Operational Solution of Some Problems in Viscous Fluid Motion. By A. F. CROSSLEY, B.A., St John's College.

[*Received* 17 February, *read* 27 February, 1928.]

Most of the results of this paper have been given before by Professor T. H. Havelock*, who obtained them by the solution of a difficult integral equation. The method used here is, however, much easier to handle than that given by Professor Havelock.

§ 1. The first problem is that of determining the motion arising from a heavy infinite vertical thin lamina falling through a viscous liquid confined between two fixed planes parallel to and equidistant from the moving one. Take the fixed boundaries to be the planes $x = \pm h$, and the axis of y downwards. If gravity is the only external force, ν the kinematic viscosity, the equation of fluid motion, which is two-dimensional, is

$$\frac{\partial v}{\partial t} = \nu \frac{\partial^2 v}{\partial x^2} \quad \text{.........................(1)},$$

with boundary conditions $v = 0$, $x = \pm h$, and $v = V$, $x = 0$, for all values of t, where V is the velocity of the lamina at any time. We shall take V zero initially. Using p as the symbolic $\frac{\partial}{\partial t}$, and writing

$$p = \nu q^2,$$

the operational form of (1) is

$$\frac{\partial^2 v}{\partial x^2} = q^2 v,$$

of which the required solution for $x > 0$ is

$$v = V \frac{\sinh q(h - x)}{\sinh qh} \quad \text{.....................(2)}.$$

If σ is the mass per unit area of the falling plate, μ the viscosity, equal to $\nu\rho$, where ρ is the density, then the velocity of the plate is given by

$$\frac{\partial V}{\partial t} = g + \frac{2\mu}{\sigma}\left(\frac{\partial v}{\partial x}\right)_{x=0} \quad \text{.....................(3)}.$$

By (2),
$$\left(\frac{\partial v}{\partial x}\right)_{x=0} = -Vq \coth qh,$$

therefore
$$V\left(p + \frac{2\mu}{\sigma} q \coth qh\right) = g \quad \text{..................(4)}.$$

* *Phil. Mag.* vol. 42, Nov. 1921.

If we write $$qh = i\lambda,\ \frac{2\rho h}{\sigma} = k,$$

this becomes $$V = \frac{-gh^2 \sin\lambda}{\nu\lambda(\lambda\sin\lambda - k\cos\lambda)} \quad \text{..................(5)},$$

which interprets as

$$V = \frac{gh\sigma}{2\mu} - \frac{4g\rho h^3}{\nu\sigma}\sum\frac{e^{-\nu\lambda^2 t/h^2}}{\lambda^2\{\lambda^2 + k(1+k)\}} \quad \text{.........(6)},$$

where summation is in regard to the positive roots of the equation

$$\lambda\tan\lambda = k \quad \text{..........................(7)}.$$

The restriction to positive roots arises from the substitution $p = \nu q^2 = -\nu\lambda^2/h^2$. The first term on the right of (6) represents the limiting steady velocity acquired.

The operational expression for the fluid velocity is obtained from (2) and (5), and is

$$v = \frac{-gh^2\sin\lambda\left(1 - \frac{x}{h}\right)}{\nu\lambda(\lambda\sin\lambda - k\cos\lambda)},$$

which interprets as

$$v = \frac{gh^2}{\nu k}\left(1 - \frac{x}{h}\right) - \frac{2gh^2k}{\nu}\sum\frac{e^{-\nu\lambda^2 t/h^2}\sin\lambda\left(1 - \frac{x}{h}\right)}{\lambda^2\{\lambda^2 + k(1+k)\}\sin\lambda},$$

the values of λ being the positive roots of (7), as before.

§ 2. When the fluid extends to infinity on both sides of the plate, the velocity of the plate is obtained by making h infinite in equation (4); thus

$$V = \frac{g}{p + \frac{2\mu q}{\sigma}} = \frac{g}{\nu}\frac{1}{q(q+\alpha)}$$

$$= \frac{g}{\alpha\nu}\left(\frac{1}{q} - \frac{1}{q+\alpha}\right) \quad \text{..........................(8)},$$

where $$\alpha = \frac{2\rho}{\sigma}.$$

We have $$\frac{1}{q} = 2\sqrt{\frac{\nu t}{\pi}},$$

and, putting $x = 0$ in equation (9) below,

$$\frac{1}{q+\alpha} = \frac{1}{\alpha}[1 - \exp\gamma^2 . (1 - \operatorname{Erf}\gamma)],$$

where $$\gamma = \alpha\sqrt{\nu t} = \frac{2\rho}{\sigma}\sqrt{\nu t};$$

whence $$V = \frac{g}{\alpha^2\nu}\left[\frac{2\gamma}{\sqrt{\pi}} - 1 + \exp\gamma^2 . (1 - \operatorname{Erf}\gamma)\right].$$

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Making h infinite in (2), and substituting for V from (8), we find that the fluid velocity is given by

$$v = \frac{g}{\alpha\nu}\left(\frac{1}{q} - \frac{1}{q+\alpha}\right)e^{-qx}.$$

Write

$$\xi = \frac{x}{2\sqrt{\nu t}}.$$

Then*

$$\frac{e^{-qx}}{q+\alpha} = \frac{1}{\alpha}\left[1 - \operatorname{Erf}\xi - \exp(\gamma^2 + \alpha x) . \{1 - \operatorname{Erf}(\xi+\gamma)\}\right] \ldots (9).$$

Taking the limit of both sides as $\alpha \to 0$,

$$\frac{e^{-qx}}{q} = -x(1 - \operatorname{Erf}\xi) + 2\sqrt{\frac{\nu t}{\pi}}\exp(-\xi^2),$$

therefore

$$v = \frac{g}{\alpha^2\nu}\left[-(1 - \operatorname{Erf}\xi)(1 + \alpha x) + \frac{2\gamma}{\sqrt{\pi}}\exp(-\xi^2) + \exp(\gamma^2 + \alpha x) . \{1 - \operatorname{Erf}(\xi+\gamma)\}\right].$$

§ 3. If there is a variable acceleration acting on the plate, say $a\cos nt$, then the equation of motion (4) may be replaced by

$$V\left[p + \frac{2\mu}{\sigma} q \coth qh\right] = a\cos nt = \frac{ap^2}{p^2+n^2},$$

or

$$V = \frac{-ah^2\lambda^3\sin\lambda}{\nu\left(\lambda^4 + \dfrac{h^4n^2}{\nu^2}\right)(\lambda\sin\lambda - k\cos\lambda)}.$$

The interpretation of this function by the partial-fraction rule leads to the result

$$V = \frac{ah^2}{2\nu}(A\cos nt + B\sin nt) - \frac{4\nu\rho a h^3}{\sigma}\Sigma\frac{\lambda^2 e^{-\nu\lambda^2 t/h^2}}{(\nu^2\lambda^4 + h^4n^2)\{\lambda^2 + k(1+k)\}},$$

where

$$A = \frac{k}{\Delta}(\sinh 2\beta + \sin 2\beta),$$

$$B = -\frac{1}{\Delta}[2\beta(\cosh 2\beta - \cos 2\beta) + k(\sinh 2\beta - \sin 2\beta)],$$

$$\Delta = \beta\,[(k^2 + 2\beta^2)\cosh 2\beta + (k^2 - 2\beta^2)\cos 2\beta + k\beta(\sinh 2\beta - \sin 2\beta)],$$

$$\beta^2 = \frac{h^2 n}{2\nu}.$$

* H. Jeffreys, *Operational Methods in Mathematical Physics* (Cambridge Tracts), pp. 59—60.

The fluid velocity may be obtained similarly by interpreting

$$v = \frac{-ah^2\lambda^3 \sin\lambda\left(1-\frac{x}{h}\right)}{\nu\left(\lambda^4+\frac{h^4n^2}{\nu^2}\right)(\lambda\sin\lambda - k\cos\lambda)}.$$

§ 4. The problem of a rotating circular cylinder filled with viscous fluid, the motion being symmetrical about the axis, is solved in a similar way to those already considered. If r is the distance from the axis, the equation of fluid motion is

$$\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} - \frac{v}{r^2}\right),$$

or

$$\frac{\partial^2 v}{\partial r^2} + \frac{1}{r}\frac{\partial v}{\partial r} + \left(q^2 - \frac{1}{r^2}\right)v = 0,$$

if

$$p = -\nu q^2,$$

whence

$$v = r\omega = AJ_1(qr),$$

the velocity remaining finite on the axis. The boundary condition is $\omega = \Omega$ at $r = a$, where Ω is the angular velocity of the cylinder at any time. This gives

$$\omega = \frac{a}{r}\Omega\frac{J_1(qr)}{J_1(qa)} = \frac{a}{r}\Omega\frac{J_1\left(\frac{\lambda r}{a}\right)}{J_1(\lambda)} \quad \text{............(10)},$$

where $qa = \lambda$. The frictional couple acting on unit length of the cylinder is

$$\left[2\pi\mu r^3\frac{\partial\omega}{\partial r}\right]_{r=a} = 2\pi\mu a^2\Omega\left[\frac{\lambda J_1'(\lambda)}{J_1(\lambda)} - 1\right].$$

If the cylinder rotates from rest under the action of a constant couple N, and if I be its moment of inertia, both of these per unit length, the equation of motion is

$$I\frac{\partial\Omega}{\partial t} = N + 2\pi\mu a^2\Omega\left[\frac{\lambda J_1'(\lambda)}{J_1(\lambda)} - 1\right],$$

whence

$$\Omega = \frac{-Na^2/I\nu}{\lambda^2 + k\left\{1 - \frac{\lambda J_1'(\lambda)}{J_1(\lambda)}\right\}}$$

$$= \frac{-Na^2/I\nu \,.\, J_1(\lambda)}{\lambda^2 J_1(\lambda) + k\lambda J_2(\lambda)} \quad \text{....................(11)},$$

where $k = 2\pi\rho a^4/I$. This function has a double pole at $\lambda = 0$. Since

$$J_1(\lambda) = \frac{\lambda}{2} - \frac{\lambda^3}{16} + \ldots, \quad J_2(\lambda) = \frac{\lambda^2}{8} - \frac{\lambda^4}{96} + \ldots \quad \text{...(12)},$$

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it is found by division that near $\lambda = 0$, Ω takes the form

$$-\frac{Na^2}{I\nu}\left\{\frac{1}{\lambda^2}\frac{4}{k+4} - \frac{k}{6(k+4)^2}\right\} + O(\lambda^2),$$

so that, on evaluating the exponential terms in the usual way, we have the interpretation of Ω,

$$\Omega = \frac{Na^2}{I\nu}\left[\frac{4\nu t}{a^2(k+4)} + \frac{k}{6(k+4)^2} - \Sigma\frac{2ke^{-\nu\lambda^2 t/a^2}}{\lambda^2(\lambda^2+4k+k^2)}\right],$$

where summation is in regard to the non-zero positive roots of

$$\lambda J_1(\lambda) + kJ_2(\lambda) = 0.$$

The operational solution for the angular velocity of the fluid is, from (10) and (11),

$$\omega = -\frac{Na^3}{I\nu r}\frac{J_1\left(\frac{\lambda r}{a}\right)}{\lambda^2 J_1(\lambda) + k\lambda J_2(\lambda)}.$$

Using (12) again, we find that near $\lambda = 0$,

$$\omega = -\frac{4Na^2}{I\nu(k+4)}\left[\frac{1}{\lambda^2} + \frac{k+6}{12(k+4)} - \frac{1}{8}\frac{r^2}{a^2}\right] + O(\lambda^2),$$

giving, with the exponential terms,

$$\omega = \frac{4N}{I(k+4)}\left[t - \frac{k+6}{12(k+4)}\frac{a^2}{\nu} + \frac{r^2}{8\nu}\right] - \frac{2a^3Nk}{\nu Ir}\Sigma\frac{J_1(\lambda r/a)\,.\,e^{-\nu\lambda^2 t/a^2}}{\lambda^2(\lambda^2+4k+k^2)J_1(\lambda)}.$$

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