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The spectrum of the mean curvature operator

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We show that the spectrum of the relativistic mean curvature operator on a bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ having smooth boundary, subject to the homogeneous Dirichlet boundary condition, is exactly the interval $(\lambda_1(2), \infty)$, where $\lambda_1(2)$ stands for the principal frequency of the Laplace operator in Ω.

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1. Introduction

1.1. Statement of the problem and motivation

The goal of this paper is to characterize the spectrum of the *relativistic mean curvature operator*, i.e.

$$\mathcal{M}u := -\mathrm{div}\left(\frac{\nabla u}{\sqrt{1-|\nabla u|^2}}\right),$$

acting on maps u defined in an open, bounded domain $\Omega \subset \mathbb{R}^N$ $(N \geqslant 1)$ with smooth boundary $\partial\Omega$, subject to the homogeneous Dirichlet boundary condition. More precisely, our goal is to analyse the problem

$$\begin{cases} \mathcal{M}u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Note that problem (1.1) is a *nontypical eigenvalue problem* since the mean curvature operator is inhomogeneous. However, its formulation is similar with those of a typical eigenvalue problem. Roughly speaking, by an eigenvalue of problem (1.1) we will understand a real number λ for which problem (1.1) has a nontrivial solution u whose sense will be made precise in the next subsection of the Introduction. Moreover, keeping in mind the terminology used in the case of typical eigenvalue problems, in this paper the *spectrum* of the mean curvature operator represents the set of all eigenvalues of problem (1.1).

Several research directions are assembled behind the study of the present work. The initial motivation comes from the fact that the *relativistic mean curvature operator* is an essential object in Geometry and Physics. More precisely, $\mathcal{M}$ appears naturally in the *Riemannian Geometry*—where it is involved in the determination of the maximal or constant mean curvature hypersurfaces in the Lorentz–Minkowski space (see, e.g., Cheng & Yau [**8**], Flaherty [**16**], Bartnik & Simon [**2**], Kiessling [**18**], Corsato *et al.* [**10**])—and in *classical relativity*—for instance in the analysis of the *forced relativistic pendulum* (see, e.g., Brezis & Mawhin [**7**]), in the study of the *Born-Infeld theory of electrodynamics* (see, e.g., Bonheure *et al.* [**5**, **6**]) or in some investigations related with the *Lorentz force equation* (see, e.g., Arcoya *et al.* [**1**]).

On the other hand, our study complements earlier investigations on problem (1.1). More precisely, partial results concerning the description of the spectrum of $\mathcal{M}$ can be found for example in [**3**, corollary 1] and [**11**, proposition 2.7 (ii)] where the existence of nontrivial solutions for problem (1.1) was established in the case when λ is sufficiently large with no control on a lower bound of λ (see also [**9**] for the case of the radial problem). The main argument used in [**3**] was the Direct Method in the Calculus of Variations while in [**11**] an approach based on the Leray–Schauder degree was considered. In this paper we are able to give the complete description of the spectrum of problem (1.1) as being exactly the interval $(\lambda_1(2), \infty)$, where $\lambda_1(2)$ stands for the principal frequency of the Laplace operator (see the statement of theorem 1.1 below). The main argument that will be used here is an approximation technique based on a Γ-convergence argument.

1.2. Preliminaries and main result

The first step in making precise the rigorous mathematical sense in which the notion of *eigenvalue* will be understood throughout this paper is to explain the function space framework that will be considered in the sequel. In this regard we note that the structure of the mean curvature operator asks for a condition of type $|\nabla u(x)| \leqslant 1$ for a.e. $x \in \Omega$. That simple observation and the homogeneous Dirichlet boundary condition involved in problem (1.1) imply that a good candidate for the functional space framework would be a subset of

$$W_0^{1,\infty}(\Omega) := \{u \in W^{1,\infty}(\Omega) : \ u = 0, \text{ on } \partial\Omega\},$$

namely

$$K_0 := \{u \in W_0^{1,\infty}(\Omega) : \ |\nabla u(x)| \leqslant 1, \text{ a.e. } x \in \Omega\}.$$

Note that K_0 is a convex and closed subset of $W^{1,\infty}(\Omega)$ which is the dual of a separable Banach space (see, e.g., the proof of [**3**, lemma 2]). This leads to the idea of constructing the Euler–Lagrange functional associated to the mean curvature operator as $I : W^{1,\infty}(\Omega) \to [0, \infty]$ defined by

$$I(u) := \begin{cases} \displaystyle\int_\Omega F(|\nabla u|)\,\mathrm{d}x & \text{if } u \in K_0 \\ +\infty & \text{if } u \in W^{1,\infty}(\Omega) \backslash K_0, \end{cases}$$

where $F : [-1, 1] \to \mathbb{R}$ is given by $F(t) := 1 - \sqrt{1 - t^2}$ for all $t \in [-1, 1]$. Then, the Euler–Lagrange functional associated to the problem (1.1) is $J_\lambda : W^{1,\infty}(\Omega) \to \mathbb{R}$

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defined by

$$J_\lambda(u) := I(u) - \frac{\lambda}{2}\int_\Omega u^2\,\mathrm{d}x, \quad \forall\, u \in W^{1,\infty}(\Omega).$$

Next, we note that J_λ is the sum of a convex, lower semi-continuous function and a C^1-functional, and, consequently, it has the structure required by *Szulkin's critical point theory* (see [**21**]). More precisely, the functional J_λ is the sum of the functional $h_\lambda : W^{1,\infty}(\Omega) \to \mathbb{R}$ defined by

$$h_\lambda(u) := -\frac{\lambda}{2}\int_\Omega u^2\,\mathrm{d}x,$$

which belongs to $C^1(W^{1,\infty}(\Omega), \mathbb{R})$ (see, e.g. [**3**] for the proof) and has the derivative given by

$$\langle h_\lambda'(u), v\rangle = -\lambda\int_\Omega uv\,\mathrm{d}x, \quad \forall\, u,\ \ v \in W^{1,\infty}(\Omega),$$

with the functional I which is convex and weakly* lower semicontinuous (see, [**3**, lemma 4]). Then, following Szulkin, we will work with a reformulation of problem (1.1) as a *variational inequality*, namely

$$\begin{cases} I(v) - I(u_\lambda) + \langle h_\lambda'(u_\lambda), v - u_\lambda\rangle \geqslant 0 \quad \text{for all } v \in W^{1,\infty}(\Omega), \\ u_\lambda \in W^{1,\infty}(\Omega). \end{cases} \tag{1.2}$$

or, equivalently,

$$\begin{cases} I(v) - I(u_\lambda) + \langle h_\lambda'(u_\lambda), v - u_\lambda\rangle \geqslant 0 \quad \text{for all } v \in K_0, \\ u_\lambda \in K_0. \end{cases} \tag{1.3}$$

In this context a real number $\lambda \in \mathbb{R}$ is called an *eigenvalue* for problem (1.1) if problem (1.3) has a nontrivial solution $u_\lambda \in K_0$. u_λ will be called an *eigenfunction* corresponding to the eigenvalue λ. According to the terminology from [**21**], we refer to u_λ as being a *critical point* of functional J_λ.

Our main result is given by the following theorem.

THEOREM 1.1. *The set of eigenvalues for problem* (1.1) *is the open interval* $(\lambda_1(2), \infty)$ *where* $\lambda_1(2)$ *stands for the principal frequency of the Laplace operator in* Ω *subject to the homogeneous Dirichlet boundary condition, i.e.*

$$\lambda_1(2) := \inf_{u \in C_0^\infty(\Omega)\setminus\{0\}} \frac{\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\displaystyle\int_\Omega u^2\,\mathrm{d}x}.$$

Moreover, for each eigenvalue λ *we can choose a corresponding eigenfunction* $u_\lambda \in K_0$ *which is nonnegative on* Ω *and minimizes* J_λ.

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2. Proof of the main result

Following the classical approach used in the analysis of the principal frequency of the Laplace operator, $\lambda_1(2)$, it is natural to associate to the problem (1.1) the quantity Λ_1 defined below, involving the corresponding Rayleigh-type quotient suggested by the formulation of problem (1.1), namely

$$\Lambda_1 := \inf_{u\in K_0\setminus\{0\}} \frac{\displaystyle\int_\Omega F(|\nabla u|)\,\mathrm{d}x}{\displaystyle\frac{1}{2}\int_\Omega u^2\,\mathrm{d}x},$$

where $F:[-1,1]\to\mathbb{R}$ is given by $F(t):=1-\sqrt{1-t^2}$ for all $t\in[-1,1]$. Our first key observation is given by the following lemma which relates the spectrum of problem (1.1) with $\lambda_1(2)$.

Lemma 2.1. *The following equality holds true*

$$\Lambda_1 = \lambda_1(2). \tag{2.1}$$

Proof. Since

$$F(t) \geqslant \frac{t^2}{2}, \quad \forall\, t\in[-1,1],$$

we deduce that

$$\lambda_1(2) \leqslant \frac{\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\displaystyle\int_\Omega u^2\,\mathrm{d}x} \leqslant 2\frac{\displaystyle\int_\Omega F(|\nabla u|)\,\mathrm{d}x}{\displaystyle\int_\Omega u^2\,\mathrm{d}x}, \quad \forall\, u\in K_0\setminus\{0\}.$$

Consequently $\lambda_1(2)\leqslant\Lambda_1$. Next, let e_1 be a positive minimizer of $\lambda_1(2)$. It is well-known that such a minimizer is an eigenfunction of $-\Delta$ over Ω subject to the homogeneous Dirichlet boundary condition. Actually, $e_1\in W_0^{1,2}(\Omega)\cap C^{1,\alpha}(\overline{\Omega})$ for some $\alpha\in(0,1)$. Let $u_1:=e_1\|\,|\nabla e_1|\,\|^{-1}_{L^\infty(\Omega)}$. It is clear that $u_1\in K_0\setminus\{0\}$. Then for each $t\in(0,1)$ we have

$$\Lambda_1 \leqslant \frac{\displaystyle\int_\Omega F(|\nabla(tu_1)|)\,\mathrm{d}x}{\displaystyle\frac{1}{2}\int_\Omega (tu_1)^2\,\mathrm{d}x} = \frac{\displaystyle\int_\Omega \left(1-\sqrt{1-t^2|\nabla u_1|^2}\right)\mathrm{d}x}{\displaystyle\frac{t^2}{2}\int_\Omega u_1^2\,\mathrm{d}x} =: g(t).$$

Simple computations imply

$$\lim_{t\to 0^+} g(t) = \lim_{t\to 0^+}\frac{\displaystyle\int_\Omega \frac{t|\nabla u_1|^2}{\sqrt{1-t^2|\nabla u_1|^2}}\,dx}{\displaystyle t\int_\Omega u_1^2\,\mathrm{d}x} = \frac{\displaystyle\int_\Omega |\nabla u_1|^2\,\mathrm{d}x}{\displaystyle\int_\Omega u_1^2\,\mathrm{d}x} = \lambda_1(2).$$

It follows that $\Lambda_1\leqslant\lambda_1(2)$, and, consequently (2.1) holds true. □

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The next step in our approach goes back to the elementary observation that the function F admits the following extension into power series

$$F(t) = \frac{1}{2}t^2 + \sum_{n \geqslant 2} a_n t^{2n}, \quad \forall\, t \in [-1, 1],$$

where for each integer $n \geqslant 2$ we let $a_n := (2n-3)!!/2^n n!$. Thus, if for each integer $n \geqslant 2$ we define $F_n : [-1, 1] \to \mathbb{R}$ by

$$F_n(t) := \frac{1}{2}t^2 + \sum_{k=2}^{n} a_k t^{2k}, \quad \forall\, t \in [-1, 1],$$

then we have

$$\lim_{n\to\infty} F_n(t) = F(t), \quad \forall\, t \in [-1, 1].$$

This simple remark suggests to us a Γ-convergence result which will play a crucial role in our approach. More precisely, we prove a Γ-convergence result which shows that the energy functional associated to the leading differential operator, $u \mapsto -\mathrm{div}(\nabla u/\sqrt{1-|\nabla u|^2})$, on the left-hand side of the PDE in (1.1) can be obtained, via De Giorgi's Γ-convergence, as the limit of the sequence of energy functionals associated to the differential operators $-\Delta u - \sum_{k=2}^{n}((2k-3)!!/2^{k-1}(k-1)!)\Delta_{2k}u$, where $\Delta_{2k}u$ stands for the $2k$-Laplacian of u (i.e. $\Delta_{2k}u = \mathrm{div}(|\nabla u|^{2k-2}\nabla u)$), for each positive integer k. We begin by recalling the definition of Γ-convergence (introduced in [**14**], [**15**]) in metric spaces. The reader is referred to [**13**] for a comprehensive introduction to this topic.

Definition 2.2. Let X be a metric space. A sequence $\{S_n\}$ of functionals $S_n : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ is said to $\Gamma(X)$-converge to $S_\infty : X \to \overline{\mathbb{R}}$, and we write $\Gamma(X) - \lim_{n\to\infty} S_n = S_\infty$, if the following hold:

(i) for every $u \in X$ and $\{u_n\} \subset X$ such that $u_n \to u$ in X, we have

$$S_\infty(u) \leqslant \liminf_{n\to\infty} S_n(u_n);$$

(ii) for every $u \in X$ there exists a sequence $\{u_n\} \subset X$ (called a recovery sequence) such that $u_n \to u$ in X and

$$S_\infty(u) \geqslant \limsup_{n\to\infty} S_n(u_n).$$

Next, for each integer $n \geqslant 2$ define $I_n : L^1(\Omega) \to \overline{\mathbb{R}}$ by

$$I_n(u) := \begin{cases} \displaystyle\int_\Omega F_n(|\nabla u|)\,\mathrm{d}x & \text{if } u \in W^{1,2n}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

and let $I_\infty : L^1(\Omega) \to \overline{\mathbb{R}}$ be given by

$$I_\infty(u) := \begin{cases} I(u) & \text{if } u \in W^{1,\infty}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

THEOREM 2.3. $\Gamma(L^1(\Omega)) - \lim_{n\to\infty} I_n = I_\infty$.

Proof. We will first establish the existence of a recovery subsequence for the Γ-limit. Let $u \in L^1(\Omega)$ be arbitrary. We only need to consider the case where $I_\infty(u) < +\infty$. Thus, $u \in K_0$, and

$$I_\infty(u) = \int_\Omega F(|\nabla u|)\,\mathrm{d}x = I(u).$$

For $n \geqslant 2$ define $u_n := u$. We have $u_n \in K_0 \subset W_0^{1,2n}(\Omega)$ and thus, for each $n \geqslant 2$,

$$I_n(u_n) = I_n(u) = \int_\Omega F_n(|\nabla u|)\,\mathrm{d}x.$$

In view of Beppo–Levi's Monotone Convergence Theorem we have $\limsup_{n\to\infty} I_n(u_n) = \limsup_{n\to\infty} I_n(u) = I(u) = I_\infty(u)$. Hence, the constant sequence $\{u_n\} = \{u\}$ is a recovery sequence for the Γ-limit.

It remains to show that for any $u \in L^1(\Omega)$ and $\{u_n\} \subset L^1(\Omega)$ such that $u_n \to u$ in $L^1(\Omega)$ we have

$$I_\infty(u) \leqslant \liminf_{n\to\infty} I_n(u_n). \tag{2.2}$$

After eventually extracting a subsequence (not relabelled), we may assume, without loss of generality, that $u_n \in W_0^{1,2n}(\Omega)$ and

$$L := \liminf_{n\to\infty} I_n(u_n) = \lim_{n\to\infty} I_n(u_n) < \infty. \tag{2.3}$$

Let $q \geqslant 1$ be an arbitrary real number. If $n \in \mathbb{N}$ is sufficiently large and, in particular, such that $n \geqslant [q]+1$, where $[q]$ stands for the integer part of q, then $u_n \in W_0^{1,2n}(\Omega) \subset W_0^{1,2([q]+1)}(\Omega)$, and

$$I_{[q]+1}(u_n) \leqslant I_n(u_n) \leqslant L+1. \tag{2.4}$$

It follows that

$$\int_\Omega |\nabla u_n|^{2([q]+1)}\,\mathrm{d}x \leqslant (L+1)a_{[q]+1}^{-1}, \quad \forall\, n \geqslant [q]+1 \text{ sufficiently large.}$$

We deduce that $\{\nabla u_n\}$ is bounded in $L^{2([q]+1)}(\Omega)$, and thus in $L^q(\Omega)$ (by Hölder's inequality). Since $u_n \to u$ in $L^1(\Omega)$, after eventually extracting a further subsequence (again, not relabelled), we have $u_n \rightharpoonup u$ weakly in $W_0^{1,q}(\Omega)$. Taking into account the fact that the functionals $I_{[q]+1}$ are sequentially weakly lower semicontinuous in $W_0^{1,2([q]+1)}(\Omega)$, we obtain

$$I_{[q]+1}(u) \leqslant \liminf_{n\to\infty} I_{[q]+1}(u_n).$$

On the other hand, (2.4) gives $\liminf_{n\to\infty} I_{[q]+1}(u_n) \leqslant \lim_{n\to\infty} I_n(u_n)$. Hence, $I_{[q]+1}(u) \leqslant \lim_{n\to\infty} I_n(u_n)$ for every $q \geqslant 2$. Letting $q \to \infty$, and using again the Monotone Convergence Theorem, the conclusion follows. □

We deduce immediately the following corollary.

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Corollary 2.4. *Let $\lambda \in \mathbb{R}$ be fixed. For each integer $n \geqslant 2$ define $H_n : L^1(\Omega) \to \overline{\mathbb{R}}$ by*

$$H_n(u) := \begin{cases} \displaystyle\int_\Omega F_n(|\nabla u|)\,\mathrm{d}x + h_\lambda(u) & \text{if } u \in W_0^{1,2n}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega), \end{cases}$$

and let $H_\infty : L^1(\Omega) \to \overline{\mathbb{R}}$ be given by

$$H_\infty(u) := \begin{cases} J_\lambda(u) & \text{if } u \in W^{1,\infty}(\Omega) \\ +\infty & \text{otherwise in } L^1(\Omega). \end{cases}$$

Then

$$\Gamma(L^1(\Omega)) - \lim_{n\to\infty} H_n = H_\infty.$$

For each integer $n \geqslant 2$ define the differential operator

$$\mathcal{M}_n u := -\Delta u - \sum_{k=2}^{n} b_k \Delta_{2k} u,$$

where $b_k := (2k-3)!!/2^{k-1}(k-1)!$, for all $k \geqslant 2$, and consider the eigenvalue problem

$$\begin{cases} \mathcal{M}_n u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

It is natural to analyse this problem in the Sobolev space $W_0^{1,2n}(\Omega)$. We say that $\lambda \in \mathbb{R}$ is an *eigenvalue* for the problem (2.5) if there exists $u_\lambda \in W_0^{1,2n}(\Omega)\backslash\{0\}$ such that

$$\int_\Omega \nabla u_\lambda \nabla\varphi\,\mathrm{d}x + \sum_{k=2}^{n} b_k \int_\Omega |\nabla u_\lambda|^{2k-2}\nabla u_\lambda \nabla\varphi\,\mathrm{d}x - \lambda\int_\Omega u_\lambda\varphi\,\mathrm{d}x = 0, \quad \forall\, \varphi \in W_0^{1,2n}(\Omega). \tag{2.6}$$

A function $u_\lambda \in W_0^{1,2n}(\Omega)\backslash\{0\}$ such that (2.6) holds will be called an *eigenfunction* corresponding to the eigenvalue λ. Standard regularity arguments (see, e.g. [**19**, theorem 4.5]) show that $u_\lambda \in C^{1,\beta}(\Omega)$, for some $\beta \in (0,1)$. For each $n \geqslant 2$ define

$$\mu_1(n) := \inf_{u\in C_0^\infty(\Omega)\backslash\{0\}} \frac{\displaystyle\int_\Omega F_n(|\nabla u|)\,\mathrm{d}x}{\displaystyle\frac{1}{2}\int_\Omega u^2\,\mathrm{d}x} = \inf_{u\in W_0^{1,2n}(\Omega)\backslash\{0\}} \frac{\displaystyle\int_\Omega F_n(|\nabla u|)\,\mathrm{d}x}{\displaystyle\frac{1}{2}\int_\Omega u^2\,\mathrm{d}x}. \tag{2.7}$$

and

$$\nu_1(n) := \inf_{u\in C_0^\infty(\Omega)\backslash\{0\}} \frac{\displaystyle\int_\Omega F_n'(|\nabla u|)|\nabla u|\,\mathrm{d}x}{\displaystyle\int_\Omega u^2\,\mathrm{d}x} = \inf_{u\in W_0^{1,2n}(\Omega)\backslash\{0\}} \frac{\displaystyle\int_\Omega F_n'(|\nabla u|)|\nabla u|\,\mathrm{d}x}{\displaystyle\int_\Omega u^2\,\mathrm{d}x}. \tag{2.8}$$

Lemma 2.5. *For each integer $n \geqslant 2$ we have $\mu_1(n) = \nu_1(n) = \lambda_1(2)$.*

Proof. The proof of this lemma follows some ideas from the proof of [**4**, proposition 1]. We recall it for readers' convenience.

We will check only the fact that $\mu_1(n) = \lambda_1(2)$. The identity $\nu_1(n) = \lambda_1(2)$ can be obtained similarly and, consequently, we will omit its proof. Thus, we start by observing that

$$\lambda_1(2) \leqslant \frac{\dfrac{1}{2}\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\dfrac{1}{2}\displaystyle\int_\Omega u^2\,\mathrm{d}x} \leqslant \frac{\displaystyle\int_\Omega F_n(|\nabla u|)\,\mathrm{d}x}{\dfrac{1}{2}\displaystyle\int_\Omega u^2\,\mathrm{d}x}, \quad \forall\, u \in C_0^\infty(\Omega)\backslash\{0\}.$$

Passing to the infimum over all $u \in C_0^\infty(\Omega)\backslash\{0\}$ we obtain that $\lambda_1(2) \leqslant \mu_1(n)$. Next, note that for every $t > 0$ and $u \in C_0^\infty(\Omega)\backslash\{0\}$ we have

$$\mu_1(n) \leqslant \frac{\displaystyle\int_\Omega F_n(|\nabla(tu)|)\,\mathrm{d}x}{\dfrac{1}{2}\displaystyle\int_\Omega (tu)^2\,\mathrm{d}x} = \frac{\displaystyle\sum_{k=2}^n a_k t^{2k-2}\int_\Omega |\nabla u|^{2k}\,\mathrm{d}x}{\dfrac{1}{2}\displaystyle\int_\Omega u^2\,\mathrm{d}x} + \frac{\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\displaystyle\int_\Omega u^2\,\mathrm{d}x}.$$

Letting $t \to 0^+$, we find

$$\mu_1(n) \leqslant \frac{\displaystyle\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\displaystyle\int_\Omega u^2\,\mathrm{d}x}, \quad \forall\, u \in C_0^\infty(\Omega)\backslash\{0\}.$$

Passing to the infimum over all $u \in C_0^\infty(\Omega)\backslash\{0\}$ we obtain $\mu_1(n) \leqslant \lambda_1(2)$. Consequently, we arrive to the conclusion that $\mu_1(n) = \lambda_1(2)$. □

Theorem 2.6. *For each integer $n \geqslant 2$ the set of eigenvalues for the problem (2.5) is the interval $(\lambda_1(2), \infty)$. Moreover, each eigenvalue λ possesses a positive corresponding eigenfunction in Ω.*

Proof. The proof of this result uses similar ideas with the one of [**4**, theorem 2]. We recall it for readers' convenience.

Assume that λ is an eigenvalue of problem (2.5) with $u_\lambda \in W_0^{1,2n}(\Omega)\backslash\{0\}$ a corresponding eigenfunction. Testing with $\varphi = u_\lambda$ in (2.6) and taking into account lemma 2.5 we deduce that $\lambda \geqslant \lambda_1(2)$. It follows that any $\lambda \in (-\infty, \lambda_1(2))$ is not an eigenvalue for (2.5). Moreover, if $\lambda_1(2)$ were an eigenvalue of (2.5), then by (2.6) with $\varphi = u_{\lambda_1(2)}$, where $u_{\lambda_1(2)}$ stands for an eigenfunction associated to $\lambda_1(2)$, and

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in view of Poincaré's inequality, we would have

$$\begin{aligned}\sum_{k=2}^{n} b_n \int_\Omega |\nabla u_{\lambda_1(2)}|^{2n}\,\mathrm{d}x + \lambda_1(2)\int_\Omega u_{\lambda_1(2)}^2\,\mathrm{d}x \\ \leqslant \sum_{k=2}^{n} b_n \int_\Omega |\nabla u_{\lambda_1(2)}|^{2n}\,\mathrm{d}x + \int_\Omega |\nabla u_{\lambda_1(2)}|^{2}\,\mathrm{d}x \\ = \lambda_1(2)\int_\Omega u_{\lambda_1(2)}^2\,\mathrm{d}x.\end{aligned}$$

It follows that $\int_\Omega |\nabla u_{\lambda_1(2)}|^{2n}\,\mathrm{d}x = 0$, or $u_{\lambda_1(2)} = 0$, a contradiction.

Next, we show that every $\lambda \in (\lambda_1(2), \infty)$ is an eigenvalue for (2.5). Let $\lambda \in (\lambda_1(2), \infty)$ and $n \geqslant 2$ be arbitrary, and define $J_{n,\lambda} : W_0^{1,2n}(\Omega) \to \mathbb{R}$ by

$$J_{n,\lambda}(u) = \int_\Omega F_n(|\nabla u|)\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega u^2\,\mathrm{d}x. \tag{2.9}$$

As it can be easily seen, $J_{n,\lambda} \in C^1(W_0^{1,2n}(\Omega), \mathbb{R})$, with the Gateaux derivative given by

$$\langle J'_{n,\lambda}(u), \varphi\rangle = \int_\Omega \nabla u \nabla \varphi\,\mathrm{d}x + \sum_{k=2}^{n} \frac{(2k-3)!!}{2^{k-1}(k-1)!}\int_\Omega |\nabla u|^{2k-2}\nabla u \nabla\varphi\,\mathrm{d}x - \lambda\int_\Omega u\varphi\,\mathrm{d}x$$

for all u, $\varphi \in W_0^{1,2n}(\Omega)$. It is now standard to check that $J_{n,\lambda}$ is coercive and weakly lower semicontinuous on $W_0^{1,2n}(\Omega)$, and thus the Direct Method in the Calculus of Variations (see, e.g., [**12**] or [**20**, theorem 1.2]) implies the existence of a global minimum point $w_\lambda \in W_0^{1,2n}(\Omega)$ of $J_{n,\lambda}$, i.e., $J_{n,\lambda}(w_\lambda) = \min_{W_0^{1,2n}(\Omega)} J_{n,\lambda}$. Using lemma 2.5, we deduce that since $\lambda > \lambda_1(2)$ there exists $v_\lambda \in W_0^{1,2n}(\Omega)$ such that $J_{n,\lambda}(v_\lambda) < 0$. Hence, $J_{n,\lambda}(w_\lambda) \leqslant J_{n,\lambda}(v_\lambda) < 0 = J_{n,\lambda}(0)$ which means that $w_\lambda \neq 0$. On the other hand, we have $\langle J'_{n,\lambda}(w_\lambda), \varphi\rangle = 0$, $\forall\ \varphi \in W_0^{1,2n}(\Omega)$, hence λ is an eigenvalue for problem (2.5).

Finally, note that if w_λ is a global minimum point of $J_{n,\lambda}$ so is $|w_\lambda|$. Next, the fact that $|w_\lambda| > 0$ in Ω can be obtained in a standard manner using as main argument Harnack's inequality *à la Trudinger* [**22**] and a covering technique (see, e.g. [**19**, lemma 5.3] for details). □

The next result emphasizes the asymptotic behaviour of the sequence u_n given by theorem 2.6, as $n \to \infty$. This result will be essential in the proof of the main result of the paper.

LEMMA 2.7. *Fix $\lambda \in (\lambda_1(2), \infty)$. For each integer $n > N$, let $u_n \in W_0^{1,2n}(\Omega)$ be a positive eigenfunction corresponding to the eigenvalue λ of the problem* (2.5). *Then there exists a subsequence of $\{u_n\}$ that converges uniformly in Ω to some nonnegative function $u_\lambda \in C(\overline{\Omega})$.*

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Proof. First, let us note that the sequence $\{a_n\}$ with $a_n := (2n-3)!!/2^n n!$ satisfies the following relation

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{2n-1}{2(n+1)} = 1. \tag{2.10}$$

Next, note that in view of the proof of theorem 2.3 we know that for each $n > N$ the eigenfunction $u_n \in W_0^{1,2n}(\Omega)$ of the eigenvalue λ of the problem (2.5) is a minimizer of $J_{n,\lambda}$, given by relation (2.9), in $W_0^{1,2n}(\Omega)$ and $J_{n,\lambda}(u_n) < 0$. Thus, for each $n > N$ we have

$$\int_\Omega F_n(|\nabla u_n|)\,\mathrm{d}x - \frac{\lambda}{2}\int_\Omega u_n^2\,\mathrm{d}x < 0,$$

and consequently

$$a_n \int_\Omega |\nabla u_n|^{2n}\,\mathrm{d}x \leqslant \frac{\lambda}{2}\int_\Omega u_n^2\,\mathrm{d}x \leqslant \frac{\lambda}{2\lambda_1(2)}\int_\Omega |\nabla u_n|^2\,\mathrm{d}x, \quad \forall\, n > N. \tag{2.11}$$

Fix $q > N$. For each integer $n > q$ a simple application of Holder's inequality yields

$$\int_\Omega |\nabla u_n|^q\,\mathrm{d}x \leqslant \left(\int_\Omega |\nabla u_n|^{2n}\,\mathrm{d}x\right)^{q/(2n)} |\Omega|^{1-q/(2n)}.$$

Combining this estimate with relation (2.11) and applying again Holder's inequality we deduce that for each integer $n > q$ we have

$$\begin{aligned}
\left(\int_\Omega |\nabla u_n|^q\,\mathrm{d}x\right)^{1/q} &\leqslant \left(\int_\Omega |\nabla u_n|^{2n}\,\mathrm{d}x\right)^{1/(2n)} |\Omega|^{1/q-1/(2n)} \\
&\leqslant \left(\frac{\lambda}{2a_n\lambda_1(2)}\int_\Omega |\nabla u_n|^2\,\mathrm{d}x\right)^{1/(2n)} |\Omega|^{1/q-1/(2n)} \\
&\leqslant \left(\frac{\lambda}{2a_n\lambda_1(2)}\right)^{1/(2n)} \left(\int_\Omega |\nabla u_n|^q\,\mathrm{d}x\right)^{1/(nq)} \\
&\quad \times |\Omega|^{(1-2/q)1/(2n)} |\Omega|^{1/q-1/(2n)}.
\end{aligned}$$

Simple computations imply

$$\left(\int_\Omega |\nabla u_n|^q\,\mathrm{d}x\right)^{1/q} \leqslant \left(\frac{\lambda}{2a_n\lambda_1(2)}\right)^{1/(2n-2)} |\Omega|^{1/q}, \quad \forall\, n > q.$$

This estimate and relation (2.10) show that the sequence $\{u_n\}$ is bounded in $W_0^{1,q}(\Omega)$ for each $q > N$. This fact and the compact embedding of $W_0^{1,q}(\Omega)$ into $C(\overline{\Omega})$ implies that there exists a subsequence (not relabelled) of $\{u_n\}$ and a function $u_\lambda \in C(\overline{\Omega})$ such that $u_n \rightharpoonup u_\lambda$ weakly in $W_0^{1,q}(\Omega)$ and $u_n \to u_\lambda$ uniformly in Ω. □

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2.1. Proof of theorem 1.1 (concluded)

We start by recalling a well-known result that can be found, for example, in [**17**, corollary 6.1.1]. It plays an essential role in the proof of theorem 1.1.

PROPOSITION 2.8. *Let Y be a topological space satisfying the first axiom of countability, and assume that the sequence $\{G_n\}$ of functionals $G_n : Y \to \overline{\mathbb{R}}$ $\Gamma-$converge to $G : Y \to \overline{\mathbb{R}}$. Let z_n be a minimizer for G_n. If $z_n \to z$ in X, then z is a minimizer of G, and $G(z) = \liminf_{n\to\infty} G_n(z_n)$.*

Next, we fix $\lambda \in (\lambda_1(2), \infty)$ arbitrary and we show that it is an eigenvalue of problem (1.1) in the sense that there exists $u_\lambda \in K_0\backslash\{0\}$ satisfying relation (1.3).

Indeed, taking into account corollary 2.4 and lemma 2.7, we can apply proposition 2.8 with $Y = L^1(\Omega)$, $G_n = H_n$, $G = H_\infty$, $z_n = u_n$ to deduce the existence of a minimizer for H_∞ and consequently a minimizer for J_λ on K_0, say u_λ. We claim that $u_\lambda \neq 0$. Indeed, since by lemma 2.1 we have

$$\inf_{u\in K_0\backslash\{0\}} \frac{\displaystyle\int_\Omega F(|\nabla u|)\,\mathrm{d}x}{\displaystyle\frac{1}{2}\int_\Omega u^2\,\mathrm{d}x} = \lambda_1(2),$$

we deduce that for each $\lambda > \lambda_1(2)$ there exists $w_\lambda \in K_0\backslash\{0\}$ such that

$$\frac{\displaystyle\int_\Omega F(|\nabla w_\lambda|)\,\mathrm{d}x}{\displaystyle\frac{1}{2}\int_\Omega w_\lambda^2\,\mathrm{d}x} < \lambda,$$

which implies

$$J_\lambda(u_\lambda) \leqslant J_\lambda(w_\lambda) < 0 = J_\lambda(0),$$

which proves the claim.

Further, note that from the fact that

$$J_\lambda(v) \geqslant J_\lambda(u_\lambda), \quad \forall\, v \in K_0,$$

we deduce that

$$I(v) - I(u_\lambda) \geqslant \frac{\lambda}{2}\int_\Omega (v^2 - u_\lambda^2)\,\mathrm{d}x \geqslant \lambda \int_\Omega u_\lambda(v - u_\lambda)\,\mathrm{d}x, \quad \forall\, v \in K_0.$$

Since $u_\lambda \in K_0\backslash\{0\}$ the above relation shows that each $\lambda > \lambda_1(2)$ is an eigenvalue of problem (1.1).

Finally, we check that each $\lambda \in (-\infty, \lambda_1(2)]$ can not be an eigenvalue of problem (1.1).

Let $\lambda \in \mathbb{R}$ be an eigenvalue for (1.1). Then there exists $u \in K_0\backslash\{0\}$ such that the inequality in (1.3) holds for all $v \in K_0$. For each $t \in (0, 1)$ it is clear that $tu \in K_0$.

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Thus, testing in (1.3) with $v = tu$ we get

$$\int_\Omega F(|\nabla(tu)|)\,\mathrm{d}x - \int_\Omega F(|\nabla u|)\,\mathrm{d}x - \lambda(t-1)\int_\Omega u^2\,\mathrm{d}x \geqslant 0, \quad \forall\ t \in (0,1),$$

or, equivalently

$$\frac{t^2-1}{2}\int_\Omega |\nabla u|^2\,\mathrm{d}x + \sum_{n\geqslant 2} a_n(t^{2n}-1)\int_\Omega |\nabla u|^{2n}\,\mathrm{d}x - \lambda(t-1)\int_\Omega u^2\,\mathrm{d}x \geqslant 0, \quad \forall\ t\in(0,1),$$

where $a_n := (2n-3)!!/2^n n!$. Dividing both sides above by $(t-1)\int_\Omega u^2\,\mathrm{d}x < 0$, we are led to

$$\frac{t+1}{2}\frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x} + \sum_{n\geqslant 2} a_n \frac{1-t^{2n}}{1-t}\frac{\int_\Omega |\nabla u|^{2n}\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x} \leqslant \lambda, \quad \forall\ t\in(0,1).$$

Letting $t \to 1^-$ we find

$$\lambda_1(2) \leqslant \frac{\int_\Omega |\nabla u|^2\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x} + \sum_{n\geqslant 2} 2na_n \frac{\int_\Omega |\nabla u|^{2n}\,\mathrm{d}x}{\int_\Omega u^2\,\mathrm{d}x} \leqslant \lambda.$$

Therefore $\lambda \geqslant \lambda_1(2)$. To conclude the proof we still need to show that $\lambda_1(2)$ is not an eigenvalue for problem (1.1). Indeed, if we assume the contrary, then there exists $u \in K_0\backslash\{0\} \subset W_0^{1,2}(\Omega)\backslash\{0\}$ such that the inequality in (1.3) is true for all $v \in K_0$. Repeating the above arguments we deduce that

$$\int_\Omega |\nabla u|^2\,\mathrm{d}x + \sum_{n\geqslant 2} 2na_n\int_\Omega |\nabla u|^{2n}\,\mathrm{d}x \leqslant \lambda_1(2)\int_\Omega u^2\,\mathrm{d}x \leqslant \int_\Omega |\nabla u|^2\,\mathrm{d}x.$$

Thus,

$$\int_\Omega |\nabla u|^{2n}\,\mathrm{d}x = 0, \quad \forall\ n \geqslant 2.$$

Since by the definition of K_0 we have $u \in K_0 \subset W_0^{1,2n}(\Omega)$, it follows that $u = 0$, a contradiction. The proof of theorem 1.1 is complete.

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