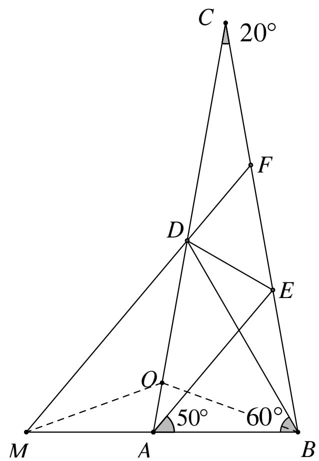
Feedback

On 103.39: Nguyen Xuan Tho writes: Here is a purely geometric proof of Chen's lemma, the statement of which is repeated below for convenience:

Let ABC be an isosceles triangle with $CA = CB$ and $\angle BCA = 20°$, let D be on AC such that $\angle ABD = 60°$, let E be on BC such that $\angle BAE = 50°$ and let F be on BC such that DF is parallel to AE. Then $CF = AB = BE$.



Let FD meet BA produced at M. Then $\angle BMD = 50°$, $MBD = 60°$ and $BDM = 70°$, and so ΔBDM is the triangle studied in [1].

We now locate the circumcentre O of ΔBDM. We have

$$\angle BOD = 2\angle BMD = 100°$$

and $\angle ODB = 40°$, so O lies on DA.

Now ΔOMB is isosceles with vertex angle 140°, as is $\angle DBC$. Hence these two triangles are similar and so $\frac{CD}{CB} = \frac{BO}{BM}$.

Other isosceles triangles in the figure are ΔBOA, ΔCAB, ΔBEA and ΔBFM, so $BO = BA$, $CB = CA$, $BA = BE$ and $BM = BF$. Hence $\frac{CD}{CA} = \frac{BE}{BF}$.

Now, by the intercept theorem on the parallel lines DF and AE, $\frac{CD}{CA} = \frac{CF}{CE}$ so $\frac{CF}{CE} = \frac{BE}{BF}$.

After a little algebra, we have $CF = BE = BA$ as required.

Reference

1. Jayant V. Narlikar, Beetham's Triangle, *Math. Gaz*. **104** (July 2020) pp. 327-330.

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On 104.19: Alan Beardon writes: Paul Stephenson uses Viète's infinite product formula

$$\prod_{n=1}^{\infty} \cos\left(\frac{\theta}{2^n}\right) = \frac{\sin\theta}{\theta}$$

to show that the centre of gravity of a uniform sector of angle 2θ of a circle is symmetrically placed at a distance $\dfrac{2\sin\theta}{3\theta}$ from its vertex. However, if this latter result is first derived directly, then Paul's argument can be 'reversed' to provide a (physical) *proof* of Viète's formula. If we regard the sector as a union of 'infinitesimal sector's' (as Paul does), then it is also clear that the sector of radius 1 (the shaded region in Figure 1) has the same centre of gravity as a uniform wire (the thick line in Figure 1) of radius $\frac{2}{3}$.

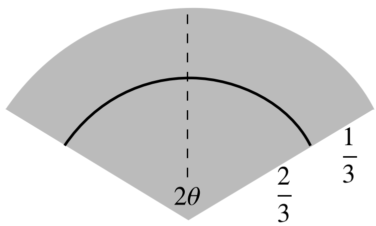

FIGURE 1: The sector and the wire

We may assume that the sector has a vertical line of symmetry and that its vertex is the origin. Then, by considering infinitesimal arcs of the wire, typically located at $\frac{2}{3}(\cos t, \sin t)$, where $\left|t - \frac{1}{2}\pi\right| \leqslant \theta$, we see that the centre of gravity of the wire is at $(\bar{x}, \bar{y})$, where $\bar{x} = 0$ and

$$2\theta\bar{y} = \frac{2}{3}\int_{\pi/2-\theta}^{\pi/2+\theta} \sin t\, dt = \frac{4\sin\theta}{3}.$$

This is shorter than many proofs but, more importantly, it raises for discussion of why, when computing the location of the centre of gravity, we may replace a part of the object by the total mass of the part located as a point mass at the centre of gravity of the part.

On 104.18, P. G. T. Lewis writes: For any triangle ABC, circumcentre O, with lines AO, BO, CO meeting the the sides at F, D, E:

If DE bisects $\angle ADB$ then DF bisects $\angle CDB$.

This is shown in the Note, being part of the proof. But it is independent of any other features in that configuration and seems worthy of comment.

Furthermore, $\angle EDF$ will be a right angle; which follows immedidately.

On 'Two plane geometry problems approached through analytic geometry', Graham Howlett writes: Looking at the diagram for Problem 2 in [1], it is seen that two important lines pass through A, namely AD and AE. While keeping with the spirit of the article, taking the origin of the coordinate system to be at A offers greater simplicity. We can further simplify by setting $B(-1, b)$ and $C(-2, 0)$, where $b > 0$.

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If the centre of the circle k is at $O(-1, c)$, then the opposite end of the diameter through A is at $(-2, 2c)$ and k is given by $x(x+2) + y(y-2c) = 0$. The condition that this passes through B is $-1 + b(b - 2c) = 0$ or $c = \frac{b^2 - 1}{2b}$. Note that $bc + 2 = \frac{b^2 + 3}{2} > 0$ (importantly, $\neq 0$).

Since $AD \perp OA$, line AD is $cy = x$ and so we have $D(bc, b) = \left(\frac{1}{2}(b^2 - 1), b\right)$.

Line CD is therefore $y = \frac{b}{bc + 2}(x + 2) = \frac{2b}{b^2 + 3}(x + 2)$. Its intersections with the circle k are given by

$$x(x + 2) + \frac{2b}{b^2 + 3}(x + 2)\left(\frac{2b}{b^2 + 3}(x + 2) - \frac{b^2 - 1}{b}\right) = 0.$$

Either $x + 2 = 0$, giving the point C, and entirely expected, or, on clearing fractions,

$$\left(b^2 + 3\right)^2 x + 4b^2(x + 2) - 2\left(b^2 - 1\right)\left(b^2 + 3\right) = 0$$

or $$\left(b^4 + 10b^2 + 9\right)x - 2b^4 + 4b^2 + 6 = 0$$

or $$\left(b^2 + 1\right)\left(\left(b^2 + 9\right)x - 2\left(b^2 - 3\right)\right) = 0.$$

So we have $E\left(\frac{2(b^2 - 3)}{b^2 + 9}, \frac{8b}{b^2 + 9}\right)$, and the equation of line AE is $\left(b^2 - 3\right)y = 4bx$. Hence we have $M\left(\frac{1}{4}\left(b^2 - 3\right), b\right)$, which is plainly half way between $B(-1, b)$ and $D\left(\frac{1}{2}\left(b^2 - 1\right), b\right)$.

Reference

1. Dina Kamber Hamzić and Zenan Šabanac, Two plane geometry problems approached through analytic geometry, *Math. Gaz.* 104 (July 2020) pp. 255-261.

On 'Paying for the end of life care in the UK', the authors write: Stubbs and Adetunji [1] showed that the amount A_N of capital C, invested at r_a per cent per annum compound K times per annum, that remains after N years is given by:

$$A_{Nk} = C\left(1 + \frac{r_a}{K}\right)^{NK} - \frac{E_X}{k}\left\{\frac{\left(1 + \frac{r_a}{K}\right)^K - 1}{\left(1 + \frac{r_a}{K}\right)^{K/k} - 1}\right\}\left\{\frac{(1 + R_E)^N - \left(1 + \frac{r_a}{K}\right)^{NK}}{(1 + R_E) - \left(1 + \frac{r_a}{K}\right)^K}\right\}$$

$$+ \frac{I_N}{l}\left\{\frac{\left(1 + \frac{r_a}{K}\right)^K - 1}{\left(1 + \frac{r_a}{K}\right)^{K/l} - 1}\right\}\left\{\frac{(1 + R_I)^N - \left(1 + \frac{r_a}{K}\right)^{NK}}{(1 + R_I) - \left(1 + \frac{r_a}{K}\right)^K}\right\} \qquad (1)$$

where k is the periodicity (typically $k = 12$) of total annual expenditure

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annuity E_X which increases at R_E per cent per annum and l is the periodicity (again, typically $l = 12$) of receipt of total annual income annuity I_N. This annual income I_N increases at R_I per cent per annum and it is expected, but necessarily required, that $E_X > I_N$ and $R_E > R_I$.

A requirement of (1), though is that $r_a \neq 0$ as is usually the case in practice. However, in the latter half of 2019 some economic predictions for the UK indicated that interest rates might decline to zero or even become negative [2], [3] while in March 2020 the IMF reported that the European Central Bank and the central banks of Denmark, Japan, Sweden, and Switzerland, had already started experimenting with negative interest rates [4]. Global and national economic uncertainty in the light of the Covid-19 pandemic, has only reinforced these predictions [5]. By the autumn of 2020, in both the UK and USA, the Bank of England [6] and the Federal Reserve Bank [7], respectively, had already reduced their lending rates to very close to zero. While (1) is still valid for $r_a < 0$ it does specifically require that $r_a \neq 0$. In this addendum to the model derived in [1], the case for $r_a = 0$ is developed and followed by a numerical illustration.

Using the same notation as in [1], and with $r_a = 0$, then the periodicity of expenditures and incomes per annum (k and l respectively) and the period of interest compounding K are of no consequence, so in this case, and using exactly the same procedure as in [1], for years, $0, 1, 2, 3, \ldots, N$:

$$A_0 = C$$
$$A_1 = C - E_X + I_N$$
$$A_2 = C - E_X\{1 + (1 + R_E)\} + I_N\{1 + (1 + R_I)\}$$
$$A_3 = C - E_X\{1 + (1 + R_E) + (1 + R_E)^2\} + I_N\{1 + (1 + R_I) + (1 + R_I)^2\}$$

and after N years:

$$A_N = C - E_X\left\{\frac{(1 + R_E)^N - 1}{R_E}\right\} + I_N\left\{\frac{(1 + R_I)^N - 1}{R_I}\right\}$$

which after rearrangement becomes:

$$(1 + R_E)^N - \phi(1 + R_I)^N - \varphi = 0 \tag{2}$$

$$\text{where } \phi = \frac{I_N R_E}{E_X R_I} \quad \text{and} \quad \varphi = \frac{E_X R_I - I_N R_E + R_E R_I (C - A_N)}{E_X R_I}. \tag{3}$$

Differentiating (2) with respect to N gives:

$$\{\ln(1 + R_E)\}(1 + R_E)^N - \phi\{\ln(1 + R_I)\}(1 + R_I)^N = 0. \tag{4}$$

Solving (2) for N by the Newton-Raphson method after γ iterations gives, from (2) and (4):

$$N_{\gamma+1} = N_\gamma - \frac{(1 + R_E)^N - \phi(1 + R_I)^N - \varphi}{\{\ln(1 + R_E)\}(1 + R_E)^N - \phi\{\ln(1 + R_I)\}(1 + R_I)^N}. \tag{5}$$

Numerical example

The data in Table 1 is used and is the same as that used originally in [1]. With zero annual interest on capital $\left(r_a = 0\right)$, it is wanted to know for how many years the capital will last for until it is reduced to £23,250.

	Symbol	Numerical Value
Amount of capital (£s) remaining after N years	A_N	23,250
Initial level of capital (£s) at time $N = 0$	C	250,000
Total annual expenditure (£s) on care home	E_X	36,000
Total annual income (£s)	I_N	15,000
Annual rates of interest on capital C	r_a	0
Annual rate of increase on expenditures E_X	R_E	0.06
Annual rate of increase (inflation index) on income I_N	R_I	0.03

TABLE 1: data values for numerical example

Substituting the values in Table 1 into (3) gives:

$$\phi = 0.833 \qquad \varphi = 0.545$$

and so (2) becomes:

$$1.06^N - 0.833 \times 1.03^N - 0.545 = 0. \tag{6}$$

Then using (5) to solve (6) for N gives $N = 8.103$ years.

For comparative purposes, Table 2 shows the time duration of annuities for annual interest rates of 1.0 and −1.0 per cent as found from solving (1) for N.

Annual interest rate	Time duration of annuities in years
$r_a = 0.01$	7.534
$r_a = 0$	8.103
$r_a = 0.01$	8.441

TABLE 2: Time duration of annuities arising from different interest rates

Table 2 shows that a reduction from 1 per cent to −1 per cent in annual interest rates, results in almost one year less time for which care home expenditures can be made.

Aside from declining interest rates, the gravity of the situation for self or family financed end of life care has been exacerbated by further rises in care home costs in 2020 consequent upon the Covid-19 pandemic. According to the charity Age UK, the costs of dealing with Coronavirus in care homes is being passed on to residents resulting in additional fee

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increases beyond what might otherwise be expected [8]. The ultimate conclusion can only be that, due to falling interest rates and rising costs, paying for end of life care in the UK will become an even more onerous financial burden for residents and their families than originally argued by Stubbs and Adetunji [1].

References

1. J. Stubbs and J. Adetunji, Paying for End of Life Care in the UK, *Math. Gaz*. **104** (November 2020) pp. 495-506.
2. No one wins in the rabbit-hole world of negative interest rates, *The Financial Times*, Monday 5th August 2019. Accessible at: https://www.ft.com/content/8081ff4c-b52f-11e9-bec9-fdcab53d6959
3. Savings: Are interest rates heading towards zero, *The Guardian*, Saturday 27th September 2019. Accessible at: https://www.theguardian.com/money/2019/sep/07/why-uk-savings-rates-may-be-heading-for-zero-what-to-do
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6. Our response to Coronavirus (Covid-19), (2020) Bank of England. Accessible at: https://www.bankofengland.co.uk/coronavirus
7. J. Cheng, D. Skidmore and D. Wessel, (2020) 'What's the Fed doing in response to the COVID-19 crisis? What more could it do? Brookings Institute, USA. Accessible at: https://www.brookings.edu/research/fed-response-to-covid19/
8. 'Care home residents are being clobbered with a 'Coronavirus bill'' (2020), Age UK, London. Accessible at: https://www.ageuk.org.uk/latest-press/articles/2020/06/care-home-residents-are-being-clobbered-with-a-coronavirus-bill/

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