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The harmonic functions of (A_t, B_t)[1]

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1. *Introduction*

The non-negative harmonic functions of a transient Markov process yield a great deal of information about the 'behaviour at infinity' of the process, and can be used to h-transform the process to behave in a certain way at infinity. The traditional analytic way of studying the non-negative harmonic functions is to construct the Martin boundary of the process (see, for example, Meyer[**4**], Kunita and T. Watanabe[**3**], and Kemeny, Snell & Knapp[**2**], Williams[**7**] for the chain case). However, certain conditions on the process need to be satisfied, one of the most basic of which is that there exists a reference measure η such that $U_\lambda(x, \cdot) \ll \eta$ for all $\lambda > 0$, all $x \in E$, the state space of the Markov process. (Here, $(U_\lambda)_{\lambda>0}$ is the resolvent of the process.)

An interesting example proposed by Erwin Bolthausen arises when we take a standard one-dimensional Brownian motion $(B_t)_{t\geqslant 0}$, and define

$$A_t \equiv \int_0^t I_{[0,\infty)}(B_s)\,ds.$$

Then the process $(X_t)_{t\geqslant 0} \equiv ((A_t, B_t))_{t\geqslant 0}$ with values in $E = \mathbb{R}^+ \times \mathbb{R}$ is a Feller–Dynkin process, and is transient. However, if this process starts at any point (a, y) with $y < 0$, then for any $t > 0$ we have $\mathbb{P}^{(a,y)}(A_t = a) > 0$, while $\mathbb{P}^{(a,y)}(A_t = b) = 0$ for $b \neq a$. It is easy then to see that for this example there can be no reference measure η with respect to which all the resolvent kernels have a density, so the question of discovering the Martin boundary is ill-posed. Nonetheless, there are harmonic functions for X, and invariant functions too. We shall characterize all invariant functions h, and shall give representations of all harmonic functions, though the exact class remains mysterious.

To be precise about our definitions, if (P_t) denotes the semigroup of X, we shall say that a function $h: E \to \mathbb{R}^+$ is

invariant if $P_t h = h$ for all $t \geqslant 0$; (1·i)

harmonic if $h(X_t)$ is a P^x-local martingale for all $x \in E$; (1·ii)

excessive if $P_t h \leqslant h$ for all $t \geqslant 0$ and $P_t h \uparrow h$ as $t \downarrow 0$. (1·iii)

We have always invariant $\Rightarrow$ harmonic $\Rightarrow$ excessive, because h is invariant if and only if $h(X_t)$ is a P^x-martingale for all x, and h is excessive if and only if $h(X_t)$ is a P^x-supermartingale for all x.

[1] It is a pleasure to thank Erwin Bolthausen for inviting me to visit Zürich in February 1992 under Swiss NSF grant 21-29833.90, during which time this work was undertaken.

The main result is the following:

THEOREM 1. *Suppose that $h : E \to \mathbb{R}^+$ is harmonic. Then there exists some measurable $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $t \geqslant 0 \geqslant x$*

$$h(t,x) = h(t,0) - x\rho(t). \tag{2}$$

The function ρ satisfies the integrability condition

$$g(a,0) \equiv \int_0^\infty \rho(a+s)\frac{ds}{\sqrt{(2\pi s)}} < \infty \quad \text{for all} \quad a \geqslant 0. \tag{3}$$

Defining

$$g(t,x) \equiv \begin{cases} \displaystyle\int_0^\infty \rho(t+s)\, e^{-x^2/2s}\frac{ds}{\sqrt{(2\pi s)}} & (x > 0) \\ g(t,0) - x\rho(t) & (x \leqslant 0) \end{cases} \tag{4}$$

the function g is invariant, and

(i) *if h is invariant, then we have the representation*

$$h(t,x) - g(t,x) = h_i(t,x) \equiv \int_0^\infty \exp(-\theta^2 t/2)\cosh(\theta x^+)\,\mu(d\theta) \tag{5}$$

for some non-negative measure μ on $\mathbb{R}^+$ satisfying the integrability condition

$$\int_0^\infty \exp(c\theta)\,\mu(d\theta) < \infty \quad \text{for all} \quad c \in \mathbb{R}; \tag{6}$$

(ii) *if h is harmonic, then*

$$\begin{aligned} h(t,x) - g(t,x) &= h_i(t,x) + \iint_{(t,\infty)\times\mathbf{R}} p_{s-t}(x^+, y)\,\nu(ds,dy) \\ &= h_i(t,x) + G\nu(t,x), \end{aligned} \tag{7}$$

where h_i is as at (5), and ν is a non-negative measure on E, symmetric under the map $(t, x) \mapsto (t, -x)$, and $p_\cdot(\cdot\,,\cdot)$ is the Brownian transition density.

Any function h represented as $h = g + h_i$, with g given by (4) and h_i by (5) is an invariant function.

Remarks. (i) Not every function $h = h_i + g + G\nu$ is necessarily harmonic, because the potential $G\nu$ is in general excessive but not harmonic. It seems to be hard in general to describe the measures ν for which $G\nu$ is harmonic, but certainly if ν is concentrated on a finite set, then $G\nu$ is harmonic.

(ii) The description of the h-transformed process is peculiar. The representation (5) corresponds to picking a drift θ according to law $\mu(d\theta)/\mu(\mathbb{R}^+)$, and h-transforming according to that while B is positive. The effect of g is to transform the process below 0 into a Bessel (3) process, pushed away from the origin at $g(A_t, 0)\,\rho(A_t) > 0$. Since both the upward-drifting Brownian motion and the Bessel (3) process are transient, eventually one or other prevails, and the particle either drifts off to $+\infty$ at a linear rate, or else goes out to $-\infty$ like a Bessel (3) process, with the value of A frozen forever.

(iii) Notice that, while g will certainly be continuous in $[0,\infty)\times(0,\infty)$, there is no reason why it need be continuous in $[0,\infty)\times[0,\infty)$ as ρ may be quite badly behaved. However, if ρ is continuous at t, it is easy to show that

$$\lim_{\epsilon\downarrow 0}\frac{\partial g}{\partial x}(t,\epsilon) = -\rho(t) = \frac{\partial g}{\partial x}(t,0-)$$

so that the x-derivative of g is continuous across the boundary.

The plan of the proof of Theorem 1 is as follows.

In Section 2, we establish (2), and that g defined by (3)–(4) is invariant and dominated by h. We thereby reduce the problem to a situation where the harmonic function $\tilde{h}\equiv h-g$ is constant along any line $\{a\}\times(-\infty,0)$. Then $\tilde{h}(A_t,B_t)=\tilde{h}(A_t,B_t^+)$, and we time change by the inverse of A to obtain the result that $\tilde{h}(t,|B_t|)$ is a local martingale; this reduces the problem to a characterization of harmonic functions for space-time Brownian motion, and we deal with this in Section 3.

However, it turns out that in the case where h is invariant, not only is $\tilde{h}(A,|B_t|)$ a local martingale, but it is also a martingale. We prove this in Section 4.

2. *The basic decomposition*

Let us observe that if we start at (a,y) with $y<0$ and stop at $H_0=\inf\{u:B_u=0\}$, then

$$h(A_{t\wedge H_0},B_{t\wedge H_0}) = h(a,B_{t\wedge H_0}) \quad \text{is a martingale,}$$

and so

$$h(a,y) = h(a,0)-y\rho(a),\ y\leqslant 0,$$

for some $\rho(a)\geqslant 0$.

We now define a function

$$g(a,y) = \lim_{n\to\infty} E^{(a,y)}[n\rho(A(H_{-n}))].$$

Is this well defined? First, observe that if we set

$$g_n(a,y) = E^{(a,y)}[n\rho(A(H_{-n}))]$$

then

$$0\leqslant g_n(a,y)\leqslant E^{(a,y)}[h(A(H_{-n}),-n)]\leqslant h(a,y)$$

using Fatou's Lemma for the last inequality: so there is no problem about finiteness of the g_n. Next, for $y\geqslant 0$,

$$E^{(a,y)}\phi(A(H_{-n})) = \int_0^\infty \frac{y\,e^{-y^2/2t}}{\sqrt{(2\pi t^3)}}dt\int_0^\infty\frac{d\nu}{n}e^{-\nu/n}\int_0^\infty\frac{\nu\,e^{-\nu^2/2s}}{\sqrt{(2\pi s^3)}}ds\,\phi(a+t+s) \tag{8}$$

because the Brownian motion has to get down to 0 (at time t) and then keep going till it hits $-n$. The local time at 0 when this happens is $V\sim\exp(1/2n)$, so the amount of time spent above 0 at that time is the same in law as the first passage time to $V/2$ for BM. That is where formula (8) above comes from!

Thus for $y\geqslant 0$

$$E^{(a,y)}[n\rho(A(H_{-n}))] = \int_0^\infty q(y,t)\,dt\int_0^\infty d\nu\,e^{-\nu/n}\int_0^\infty q(\nu,s)\,ds\,\rho(a+t+s)$$

where we abbreviate the Brownian first-passage density to $q(\cdot,\cdot)$. Now it is clear that as $n \to \infty$, this remains bounded by $h(a,y)$ and increases to a limit.

$$\begin{aligned} g(a,y) &= \int_0^\infty q(y,t)\,dt \int_0^\infty \rho(a+t+s)\frac{ds}{\sqrt{(2\pi s)}} \\ &= \int_0^\infty \rho(a+u)\frac{e^{-y^2/2u}}{\sqrt{(2\pi u)}}du \\ &\uparrow \int_0^\infty \rho(a+u)\frac{du}{\sqrt{(2\pi u)}}\ (y\downarrow 0). \end{aligned} \tag{9}$$

Thus the integrability condition

$$\int_0^\infty \rho(a+s)\,ds/\sqrt{s} < \infty \quad \text{for all} \quad a \geqslant 0 \tag{10}$$

is necessary for h to be harmonic, and is sufficient for us to define a function $g(a,y)$ (at least for $y \geqslant 0$) by (9).

This justifies the definition of g for $y \geqslant 0$. For $y < 0$, let us assume $n \geqslant |y|$, and then

$$g_n(a,y) = \frac{|y|}{n}n\rho(a) + \left(1-\frac{|y|}{n}\right)g_n(a,0) \uparrow -y\rho(a)+g(a,0) \quad (\text{as } n\uparrow\infty).$$

Thus the function g is well defined, and

$$g(a,y) = g(a,0) - y\rho(a) \quad \text{for} \quad y \leqslant 0.$$

If we set $g_n(a,y) = g_n(a,-n)$ for $y \leqslant -n$, then the g_n increase everywhere, and we shall next prove that the limit g is invariant. For this,

$$\begin{aligned} E^{(a,y)}[g(A_t,B_t)] &= \uparrow \lim_n E^{(a,y)}[g_n(A_t,B_t)] \\ &= \uparrow \lim_n \{E^{(a,y)}[g_n(A_t,B_t):t \leqslant H_{-n}] + E^{(a,y)}[g_n(A_t,B_t):t > H_{-n}]\}. \end{aligned}$$

The second term is negligible since

$$E^{(a,y)}[g_n(A_t,B_t):t > H_{-n}] \leqslant E^{(a,y)}[h(A_t,B_t):t > H_{-n}] \downarrow 0 \quad \text{as} \quad n\to\infty$$

since $h(A_t,B_t)\in L^1$. Also,

$$\begin{aligned} E^{(a,y)}[g_n(A_t,B_t):t \leqslant H_{-n}] &= E^{(a,y)}[n\rho(A(H_{-n})):t \leqslant H_{-n}] \\ &= g_n(a,y) - E^{(a,y)}[n\rho(A(H_{-n})):t > H_{-n}]. \end{aligned}$$

All that is needed now is to prove that for $t > 0$ fixed

$$E^{(a,y)}[n\rho(A(H_{-n})):H_{-n} < t] \equiv \Psi_n(t) \underset{n\to\infty}{\to} 0.$$

Since $\Psi_n(\cdot)$ is clearly increasing, it will be sufficient to prove that, with $\lambda > 0$ fixed,

$$\int_0^\infty \lambda e^{-\lambda t} E^{(a,y)}[n\rho(A(H_{-n})):H_{-n} < t]\,dt \equiv E^{(a,y)}[n\rho(A(H_{-n})):H_{-n} < T] \to 0 \quad (\text{as } n\to\infty) \tag{11}$$

where T is exponentially distributed with parameter λ, independent of B. Now for $y > -n$,

$$P^{(a,y)}[H_{-n} < T] = \exp(-(y+n)\sqrt{(2\lambda)}),$$

and conditional on $\{H_{-n} < T\}$, $(B_t : 0 \leqslant t \leqslant H_{-n})$ is identical in law to

$$(B_t - \theta t : 0 \leqslant t \leqslant \sigma_{-n}),$$

where $\theta \equiv \sqrt{(2\lambda)}$ and σ_{-n} is the first time $B_t - \theta t$ reaches $-n$ (see Williams[6]).

So now we want to compute the law of

$$A(H_{-n}) \equiv \zeta_n \equiv \int_0^{\sigma_{-n}} I_{[0,\infty)}(B_t - \theta t)\, dt.$$

However, if $\psi(y) = E^y[e^{-\alpha\zeta_n}]$, then ψ must satisfy

$$\tfrac{1}{2}\psi'' - \theta\psi' - \alpha\psi = 0 \quad \text{in} \quad (0, \infty)$$
$$\tfrac{1}{2}\psi'' - \theta\psi' = 0 \quad \text{in} \quad (-n, 0)$$
$$\psi(-n) = 1$$

together with the condition that ψ is C^1 at 0. A few simple calculations yield the solution

$$\psi(y) = c\,e^{-\gamma y} \quad (y \geqslant 0)$$
$$= c\left\{1 + \frac{\alpha}{2\theta}(1 - e^{2\theta y})\right\} \quad (-n \leqslant y \leqslant 0), \tag{12}$$

where $\gamma \equiv \sqrt{(\theta^2 + 2\alpha)} - \theta$, $c^{-1} \equiv 1 + (\gamma/2\theta)(1 - e^{-2\theta n})$. For the time being, we restrict our attention to starting values $y \geqslant 0$. What (12) tells us is that the P^y-distribution of ζ_n is the same as the time taken for $B_t - \theta t$ to drop from y to $-V_n$, where V_n is exponentially distributed with mean $(1 - e^{-2\theta n})/2\theta$. This comes as no surprise to anyone who has understood the path decompositions of Williams, and the excursion theory of drifting Brownian motion (see VI.55 in Rogers and Williams[5]).

Thus for $y \geqslant 0$, with $q_n \equiv 2\theta(1 - e^{-2\theta n})^{-1}$,

$$E^{(a,y)}[n\rho(A(H_{-n})) \mid H_{-n} < T]$$
$$= \int_0^\infty q_n\, e^{-q_n x}\, dx \int_0^\infty (x + y)\, e^{-(x+y-\theta t)^2/2t} \frac{dt}{\sqrt{(2\pi t^3)}} n\rho(a + t).$$

Since the q_n decrease to 2θ, we have the upper bound

$$q_1 \int_0^\infty e^{-2\theta x}\, dx \int_0^\infty (x + y)\, e^{-(x+y-\theta t)^2/2t} \frac{dt}{\sqrt{(2\pi t^3)}} n\rho(a + t),$$

which is finite if and only if it is finite for $y = 0$. Putting $y = 0$,

$$\int_0^\infty e^{-2\theta x}\, dx \int_0^\infty x\, e^{-(x-\theta t)^2/2t} \frac{dt}{\sqrt{(2\pi t^3)}} n\rho(a + t)$$
$$= \int_0^\infty dx \int_0^\infty x\, e^{-(x+\theta t)^2/2t} \frac{dt}{\sqrt{(2\pi t^3)}} n\rho(a + t)$$
$$= \int_0^\infty \frac{dt}{\sqrt{t}} \rho(a + t) \int_0^\infty \zeta \exp(-(\zeta + \theta\sqrt{t})^2/2) \frac{d\zeta}{\sqrt{(2\pi)}}$$
$$\leqslant \int_0^\infty \frac{dt}{\sqrt{t}} \rho(a + t) \int_0^\infty (\zeta + \theta\sqrt{t}) \exp(-(\zeta + \theta\sqrt{t})^2/2) \frac{d\zeta}{\sqrt{(2\pi)}}$$
$$\leqslant \int_0^\infty \frac{dt}{\sqrt{(2\pi t)}} \rho(a + t) < \infty$$

from (3). Since $P[H_{-n} < T] \to 0$ as $n \to \infty$, we deduce (10), at least when $y \geqslant 0$. But for $y < 0$, $n > -y$,

$$E^{(a,y)}[n\rho(A(H_{-n})) : H_{-n} < T] = E^{(a,0)}[n\rho(A(H_{-n})); H_{-n} < T].P^{(a,y)}[H_0 < H_{-n} \wedge T] + n\rho(a)P^{(a,y)}[H_{-n} < H_0 \wedge T],$$

and it is a simple matter now to deduce that this goes to 0. Hence the function g is invariant for X.

3. *Representing* $h - g$

Since g is invariant and $g \leqslant h$, it follows that $\tilde{h} \equiv h - g$ is harmonic, that is, $\tilde{h}(X_t)$ is a local martingale. But since $\tilde{h}(a,x) = \tilde{h}(a,x^+)$, if we let

$$T_n \equiv \inf\{t : \tilde{h}(X_t) > n\}$$

then certainly $X(T_n)$ must be in $\mathbb{R}^+ \times \mathbb{R}^+$. So if we define the time change $\tau_t \equiv \inf\{u : A_u > t\}$ then $\tilde{h}(X(\tau_t))$ is a local martingale in the $(\mathscr{F}_{\tau_t})$-filtration, reduced by the stopping times $A(T_n)$. However, $(X(\tau_t))_{t \geqslant 0}$ has the same distribution as $((t, |B_t|))_{t \geqslant 0}$, so if we redefine

$$\tilde{h}(t,y) = \tilde{h}(t,-y) \quad (y \leqslant 0)$$

we obtain the conclusion that

$$\tilde{h}(t, B_t) \text{ is a local martingale.}$$

So we now address the task of characterizing all functions $h : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ which are symmetric in that $h(t,y) = h(t,-y)$ for all (t,y) and such that $h(t,B_t)$ is a local martingale under every P^x. Observe that this means that h is excessive.

To finish things reasonable directly, we appeal to results of Meyer [4]. Using the results III.T13, I.T16, III.T9, we may represent excessive h_e as

$$h_e(x) = h_0(x) + \int g(x,x')\,\mu(dx') \tag{13}$$

for some measure μ such that

$$u(x) \equiv \int g(x,x')\,\mu(dx') < 0 \quad \text{for all } x, \tag{14}$$

where
$$g((s,y),(s',y')) \equiv I_{\{s'>s\}}\,p_{s'-s}(y,y'),$$

with $p.(\cdot,\cdot)$ denoting the Brownian transition density. The function h_0 in the representation (13) has the property

$$h_0(x) = E^x[h_0(X(\tau_K))] \quad \text{for all compact } K \tag{15}$$

where $\tau_K \equiv \inf\{t > 0 : X_t \notin K\}$. This property is what Meyer [4] calls harmonic, but this does not agree with our definition. We do, however, have the following result.

PROPOSITION. *If h_0 satisfies* (15) *then h_0 is invariant.*

Proof. For ease of notation, we consider only starting points on $\{0\} \times \mathbb{R}$, and (for the purposes of this proof only) abbreviate $(0,y)$ to y. Thus for any $t > 0$ and $-K < y < N$,

$$\begin{aligned} h_0(y) &= E^y[h_0(X(H_{-K} \wedge H_N \wedge t))] \\ &= E^y[h_0(X_t) : t < H_{-K} \wedge H_N] + E^y[h_0(X(H_N)) : H_N < t \wedge H_{-K}] \\ &\quad + E^y[h_0(X(H_{-K})) : H_{-K} < t \wedge H_N]. \end{aligned}$$

The first of these three terms tends to $E^y[h_0(X_t)]$ as $N, K \to \infty$, so we must show that the last two tend to zero. By letting $K \to \infty$, we learn that $E^y[h_0(X(H_N)) : H_N < t] \leqslant h_0(y)$ and so $\int_0^t q(N-y,s)\,h(s,N)\,ds \leqslant h_0(y)$. But if we consider starting from $y+1$, we learn that

$$\begin{aligned} h_0(y+1) &\geqslant \int_0^t (N-y-1)\exp\{-(N-y-1)^2/2s\}\frac{h(s,N)}{\sqrt{(2\pi s^3)}}\,ds \\ &\geqslant \frac{N-y-1}{N-y}\int_0^t (N-y)\exp\{-(N-y)^2/2s+(N-y)/2s\}\frac{h(s,N)}{\sqrt{(2\pi s^3)}}\,ds \\ &\geqslant \frac{N-y-1}{N-y}e^{(N-y)/2t}\int_0^t q(N-y,s)\,h(s,N)\,ds. \end{aligned}$$

It follows immediately that

$$E^y[h_0(X(H_N)) : H_N < t] \to 0 \quad (\text{as } N \to \infty)$$

and the invariance of h_0 is established.

Now it is well known that the invariant functions h_i for space-time Brownian motion are all of the form

$$h_i(t,x) = \int \exp(\theta x - \theta^2 t/2)\,\nu(d\theta)$$

for some ν satisfying the integrability condition (6) and that h_i is symmetric in x if and only if ν is symmetric.

We have now established that any harmonic function of (A_t, B_t) can be represented as at (7) for some invariant h_i, and the potential of some measure ν. Not every measure ν will give a harmonic function, of course and it appears difficult to characterise the ν for which we do get a harmonic function. One example is where we take ν to be the unit mass at $(1,0)$, and then we have the harmonic function

$$h(a,y) = (2\pi(1-a))^{-\frac{1}{2}}\exp\{-(y^+)^2/2(1-a)\}I_{\{a<1\}}.$$

This is a particularly interesting example, because if we h-transform using h, we obtain a process (A_t, Y_t) satisfying

$$dY_t = dW_t - \frac{Y_t^+}{1-A_t}dt, \qquad dA_t = I_{\{Y_t>0\}}\,dt,$$

and it is not clear how this process behaves; it is like Brownian motion when $Y < 0$, and like Brownian bridge when $Y > 0$, but does it have finite or infinite lifetime? As A approaches 1, the excursions of Y from 0 into $(0,\infty)$ get shorter and shorter, so it is conceivable that the lifetime of the process could be infinite, as the bulk of time is spent with $Y < 0$. We shall prove that this is not in fact the case, by observing first that $Z_t \equiv Y_t(1-A_t)^{-1}$ satisfies

$$dZ_t = \frac{dWt}{1-A_t}$$

and so if σ is inverse to the continuous increasing process

$$\gamma_t \equiv \int_0^t (1-A_s)^{-2}\,ds,$$

then $\tilde{Z}_t = Z(\sigma_t)$ is a Brownian motion. We now express the time change in terms of $\tilde{Z}$. We have

$$\dot{\sigma}_t = (1-A(\sigma_t))^2 = \left(1-\int_0^{\sigma_t} I_{\{Y_u>0\}}\,du\right)^2 = \left(1-\int_0^t \dot{\sigma}_s I_{\{\tilde{Z}_u>0\}}\,du\right)^2.$$

Thus if $\beta_t \equiv I_{\{\tilde{Z}_t>0\}}$, a little elementary calculus gives us

$$\dot{\sigma}_t = \left(1+\int_0^t \beta_u\,du\right)^2.$$

Now Hobson [1] proves that if $f:\mathbb{R}^+\to\mathbb{R}^+$ is decreasing, and $\int^\infty t^{-1}\sqrt{f(t)}\,dt<\infty$, then $\liminf_{t\to\infty} A_t/tf(t) = +\infty$. Taking $f(t)=t^{-\epsilon}$, we see that $A_t \geqslant t^{1-\epsilon}$ for all large enough t, and so $\dot{\sigma}$ is integrable. The conclusion is that $\sigma_\infty<\infty$ and γ explodes in finite time, that is, A reaches 1 in finite time!

4. *The invariant case*

The first thing to prove is that the potential term $G\nu$ in (7) cannot be invariant for X. However, this is almost obvious if we re-express it as

$$G\nu(t,x) = \iint_{(t,\infty)\times(0,\infty)} \{p_{s-t}(x^+,y)+p_{s-t}(x^+,-y)\}\,\nu(ds,dy) + \int_{(t,\infty)} p_{s-t}(x^+,0)\,\nu(ds,\{0\}).$$

Indeed, $$\gamma(t,x;s,y) \equiv \{p_{s-t}(x^+,y)+p_{s-t}(x^+,-y)\}I_{(s>t)}$$

is the density with respect to Lebesgue measure of the Green's function of X, at least for $y\geqslant 0$. This is because if we only view X at times when $B\geqslant 0$, we see a reflecting space-time Brownian motion in $\mathbb{R}^+\times\mathbb{R}^+$. Hence $G\nu$ is in fact a potential with respect to the semi-group of X, and so is not invariant. Next we must prove that every function h_i of the form given in (5) is invariant for X. This is not difficult if we set it up correctly.

Take a Brownian motion W, with local time L at zero, and an independent stable $(\frac{1}{2})$ subordinator Z:

$$E\exp(-\alpha Z_t) = \exp(-t\sqrt{(2\alpha)})$$

and now consider the bivariate Markov process $(|W_t|, t+Z(L_t))_{t\geqslant 0}$.

Let $(\mathscr{G}_t)_{t\geqslant 0}$ be the filtration of this process. We then have that $M_t \equiv h_i(t,|W_t|)$ is a $(\mathscr{G}_t)$-martingale. If we now time change by

$$\sigma_t \equiv \inf\{u : u+Z(L_u)>t\}$$

then always $\sigma_t\leqslant t$, and so $M(\sigma_t)$ is a martingale in the filtration $(\mathscr{G}_{\sigma(t)})$. However, by the way it has been constructed, we have the identity in law as processes

$$(\sigma_t, |W_{\sigma(t)}|)_{t\geqslant 0} \overset{D}{=} (A_t, B_t^+),$$

and so $h_i(A_t,B_t^+) \equiv h_i(A_t,B_t)$ is a martingale, which means that h_i is also invariant for $(A_t,B_t)\equiv X_t$.

Assembling this finally, if h is invariant for X, then it is harmonic for X, and so has a representation of the form (7), by the previous section. However, we have seen that g is invariant for X, and have just proved that h_i is invariant for X, so we conclude that the potential $G\nu$ is invariant for X. This can only happen if $\nu \equiv 0$. The theorem is proved.

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