x, y)^2}{2}$$

for $(x, y) \in \mathbb{R}^2$ and its associated Hamiltonian vector field $\chi = (P, Q)$, that is,

$$P = -H_y = -ff_y - gg_y, \quad Q = H_x = ff_x + gg_x.$$

We observe that $q \in \mathbb{R}^2$ is a singular point of χ if and only if $DF(q) \cdot q = (0, 0)$, which is equivalent to $F(q) = (0, 0)$ as $\det DF(q) \neq 0$. Let U be a neighbourhood of q where F is injective. It follows that H is positive in all the points of U different from q, while $H(q) = 0$, proving that q is an isolated minimum of H. Then all the orbits of χ in a neighbourhood of q (maybe smaller than U) are closed, proving that q is a centre of X. We state this result as a lemma for further reference.

Lemma 2. *The singular points of* χ *are the zeros of* F. *Each of them corresponds to a centre of* χ, *and so has index* 1.

The following is a generalization given in [2] of a result from [17], see also [11]:

Theorem 3. *Let* $F : \mathbb{R}^2 \to \mathbb{R}^2$ *be a polynomial map satisfying (1). Assume there is* $z \in \mathbb{R}^2$ *such that* $F(z) = (0, 0)$. *Then* F *is injective if and only if the centre* z *of* χ *is global.*

In what follows we use results and notation on the *Poincaré compactification* of polynomial vector fields of $\mathbb{R}^2$. Particularly, U_i, V_i, $i = 1, 2, 3$, are the canonical local charts of the *Poincaré sphere* $\mathbb{S}^2$. For details on this technique, we refer the reader to [8, Chapter 5] or to [12]. Letting X be a polynomial vector field of $\mathbb{R}^2$, we denote by $p(X)$ its compactification. As usual, we say that q is an *infinite singular point* of X, or of $p(X)$, if q is in the equator of $\mathbb{S}^2$. We also say that a hyperbolic sector h of q is *degenerate* if its two separatrices are contained in the equator of $\mathbb{S}^2$. Finally, by the *Poincaré disc*, we mean the projection of the north hemisphere together with the equator of $\mathbb{S}^2$ on the plane $z = 0$.

Next result studies the infinite singular points of a general polynomial Hamiltonian vector field, giving necessary conditions in order to have a non-degenerate hyperbolic sector. It turns out that the present result generalizes a similar result from [5], by considering also weight-homogeneous polynomials. We recall that for a Hamiltonian vector field $X = (-H_y, H_x)$, where $H : \mathbb{R}^2 \to \mathbb{R}$ is a polynomial, the infinite singular points of $p(X)$ in the Poincaré disc are the endpoints of each straight line $ax + by = 0$, where $ax + by$ is a real linear factor of the higher homogeneous part of H.

Proposition 4. *Let* q *be an infinite singular point of a Hamiltonian system* $X = (P, Q) = (-H_y, H_x)$ *(with* $PQ \not\equiv 0$*), endpoint of the straight line* $ax + by = 0$ *in the Poincaré disc. If* q *has a non-degenerate hyperbolic sector, then* $ax + by$ *is a common factor of the higher homogeneous parts of* P *and* Q. *If* $a = 0$ *(respectively* $b = 0$*), then* y

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(respectively x) is a common factor of the higher s-weight-homogeneous parts of P and Q, for each weight-exponent $s = (s_1, s_2)$.

Proof. We clearly can assume that the degree of H is greater than 1 and that the higher homogeneous term of H has the form

$$r(x, y)(ax + by)^\tau,$$

where $\tau \geq 1$ is an integer and $r(x, y)$ is a polynomial that does not have $ax + by$ as a factor.

Clearly if $ab \neq 0$ and $\tau \geq 2$, then the higher homogeneous terms of P and Q have both the factor $ax + by$, and we are done.

So it remains to consider the following three cases concerning the higher homogeneous term of H: (i) it has the form $r(x, y)(ax + by)$ with $ab \neq 0$, i.e., $\tau = 1$; or (ii) it has the form $r(x, y)y^\tau$, i.e., $a = 0$; or (iii) it has the form $r(x, y)x^\tau$, i.e., $b = 0$. By changing x and y we do not need to consider case (iii) (observe that with a change like that, a (s_1, s_2)-weight-homogeneous polynomial is carried to a (s_2, s_1)-weight-homogeneous polynomial). Also, with a linear change of variable we can transform (i) into (ii) and consider just the later case (because the degree of H is greater than 1). Our conclusion will show that case (ii) with $\tau = 1$ (and so case (i)) cannot happen.

So we assume that case (ii) is in force. Letting $s = (s_1, s_2)$ be a given weight-exponent, we denote by m and n the weight degrees with respect to s of P and Q, respectively. In the sequel, we shall use notation on the Poincaré compactification of X. Observe that q is the origin of the local chart U_1, that we will treat with the variables (u, v), with the relation between (x, y) and (u, v) given by $(u, v) = (y/x, 1/x)$. The equator of the Poincaré sphere, i.e., the infinite of $\mathbb{R}^2$ is mapped in the straight line $v = 0$. Let r_1 and r_2 be the two separatrices of a hyperbolic sector h of $(0, 0)$ in U_1. Without loss of generality, we assume that the interior of h is contained in the region $v > 0$. Suppose that r_1 is not contained in the infinite, i.e. in the straight line $v = 0$. We *claim that r_2 is not contained in the infinite and that r_1 and r_2 have the same tangent line at* $(0, 0)$. Indeed, assume on the contrary that there exists a straight half-line $u = \lambda v$, with $v > 0$, between r_1 and r_2. Each obit of X is contained in a level set of $H(x, y) = c$ of H. So, by letting

$$\tilde{H}(u, v) = v^{d+1} H\left(\frac{1}{v}, \frac{u}{v}\right) \tag{3}$$

and $\tilde{G}(u, v) = \tilde{H}(u, v)/v^{d+1}$, points (u, v) of the compactified orbit will satisfy $\tilde{G}(u, v) = c$. Let c be the value of $\tilde{G}$ in r_1. Since h is an hyperbolic sector, each sequence $\{w_n\}$ in the interior of h such that $\lim_{n\to\infty} w_n = (0, 0)$ will satisfy $\lim_{n\to\infty} \tilde{G}(w_n) = c$. So

$$\lim_{v\to 0} \tilde{G}(\lambda v, v) = c.$$

Then writing $\tilde{H} = \sum_{i=0}^{d+1} \tilde{H}_i$, with $\tilde{H}_i$ being the homogeneous part of degree i of $\tilde{H}$, we get $\tilde{H}_0(\lambda, 1) = \cdots = \tilde{H}_d(\lambda, 1) = 0$ and $\tilde{H}_{d+1}(\lambda, 1) = c$. Hence

$$\tilde{G}(\lambda v, v) = \frac{\tilde{H}_{d+1}(\lambda v, v)}{v^{d+1}} = c,$$

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meaning that the straight half-line $u = \lambda v$ is invariant by the flow, a contradiction. This proves the claim.

Clearly $\tilde{G}(u, v)$ have the same value c in r_1 and r_2, by the continuity of $\tilde{G}$ in $v > 0$. Let $u = \lambda v$, $v > 0$, the common tangent of r_1 and r_2 at $(0, 0)$. This line is contained in the tangent cone of the algebraic variety $\tilde{H}(u, v) - cv^{d+1} = 0$, with multiplicity at least two, i.e.,

$$\tilde{H}(u, v) - cv^{d+1} = \sum_{i=k}^{d+1} \tilde{\tilde{H}}_i(u, v),$$

with $k \geq 2$ and $\tilde{\tilde{H}}_k(u, v) = (u - \lambda v)^2 R(u, v)$, where $R(u, v)$ is a homogeneous polynomial of degree $k - 2$, and $\tilde{\tilde{H}}_i$ is the homogeneous part of degree i of $\tilde{H}(u, v) - cv^{d+1}$. Therefore, from (3), it follows that

$$H(x, y) = \frac{\tilde{H}(u, v)}{v^{d+1}} = c + \sum_{i=k}^{d+1} x^{d+1-i} \tilde{\tilde{H}}_i(y, 1).$$

Note that if $k = d + 1$ then $H(x, y) = \tilde{H}_{d+1}(y)$ which is not possible because then $Q \equiv 0$. So, $k < d + 1$, and Q contains the term $(d + 1 - k)x^{d-k} \tilde{\tilde{H}}_k(y, 1)$. Since $\tilde{H}_k = (y - \lambda)^2 \tilde{R}(y, 1)$ we get that n, which is the s-weight degree of Q, satisfies

$$n \geq (d - k)s_1 + 2s_2. \tag{4}$$

The s-weight-homogeneous part of weight degree n of Q writes

$$Q_n = \sum_{\substack{i,j \\ is_1 + js_2 = n}} a_{ij} x^i y^j.$$

Since the maximum exponent of x in H is $d + 1 - k$, and so the maximum possible exponent of x in Q_n is $d - k$, it follows that if $a_{ij} \neq 0$ in the above sum, then $(d - k)s_1 + js_2 \geq n \geq (d - k)s_1 + 2s_2$, from (4), forcing that j is at least 2. This means that

$$Q_n = y^2 T(x, y),$$

where $T(x, y)$ is a suitable s-weight-homogeneous polynomial of weight degree $n - 2s_2$. Since $Q_n = \partial H_{n+s_1}/\partial x$, where here H_{n+s_1} means the s-weight-homogeneous term of weight degree $n + s_1$ of H (recall that if H has s-weight degree ℓ then $\partial H/\partial x$, if not zero, has s-weight degree $\ell - s_1$), it thus follows that

$$H_{n+s_1} = y^2 \int T(x, y) + G(y) \tag{5}$$

for some polynomial $G(y)$ which must be a factor of y^2, otherwise H_{n+s_1} is not weight-homogeneous.

Now the higher s-weight-homogeneous term, P_m, of $P = -H_y$ comes from H_{m+s_2}. Clearly $m + s_2 \geq n + s_1$ by (5). In case $m + s_2 = n + s_1$, then P_m has a factor y. On the

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other hand, if $m+s_2>n+s_1$, it follows that $H_{m+s_2}=sy^j$, with $s_2j=m+s_2$, otherwise the higher s-weight-homogeneous term of Q is not Q_n because $m+s_2-s_1>n$. In particular, P_m has a factor y.

Observe that our proof shows in particular that the higher s-weight-homogeneous term of H has the factor $(ax+by)^2$ (homogeneous if $ab\neq 0$). In particular, case (i) of the beginning of the proof is not possible. □

Corollary 5. *Let $H\colon \mathbb{R}^2\to\mathbb{R}$ be a polynomial function. Let q be an infinite singular point of the polynomial Hamiltonian vector field $\chi=(-H_y,\,H_x)$, endpoint of the straight line $ax+by=0$ in the Poincaré disc. Assume either that the higher homogeneous terms of H_x and H_y do not have real linear factors in common, or, if $a=0$, (respectively $b=0$) that y (respectively x) is not a common factor of the higher s-weight-homogeneous parts of H_x and H_y, for some weight-exponent $s=(s_1,\,s_2)$. Then the topological index of q is greater than or equal to zero. If this index is zero, then q is formed by two degenerate hyperbolic sectors.*

Proof. From Proposition 4, it follows that q have no non-degenerate hyperbolic sectors. Thus, the number of hyperbolic sectors of q is $h\leq 2$. By the index formula we conclude that the index of q is greater than or equal to the number of elliptic sectors of q, and so greater than or equal to zero. If the index is zero, it thus clearly follows that there are no elliptic sectors and also that q must have two degenerate hyperbolic sectors. □

3. Proof of Theorem 1

We consider the function $H(x,\,y)=(f(x,\,y)^2+g(x,\,y)^2)/2$ defined in $\mathbb{R}^2$, and the associated Hamiltonian vector field $\chi=(-H_y,\,H_x)$. Since $F(z)=(0,\,0)$, it follows from Theorem 3 that in order to prove that F is injective, it is enough to prove that z is a global centre of the vector field χ.

Let q be an infinite singular point of χ, endpoint of the straight line $ax+by=0$ in the Poincaré disc. If we are under assumption (a) of Theorem 1, then we are under the assumptions of Corollary 5. If we assume (b) of the Theorem and $ab\neq 0$, then $ax+by$ is not a common factor of H_x and H_y (from (2)), and we are again under the assumptions of Corollary 5. On the other hand, if $ab=0$, assumption (b) guarantees the existence of a weight-exponent s satisfying the assumptions of Corollary 5. Thus, in any case, it follows from this corollary that the topological index of any infinite singular point of χ is greater than or equal to zero.

Also, the index of each finite singular point of χ is one, by Lemma 2. Corresponding to the singular point z of χ there are two singular points of $p(\chi)$, the Poincaré compactification of χ, one in each hemisphere of the Poincaré sphere, having index 1. Thus the sum of the indices of all the singular points of $p(\chi)$ in the Poincaré sphere is at least 2. From the Poincaré–Hopf theorem (see Theorem 6.30 in[8]), this sum must be 2. So, we conclude that $p(\chi)$ does not have other finite singular points (other than the two centres corresponding to the centre of χ) and $p(\chi)$ either does not have infinite singular points or each of them has index 0, which again by Corollary 5, must be formed by two degenerate hyperbolic sectors.

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Looking at the Poincaré disc, we summarize the frame: χ is a polynomial vector field such that its Poincaré compactification $p(\chi)$ in the Poincaré disc has a centre in its only finite singular point, and $p(\chi)$ either does not have infinite singular points or they are formed by degenerate hyperbolic sectors. From this, it is not difficult to conclude that z must be a global centre of χ (see, for instance, Corollary 10 and the proof of Theorem 1 in [4]).

We end the paper with the following Lemma, that gives a relation between our Theorem and the already mentioned result of [6].

Lemma 6. *Let $F = (f, g)\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a polynomial map and let $s = (s_1, s_2)$ be a weight-exponent. Assume that the weight degrees of f and g with respect to s are equal. If the higher s-weight-homogeneous terms of f and g do not have real linear factors in common, then so do the higher s-weight-homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$.*

Proof. We let m be the weight degree of f and g and write $f = f_0 + \cdots + f_m$ and $g = g_0 + \cdots + g_m$ the weight decomposition of f and g. We first observe that $(f_m^2 + g_m^2)_x \not\equiv 0$ and $(f_m^2 + g_m^2)_y \not\equiv 0$, because if $(f_m^2 + g_m^2)_x \equiv 0$, for instance, then (we are dealing with polynomials) $f_m = a_{0\ell} y^\ell$ and $g_m = b_{0\ell} y^\ell$, with $\ell s_2 = m$ and $a_{0\ell}, b_{0\ell} \in \mathbb{R}$, a contradiction.

Thus, the higher s-weigh-homogeneous parts of $ff_x + gg_x$ and $ff_y + gg_y$ are, respectively, $(f_m^2 + g_m^2)_x/2$ and $(f_m^2 + g_m^2)_y/2$. If there is a linear factor dividing the last polynomials, it will also divide

$$s_1 x \frac{\partial \left(f_m^2 + g_m^2\right)}{\partial x} + s_2 y \frac{\partial \left(f_m^2 + g_m^2\right)}{\partial y} = m \left(f_m^2 + g_m^2\right),$$

and so this factor will be common to f_m and g_m, a contradiction. □

This lemma is no longer true without the assumption that the weight degrees of f and g are the same, as the map $F(x, y) = (x + y^3, y - x^3)$ shows. It satisfies $\det DF = 1 + 9x^2y^2$ and, with $s = (3, 1)$, the higher homogeneous terms of f and g are $x + y^3$ and $= x^3$, respectively. But the higher homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ are $3x^5$ and $-x^3$, respectively. Here it is worth to mention that with $s = (5, 3)$, the higher s-weight-homogeneous terms of $ff_x + gg_x$ and $ff_y + gg_y$ are $3x^5$ and $-x^3 + 3y^5$, respectively. That is, this map satisfies the assumption of Theorem 1.

Acknowledgements. The first named author was partially supported by the grant 2017/00136-0, São Paulo Research Foundation (FAPESP). The second named author was partially supported by FCT/Portugal through UID/MAT/04459/2013.

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https://doi.org/10.1017/S0013091521000766 Published online by Cambridge University Press