Mathematical Proceedings of the Cambridge Philosophical Society

VOL. 103 MARCH 1988 PART 2

Math. Proc. Camb. Phil. Soc. (1988), **103**, 193
Printed in Great Britain

Representation–theoretic interpretation of a formula of D. E. Littlewood

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(*Received* 27 *March* 1987)

This note is a continuation of our attempts (see [**3**]) to give a satisfactory representation–theoretic justification of the following formula of D. E. Littlewood:

$$\prod_i (1-x_i) \prod_{i<j} (1-x_i x_j) = \sum_I (-1)^{(|I|+r(I))/2} s_I(x), \tag{1}$$

where s_I is the Schur symmetric function corresponding to a partition I, $|I|$ is the weight of $I, r(I)$ is the rank of I, and the summation ranges over all self-conjugate partitions (i.e. partitions I such that $I = \tilde{I}$ where $\tilde{I}$ is the partition conjugate to I).

The inverse of the left-hand side of (1) is the character of the representation $S.(U+\Lambda^2 U)$ over $Gl(U)$, where $S.(V)$ means the symmetric algebra of vector space V over a field of characteristic zero. In [3] we used the Koszul complex over $S.(U+\Lambda^2 U)$ to express the left-hand side of (1) as the alternating character P of the representation $\Lambda(U+\Lambda^2 U)$ of $Gl(U)$. We decomposed $\Lambda(U+\Lambda^2 U)$ into irreducible representations (Schur functors) and showed that only terms corresponding to self-conjugate partitions remain in P after cancellation of other terms, thus proving (1).

The main result of this note is the following

THEOREM. *Let U be a vector space over a field K of characteristic zero and consider $U+\Lambda^2 U$ as a Lie algebra with the only non-trivial bracket $[u,v] = u \wedge v \in \Lambda^2 U$, $u, v \in U$. Let $R = R(U)$ denote the enveloping algebra of $U+\Lambda^2 U$. There exists an exact complex of free R-modules*

$$C(U): \ldots \rightarrow R \otimes_K \Big(\bigoplus_{I=\tilde{I}} S_I(U)\Big) \rightarrow \ldots \rightarrow R \otimes_K S_{2,1}(U) \rightarrow \ldots \rightarrow R \otimes_K U \rightarrow R \rightarrow K \rightarrow 0, \tag{2}$$

where S_I is the Schur functor corresponding to a partition I and the summation in the nth component of $C(U)$ ranges over all self-conjugate partitions I such that $n = (|I|+r(I))/2$.

Obviously the Theorem implies (1). Indeed, the exactness of $C(U)$ gives

$$\operatorname{char} R(U) . \operatorname{char} \Big(\sum_{I=\tilde{I}} (-1)^{(|I|+r(I))/2} S_I(U) \Big) = 1,$$

and since char $R(U) = \operatorname{char} S.(U + \Lambda^2 U)$ by the Poincaré–Birkhoff–Witt Theorem, we get (1).

To proceed with the proof of the Theorem, let us notice that R can be endowed with a grading by setting $\deg x = 1$ for $x \in U$ and $\deg x = 2$ for $x \in \Lambda^2 U$. One knows that, for R viewed as a graded K-algebra, there exists a minimal free resolution of K over R, say

$$\mathbf{E}: \ldots \to E_n \to \ldots \to E_1 \to R \to K \to 0,$$

and every two minimal resolutions are isomorphic. The minimality of $\mathbf{E}$ means that the complex $\mathbf{E} \otimes_R K$ has zero differentials, i.e. that $\operatorname{Tor}^R(K,K) = H(\mathbf{E} \otimes_R K) = \mathbf{E} \otimes_R K$. In other terms $E_n = \operatorname{Tor}^R_n(K,K) \otimes_K R$ gives the nth component of the minimal resolution. This implies that the Theorem follows from the following

PROPOSITION. *If R is the enveloping K-algebra of the Lie algebra $U + \bigwedge^2 U$, then $\operatorname{Tor}^R_n(K,K) \approx \bigoplus_I S_I(U)$, where the summation ranges over all self-conjugate partitions I such that $n = (|I| + r(I))/2$.*

To compute $\operatorname{Tor}^R_n(K,K)$ we are going to use a special resolution of K over R, its Koszul complex.

With an arbitrary Lie algebra L over a field K one can associate the Koszul complex $\mathbf{K}(L) = \{K_p, d_p\}$ over the enveloping algebra R of L:

$$K_p = R \otimes_K \Lambda^p L, \quad d_p: K_p \to K_{p-1},$$

$$d_p(r \otimes t_1 \wedge \ldots \wedge t_p) = \sum_{i=1}^{p} (-1)^{i+1} r\varphi(t_i) \otimes t_1 \wedge \ldots \wedge \hat{t}_i \wedge \ldots \wedge t_p$$

$$+ \sum_{1 \leqslant i < j \leqslant p} (-1)^{i+j} r \otimes [t_i, t_j] \wedge t_1 \wedge \ldots \wedge \hat{t}_i \wedge \ldots \wedge \hat{t}_j \wedge \ldots \wedge t_p,$$

where $r \in R, t_i \in L$, and $\varphi: L \to R$ is the canonical embedding. It turns out that this complex is acyclic for any Lie algebra (see [1], chapter XIII) and is a free resolution of K over R. Hence $\operatorname{Tor}^R(K,K) = H(\mathbf{K}(L) \otimes_R K)$.

Proof of the Proposition. To compute $\operatorname{Tor}^R(K,K)$ in the case of the Lie algebra $L = U + \Lambda^2 U$ we must analyse the complex $\mathbf{K}(L) \otimes_R K$. It is clear from the explicit formula of the differential in $\mathbf{K}(L)$ that

$$\mathbf{K}(L) \otimes_R K: \ldots \to \Lambda^p(U + \Lambda^2 U) \xrightarrow{d_p \otimes 1} \Lambda^{p-1}(U + \Lambda^2 U) \to \ldots.$$

Decomposing $\Lambda^p(U + \Lambda^2 U)$ as $\Sigma_{i+j=p} \Lambda^i(U) \otimes \Lambda^j(\Lambda^2 U)$ we see that the differential $d_p \otimes 1$ can be expressed as a composition

$$\Lambda^i(U) \otimes \Lambda^j(\Lambda^2 U) \to \Lambda^{i-2}(U) \otimes \Lambda^2(U) \otimes \Lambda^j(\Lambda^2 U) \to \Lambda^{i-2}(U) \otimes \Lambda^{j+1}(\Lambda^2 U),$$

where the first map consists in diagonalizing $\Lambda^i(U)$ into $\Lambda^{i-2}(U) \otimes \Lambda^2(U)$ and the second one is the multiplication in the algebra $\Lambda(\Lambda^2 U)$. Therefore $\mathbf{K}(L) \otimes_R K$ splits into a direct sum of the following complexes:

$$\mathbf{F}^{(2p)}(U): 0 \to \Lambda^{2p}(U) \to \Lambda^{2p-2}(U) \otimes \Lambda^2(U) \to \ldots$$
$$\to \Lambda^{2p-2i}(U) \otimes \Lambda^i(\Lambda^2 U) \to \ldots \to \Lambda^p(\Lambda^2 U) \to 0,$$

$$\mathbf{F}^{(2p+1)}(U): 0 \to \Lambda^{2p+1}(U) \to \Lambda^{2p-1}(U) \otimes \Lambda^2(U) \to \ldots$$
$$\to \Lambda^{2p+1-2i}(U) \otimes \Lambda^i(\Lambda^2 U) \to \ldots \to U \otimes \Lambda^p(\Lambda^2 U) \to 0.$$

Notice that the total degree of the term $\Lambda^i(U) \otimes \Lambda^j(\Lambda^2 U)$ in $\boldsymbol{K}(L)$ is $i+j$.

We write $$\boldsymbol{F}^+(U) = \oplus_p \boldsymbol{F}^{(2p)}(U), \quad \boldsymbol{F}^-(U) = \oplus_p \boldsymbol{F}^{(2p+1)}(U)$$

and $$\boldsymbol{F}(U) = \boldsymbol{K}(L) \otimes_R K = \boldsymbol{F}^+(U) \oplus \boldsymbol{F}^-(U);$$

hence we have $$\operatorname{Tor}^R(K,K) = H(\boldsymbol{F}(U)) = H(\boldsymbol{F}^+(U)) \oplus H(\boldsymbol{F}^-(U)).$$

It follows from the discussion on p. 4 of [**3**] that

$$H_n(\boldsymbol{F}^+(U)) \supset \bigoplus_{\substack{I=\tilde{I} \\ r(I)\,\text{even}}} S_I(U) \quad \text{and} \quad H_n(\boldsymbol{F}^-(U)) \supset \bigoplus_{\substack{I=\tilde{I} \\ r(I)\,\text{odd}}} S_I(U),$$

where the summation ranges over all self-conjugate partitions I such that $n = (|I|+r(I))/2$. Indeed, $S_I(U)$ for $I = \tilde{I}$ occurs only once in $\boldsymbol{F}^+(U)$ or $\boldsymbol{F}^-(U)$ in the term $\Lambda^i(U) \otimes \Lambda^j(\Lambda^2 U)$ where $i = r(I)$, $j = (|I|-r(I))/2$. To obtain equalities in the above inclusions we need a stronger tool.

It turns out that a computation of $H(\boldsymbol{F}^+(U))$ and $H(\boldsymbol{F}^-(U))$ is closely related to the syzygies of the ideal of 2×2 minors of a generic symmetric matrix. These syzygies were calculated in [**2**] in terms of Schur functors and we are grateful to P. Pragacz for pointing out this relationship to us.

Let us recall that the symmetric algebra $S.(S_2 U)$ can be regarded as an algebra of polynomial functions on the affine space $\operatorname{Sym}_m(K)$ of symmetric $m \times m$ matrices with entries in a field K, where $m = \dim_K U$. One knows that $S.(S_2 U) = \oplus_J S_J(U)$, summed over all partitions J with even parts. The ideal generated by all 2×2 minors corresponds in the functorial notation to the ideal generated by $S_{22}(U)$ in $S.(S_2 U)$. To quote the result we need about syzygies of this ideal we define the following family of complexes

$$\boldsymbol{G}^{(2p)}(U): 0 \to \Lambda^p(S_2 U) \to S_2 U \otimes \Lambda^{p-1}(S_2 U) \to \dots$$
$$\to S_{2p-2i}(U) \otimes \Lambda^i(S_2 U) \to \dots \to S_{2p}(U) \to 0,$$
$$\boldsymbol{G}^{(2p+1)}(U): 0 \to U \otimes \Lambda^p(S_2 U) \to S_3(U) \otimes \Lambda^{p-1}(S_2 U) \to \dots$$
$$\to S_{2p+1-2i}(U) \otimes \Lambda^i(S_2 U) \to \dots \to S_{2p+1}(U) \to 0,$$

where differentials are described similarly as in $\boldsymbol{F}^{(2p)}(U)$ and $\boldsymbol{F}^{(2p+1)}(U)$. However, the grading in $\boldsymbol{G}^{(q)}(U)$ is defined by the exterior power index; notice the difference with $\boldsymbol{F}^{(q)}(U)$. We write $\boldsymbol{G}^+(U) = \oplus_p \boldsymbol{G}^{(2p)}(U)$, $\boldsymbol{G}^-(U) = \oplus_p \boldsymbol{G}^{(2p+1)}(U)$. Now we can quote from chapter III of [**2**] the following

LEMMA. (*a*) $\operatorname{Tor}_n^{S.(S_2U)}(S.(S_2 U)/(S_{22} U), K) \approx H_n(\boldsymbol{G}^+(U)) \approx \oplus_I S_I(U)$, *where the summation is over all self-conjugate partitions such that $r(I)$ is even and $n = (|I|-r(I))/2$;*
(*b*) $\operatorname{Tor}_n^{S.(S_2U)}(M, K) \approx H_n(\boldsymbol{G}^-(U)) \approx \oplus_I S_I(U)$, *where M is the cokernel of the composition*

$$S_{21}(U) \otimes S.(S_2 U) \to U \otimes S_2 U \otimes S.(S_2 U) \to U \otimes S.(S_2 U),$$

and where the summation is over all self-conjugate partitions I such that $r(I)$ is odd and $n = (|I|-r(I))/2$.

In both cases $S_I(U)$ is contained in $S_{r(I)}(U) \otimes \Lambda^{(|I|-r(I))/2}(S_2 U)$.

The first equalities in both formulae are elementary and can be proved using the Koszul resolution of K over $S.(S_2 U)$. The hard part is the calculation of the homology of $\boldsymbol{G}^+(U)$ and $\boldsymbol{G}^-(U)$.

To compare the homology of $G^{(q)}(U)$ and $F^{(q)}(U)$ we observe that one can pass from $G^{(q)}(U)$ to $F^{(q)}(U)$ using two operations. The first operation is taking the dual complex

$$(G^{(2p)}(U))^*: 0 \to S_{2p}(U^*) \to S_{2p-2}(U^*) \otimes S_2(U^*) \to \dots \to S_{2p-2i}(U^*) \otimes \Lambda^i(S_2(U^*)) \to \dots \to \Lambda^p(S_2(U^*)) \to 0.$$

The second one is the Young duality that sends $S_I(U)$ into $S_{\tilde{I}}(U)$. This means that $S_{2j}(U^*) \otimes \Lambda^i(S_2(U^*))$ goes into $\Lambda^{2j}(U^*) \otimes \Lambda^i(\Lambda^2(U^*))$ and the complex $(G^{(2p)}(U))^*$ into $F^{(2p)}(U^*)$. Since the homology of $G^{(2p)}(U)$ consists of $S_I(U)$ with I self-conjugate (see (*a*) of the Lemma) we infer that

$$H_n(F^+(U)) \approx \oplus_I S_I(U),$$

where I is self-conjugate, $r(I)$ is even and $n = (|I| + r(I))/2$, this last equality coming from the fact that we change the grading when passing from $G^+(U)$ to $F^+(U)$.

In a similar way we interpret $H_n(F^-(U))$ using (*b*) of the Lemma thus completing the proof of the Proposition.

It would be interesting to have an explicit construction of the complex (2).

It is perhaps worth mentioning at this point that the Lemma implies the following formulae:

$$(\sum_i s_{2i}(x)) \prod_{i \leqslant j} (1 - x_i x_j) = \sum_{\substack{I = \tilde{I} \\ r(I) \text{ even}}} (-1)^{(|I| - r(I))/2} s_I(x),$$

$$(\sum_i s_{2i+1}(x)) \prod_{i \leqslant j} (1 - x_i x_j) = \sum_{\substack{I = \tilde{I} \\ r(I) \text{ odd}}} (-1)^{(|I| - r(I))/2} s_I(x),$$

because $\operatorname{char} S.(S_2 U)/(S_{22} U) = \Sigma_i s_{2i}(x)$ and $\operatorname{char} M = \Sigma_i s_{2i+1}(x)$. These formulas when added up give the formula

$$\prod_i (1 + x_i) \prod_{i<j} (1 - x_i x_j) = \sum_{I = \tilde{I}} (-1)^{(|I| - r(I))/2} s_I(x).$$

By substituting $-x_i$ in place of x_i we obtain the original formula (1) since we have $s_I(-x_1, -x_2, \dots) = (-1)^{|I|} s_I(x_1, x_2, \dots)$ for any partition I.

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