MEAN-VALUES OF THE RIEMANN ZETA-FUNCTION

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§1. *Introduction.* Let

$$I_k(T) = \int_T^{2T} |\zeta(\tfrac{1}{2} + it)|^{2k} dt.$$

Asymptotic formulae for $I_k(T)$ have been established for the cases $k = 1$ (Hardy-Littlewood, see **[13]**) and $k=2$ (Ingham, see **[13]**). However, the asymptotic behaviour of $I_k(T)$ remains unknown for any other value of k (except the trivial $k=0$, of course). Heath-Brown, **[6]**, and Ramachandra, **[10]**, **[11]**, independently established that, assuming the Riemann Hypothesis, when $0 \leqslant k \leqslant 2$, $I_k(T)$ is of the order $T(\log T)^{k^2}$. One believes that this is the right order of magnitude for $I_k(T)$ even when $k = 2$ and indeed expects an asymptotic formula of the form

$$I_k(T) = (C_k + o(1))T(\log T)^{k^2},$$

where C_k is a suitable positive constant. It is not clear what the value of C_k should be.

In **[3]**, Conrey and Ghosh showed that the Riemann Hypothesis implies

$$I_k(T) \geqslant (1 + o(1))T \sum_{n \leqslant T} \frac{d_k^2(n)}{n} = (c_k + o(1))T(\log T)^{k^2}$$

where

$$c_k = \frac{1}{\Gamma(k^2+1)} \prod_p \left\{\left(1 - \frac{1}{p}\right)^{k^2} \sum_{m=0}^{\infty} \left(\frac{k(k+1)\dots(k+m-1)}{m!}\right)^2 p^{-m}\right\}.$$

The dependency on the Riemann Hypothesis was removed by Balasubramanian and Ramachandra, **[2]**, in the special case where k is an integer. In a later paper, **[4]**, Conrey and Ghosh used results of Balasubramanian, Conrey and Heath-Brown, **[1]**, to improve this lower bound. In this paper we obtain further improvements on the results of Conrey and Ghosh. In the case when $k \geqslant 3$ is an integer our results double the previous bounds and a comparable (although smaller) improvement is obtained in the non-integral cases as well. This work is motivated and inspired by Ramachandra's proof of the fourth power moment, **[9]**.

[MATHEMATIKA, **42** (1995), 158–174]

Let $k \geqslant 3$ be an integer. Let $N = T^{\theta}$ where $\theta \in (0, 1)$ will be fixed later. Let $r = k-1$ and

$$A_r(s, P) = \sum_{n \leqslant N} \frac{d_r(n) P(\log n / \log N)}{n^s}$$

where $s = \sigma + it$ is a complex variable and P is a polynomial which will be chosen appropriately. The lower bounds of **[4]** were obtained by analysing the right-hand side of the inequality

$$0 \leqslant \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 |\zeta^r(s) - A_r(s, P)|^2 ds.$$

We 'twist' the right-hand side and consider

$$0 \leqslant \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 |\zeta^r(s) - A_r(s, P) - \chi^r(s) A_r(1-s, P)|^2 ds$$

where $\chi(s)$ is the factor arising from the asymmetric functional equation

$$\zeta(s) = \chi(s)\zeta(1-s).$$

THEOREM 1.1. *Suppose $\theta < \frac{1}{2}$ when $k=3$. Put*

$$J_4 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \chi^r(s) A_r(1-s, P)^2 ds.$$

With the above notations the bound

$$J_4 = O(T^{1-\varepsilon})$$

holds, unconditionally when $k=3$ and on the assumption of the Lindelöf Hypothesis when $k \geqslant 4$, for arbitrary fixed $\varepsilon > 0$.

COROLLARY 1.2. *The lower bound*

$$I_3 \geqslant (20{\cdot}26 + o(1)) c_3 T (\log T)^9$$

holds unconditionally. If the Lindelöf Hypothesis is assumed then the following asymptotic lower bounds for $F_k = I_k/(c_k T(\log T)^{k^2})$, $(4 \leqslant k \leqslant 6)$ hold:

$$F_4 \geqslant 410, \qquad F_5 \geqslant 6484, \qquad F_6 \geqslant 56260.$$

Observe that Corollary 1.2 improves Corollary 1 of **[4]** by a factor of 2. Conrey and Ghosh also obtain results assuming the validity of their Theorems 1 and 2 in a wider range of θ. This assumption is roughly of the same strength as the conjecture in **[1]**. Upon making this assumption we obtain corresponding improvements since our Theorem 1.1 is valid for all $\theta < 1$ when $k \geqslant 4$.

Corollary 1.3. *Assuming the Lindelöf Hypothesis and that Theorems* 1 *and* 2 *of Conrey and Ghosh,* [4], *remain valid for any* $\theta < 1$ *the improved asymptotic lower bounds*

$$I_4 \geqslant 43056 c_4 T(\log T)^{16}; \qquad I_5 \geqslant 96877600 c_5 T(\log T)^{25}$$

hold. Also as $k \to \infty$

$$I_k \geqslant 2c(ek/2)^{2k-3/2} c_k T(\log T)^{k^2}$$

where $c = 1/(e\sqrt{2\pi e})$.

If $k \geqslant 7$ then the assumption that $\theta < 1$ is permissible is necessary to obtain improvements over the bound of Conrey and Ghosh, [3]. This is a consequence of the θ^{k^2}, present in our Lemma 2.3, which rapidly goes to 0 as k becomes large. However the results of Conrey and Ghosh, [3], can be improved unconditionally by considering

$$\int_{1/2+iT}^{1/2+2iT} |\zeta^k(s) - A_k(s, P) - \chi^k(s) A_k(1-s, P)|^2 ds.$$

Let

$$K_4 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} \chi^k(s) A_k(1-s, P)^2 ds.$$

Theorem 1.4. *If* k *is an integer and* $\theta < 1$ *then, unconditionally,*

$$|K_4| = O(T^{1-\varepsilon}).$$

Theorem 1.4 may be proved along the exact same lines as Theorem 1.1. As a consequence we obtain the aforementioned improvement.

Corollary 1.5. *If* $k \geqslant 2$ *is an integer then*

$$I_k \geqslant (2 + o(1)) c_k T(\log T)^{k^2}.$$

Note that Corollary 1.5 is independent of any hypothesis. This is a consequence of the results of [2] which remove the dependency upon the Riemann Hypothesis in [3].

We now turn to the case when k is not an integer. The difficulties in this case arise from the failure of the equality

$$\frac{\zeta^k(s)}{\zeta^k(1-s)} = \chi^k(s)$$

when k is not an integer.

THEOREM 1.6. *Suppose $k>2$ and assume the truth of the Riemann Hypothesis. Put*

$$H(x,y,z)=\log\left(\frac{(r+1)\log T-2\log(xy/z)}{(r-1)\log T-2\log z}\right)$$
$$+r\left(\theta-\frac{\log y}{\log T}\right)\log\left(\frac{(r-\theta)\log T+2\log z-\log x}{(r-1-\theta)\log T-\log z}\right).$$

Then, if $\theta<\min(1/2,(r-1)/2)$,

$$|J_4|=\left|\int_{1/2+iT}^{1/2+2iT}|\zeta(s)|^2\frac{\zeta^r(s)}{\zeta^r(1-s)}A_r(1-s,P)^2ds\right|$$
$$\leqslant\frac{T\log T}{\pi}|e^{2\pi ik}-1|\sum_{u,v\leqslant T^\theta}\frac{b_u b_v}{[u,v]}H(u,v,(u,v)),$$

where $b_u=d_r(u)P(\log u/(\theta\log T))$.

COROLLARY 1.7. *Put*

$$Q(x)=P(x)\left(\log\left(\frac{r+1-2\theta}{r-1-2\theta}\right)+r(\theta-\theta x)\log\left(\frac{r}{r-1-2\theta}\right)\right)$$

and

$$G(x)=\left(\int_x^1 r(z-x)^{r-1}P(z)dz\right)\left(\int_x^1 r(z-x)^{r-1}Q(z)dz\right).$$

Then, if $\theta<\min(1/2,(r-1)/2)$,

$$|J_4|\leqslant\frac{\Gamma(k^2+1)}{\Gamma^2(r+1)\Gamma(r^2+1)}c_k\theta^{k^2-1}T(\log T)^{k^2}\left(\int_0^1 y^{r^2-1}G(y)dy\right).$$

COROLLARY 1.8. *If $\theta<1$ is permissible in Theorems 1 and 2 of Conrey and Ghosh,* [4], *and in our Theorem 1.6, then, as $k\to\infty$,*

$$I_k\geqslant\left(2+O\left(\frac{1}{k}\right)\right)c(ek/2)^{2k-3/2}c_kT(\log T)^{k^2}$$

where, as in Corollary 1.3, $c=1/(e\sqrt{2\pi e})$.

As in the integral case, when k becomes large it is necessary to assume the $\theta=1$ conjectures to obtain improvements upon the bound of [3]. Again, as in the integral case, considering

$$0 \leqslant \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} \left| \zeta^k(s) - A_k(s,P) - \frac{\zeta^k(s)}{\zeta^k(1-s)} A_k(1-s,P) \right|^2 ds$$

leads to improvements that are independent of the $\theta = 1$ conjectures.

THEOREM 1.9. *Suppose k is not an integer and assume the validity of the Riemann Hypothesis. Then, if $\theta < \min(1, k/2)$,*

$$|K_4| = \left| \int_{1/2+iT}^{1/2+2iT} \frac{\zeta^k(s)}{\zeta^k(1-s)} A_k(1-s,P)^2 ds \right|$$

$$\leqslant |e^{2\pi ik} - 1| \frac{T}{\pi} \left(\sum_{n \leqslant T^\theta} \frac{d_k(n)^2 P(\log n/(\theta \log T))^2 \log T}{n(k \log T - 2 \log n)} \right.$$

$$\left. + 2k \sum_{n \leqslant T^\theta} \frac{d_k(n)^2 P(\log n/(\theta \log T))^2}{n} \log \left(\frac{k \log T - 2 \log n}{(k-\theta) \log T - \log n} \right) \right).$$

COROLLARY 1.10. *Put $\theta = \frac{1}{2}k - \varepsilon$ if $k \leqslant 2$ and $\theta = 1 - \varepsilon$ if $k \geqslant 2$. Take*

$$P(x) = \left(1 + |e^{2\pi ik} - 1| \left(\frac{1}{2\pi(k - 2\theta x)} + \frac{k}{\pi} \log \left(\frac{k - 2\theta x}{k - \theta - \theta x} \right) \right) \right)^{-1}.$$

Then

$$I_k(x) \geqslant (1 + o(1)) 2 c_k \theta^{k^2} T (\log T)^{k^2} \left(\int_0^1 k^2 x^{k^2-1} P(x) dx \right).$$

As $k \to \infty$,

$$I_k \geqslant \left(2 + O\left(\frac{1}{k} \right) \right) c_k T (\log T)^{k^2}.$$

Theorem 1.9 may be proved using exactly the same techniques as the proof of Theorem 1.6. If $1 < k < 2$, Conrey and Ghosh, **[4]**, have explicit lower bounds for $I_k(T)$. If the $\theta = 1$ conjectures are not assumed then these bounds are never better than

$$I_k(T) \geqslant 1 \cdot 7 c_k T (\log T)^{k^2}.$$

It is evident that our Corollary 1.10 gives

$$I_k(T) \geqslant (2 - \varepsilon) c_k T (\log T)^{k^2}$$

provided k is sufficiently close to 2. In practice, this improvement is noticeable only when $k \geqslant 1 \cdot 95$. It is possible to refine the bounds of Theorems 1.6 and 1.9 by moving the line of integration to $\frac{1}{2} + c/\log T$ (instead of just $\mu > \frac{1}{2}$) with an appropriately chosen c. This would enable us to take longer Dirichlet polynomials A, when $k<2$ (in Theorem 1.9) or $r<2$ (in Theorem 1.6). However, the improvements arising from these refinements do not justify the extra efforts required.

We often write $A \leqslant B$ when we mean $A \leqslant (1+o(1))B$. This should not be cause for confusion.

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§2. *Some Lemmata.* Let

$$J_2 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \zeta^r(s) A_r(1-s, P) ds$$

and

$$J_3 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 |A_r(s, P)|^2 ds.$$

LEMMA 2.1. *With the above notation*

$$I_k \geqslant 4\Re J_2 - 2J_3 - 2\Re J_4.$$

Proof. Observe that

$$0 \leqslant \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 |\zeta^r(s) - A_r(s, P) - \chi^r(s) A_r(1-s, P)|^2 ds$$

$$= \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^{2k} ds - 4\Re \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \zeta^r(s) A_r(1-s, P) ds$$

$$+ \frac{2}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s) A_r(s, P)|^2 ds + 2\Re \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \chi^r(s) A_r(1-s, P)^2 ds.$$

Rearranging we obtain the lemma.

Let

$$K_2 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} \zeta^k(s) A_k(1-s, P) ds,$$

$$K_3 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |A_k(s, P)|^2 ds,$$

and

$$K_4 = \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} \frac{\zeta^k(s)}{\zeta^k(1-s)} A_k(1-s, P)^2 ds.$$

LEMMA 2.2. *With the above notation*

$$I^k(T) \geqslant 4\Re K_2 - 2K_3 - 2\Re K_4.$$

Proof. The proof is identical to the proof of Lemma 2.1.

LEMMA 2.3. *Let*

$$h(\alpha) = \int_0^1 (\beta - \alpha)^r P(\beta) d\beta$$

and let ${}_2F_1$ *denote the usual hypergeometric function. Then, if* $\theta < \frac{1}{2}$,

$$J_3 = (1+o(1))T(\log N)^{k^2} \frac{c_k \Gamma(k^2+1)}{\Gamma(r+1)^2 \Gamma(r^2)}$$
$$\times \int_0^1 \alpha^{r^2-1} \left(\frac{h'(\alpha)^2}{\theta} + 2rh(\alpha)h'(\alpha) \right) d\alpha$$

and

$$J_2 = (1+o(1))T(\log N)^{k^2} \frac{c_k \Gamma(k^2+1)\theta^{-1-r}}{\Gamma(r+2)\Gamma(r^2+r)}$$
$$\times \int_0^1 P(\alpha)\alpha^{k^2-r-2} {}_2F_1(-r, -r-1, k^2-r-1, -\alpha\theta) d\alpha.$$

Proof. See Theorems 1 and 2 of Conrey and Ghosh, [**4**].

LEMMA 2.4. *The following asymptotic formulae hold.*

$$K_3 = (1+o(1))T \sum_{n \leqslant T^\theta} \frac{d_k^2(n) P(\log n/(\theta \log T))^2}{n}$$

and

$$K_2 = (1+o(1))T \sum_{n \leqslant T^\theta} \frac{d_k^2(n) P(\log n/(\theta \log T))}{n}.$$

Proof. This is easily obtained by slightly modifying the results in Conrey and Ghosh, [**3**].

LEMMA 2.5. *Let x, $T>1$ and $\rho=\beta+i\gamma$ run over the complex zeros of $\zeta(s)$. Then*

$$\sum_{T\leqslant\gamma\leqslant 2T} x^{\rho}=-\frac{T}{2\pi}\Lambda(x)+O(x\log(2x)\log\log(3x)+x\log 2T)$$
$$+O\left(\log x\min\left(T,\frac{x}{\langle x\rangle}\right)+\min\left(\frac{\log T}{\log x},T\log T\right)\right)$$

where $\langle x\rangle$ is the distance from x to the nearest prime power other than x.

Proof. This is Theorem 1 of Gonek, [**5**].

LEMMA 2.6. *Let $r\geqslant 0$ and $h=\prod_p p^{h_p}$ be a positive integer. Define*

$$D_r(h,s)=\prod_{p\mid h}\left((1-p^{-s})^r\sum_{m=0}^{\infty}d_r(p^{m+h_p})p^{-ms}\right)$$

so that when $\Re s>1$,

$$D_r(h,s)\zeta^r(s)=\sum_{n=1}^{\infty}\frac{d_r(hn)}{n^s}.$$

Then

$$\sum_{n\leqslant x}\frac{d_r(hn)}{n}=\frac{D_r(h,1)(\log x)^r}{\Gamma(r+1)}+O(E(h,r)),$$

say, where

$$\sum_{h\leqslant H}|E(r,h)|=O(H(\log H)^{r-1}).$$

Also

$$\sum_{n\leqslant x}\frac{D_r(n,1)^2\varphi(n)}{n}=(1+o(1))\frac{c_{r+1}\Gamma(1+(r+1)^2)}{\Gamma(r^2)}x(\log x)^{r^2-1}.$$

Proof. These results are easily obtainable *via* the results of Selberg, [**12**]. They may also be found as Lemmata 4 and 5, and equations (36) and (37) of Conrey and Ghosh, [**4**].

§3. *Proof of Theorem* 1.1 *and its Corollaries.* We first treat the case $k\geqslant 4$ and when the Lindelöf Hypothesis is assumed. By Cauchy's theorem

$$J_4=\frac{1}{i}\left(\int_{\mu+iT}^{\mu+2iT}+\int_{1/2+iT}^{\mu+iT}-\int_{1/2+2iT}^{\mu+2iT}\right)\zeta^2(s)\chi(s)^{r-1}A_r(1-s,P)^2ds$$

where $\mu>\frac{1}{2}$ is some constant. Since $|A_r(1-s,P)|=O(T^{\theta(\sigma+\varepsilon)})$, $|\chi(\frac{1}{2}+it)|=1$ and $|\zeta(s)|=O(T^{\varepsilon})$ by the Lindelöf Hypothesis, the horizontal integrals are bounded by $O(T^{\theta+\varepsilon}) = O(T^{1-\varepsilon})$. Since $|\chi(s)| = O(T^{(1/2-\sigma)})$ it follows that the integral on the μ-line is bounded by

$$O\left(T^{(r-1)(1/2-\mu)+\varepsilon} \int_{\mu+iT}^{\mu+2iT} |A_r(1-s, P)|^2 ds\right).$$

By the mean-value theorem for Dirichlet polynomials, this is

$$O\left(T^{1+(r-1)(1/2-\mu)+\varepsilon} \sum_{n \leqslant N} \frac{d_r^2(n) P^2(\log n/\log N)}{n^{(2-2\mu)}}\right) = O(T^{1+(r-1-2\theta)(1/2-\mu)+\varepsilon})$$
$$= O(T^{1-\varepsilon}).$$

It remains now to treat the case $k=3$ and $\theta < \frac{1}{2}$. Again, by Cauchy's theorem,

$$J_4 = \frac{1}{i}\left(\int_{\mu+it_0}^{\mu+it_1} + \int_{1/2+it_0}^{\mu+it_0} - \int_{1/2+it_1}^{\mu+it_1}\right) \zeta^2(s)\chi^{r-1}(s) A_r(1-s, P)^2 ds$$
$$+ O\left(T^{\theta+\varepsilon}\left(\int_T^{t_0} + \int_{t_1}^{2T}\right)|\zeta(\tfrac{1}{2}+it)|^2 dt\right)$$

where $t_0 \in (T+T^{1/3+\varepsilon}, T+T^{1/2-\varepsilon})$ and $t_1 \in (2T-T^{1/2-\varepsilon}, 2T-T^{1/3+\varepsilon})$ will be fixed later. Since the mean-square of the ζ-function is known with an error $O(T^{1/3+\varepsilon})$ (see [13]) the error term in the above relation is

$$O(T^{\theta+\varepsilon}(t_0 - T + 2T - t_1)^{1+\varepsilon}) = O(T^{1-\varepsilon}).$$

Also the integral on the μ-line is easily seen, as before, to be small.

It suffices to estimate the horizontal integrals. We choose t_0 so that

$$\int_{1/2+it_0}^{\mu+it_0} \zeta^2(s)\chi^{r-1}(s) A_r(1-s, P)^2 ds$$
$$= O\left(\frac{T^{\varepsilon}}{T^{1/2}} \int_{T+T^{1/3+\varepsilon}}^{T+T^{1/2-\varepsilon}} \int_{1/2}^{\mu} |\zeta(\sigma+it)^2 \chi(\sigma+it)^{r-1} A_r(1-\sigma-it, P)^2| d\sigma dt\right)$$
$$= O\left(\frac{1}{T^{1/2-\varepsilon}} \int_{T+T^{1/3+\varepsilon}}^{T+T^{1/2-\varepsilon}} \int_{1/2}^{\mu} T^{2\theta\sigma+(r-1)(1/2-\sigma)} |\zeta(\sigma+it)|^2 d\sigma dt\right)$$
$$= O\left(T^{\varepsilon} \int_{1/2}^{\mu} T^{2\theta\sigma+(r-1)(1/2-\sigma)} d\sigma\right) = O(T^{1/2}).$$

A similar bound holds for the integral on it_1. This completes the proof.

To prove Corollaries 1.2 and 1.3, we use Theorem 1.1 with Lemma 2.1 to see that

$$I_k(T) \geqslant 4\Re \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \zeta^r(s) A_r(1-s, P)\,ds - \frac{2}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s) A_r(s, P)|^2\,ds.$$

The bounds of Conrey and Ghosh, **[4]**, were based on the inequality

$$I_k(T) \geqslant 2\Re \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s)|^2 \zeta^r(s) A_r(1-s, P)\,ds - \frac{1}{i} \int_{1/2+iT}^{1/2+2iT} |\zeta(s) A_r(s, P)|^2\,ds.$$

Thus making the same choice of P which leads to their Corollaries 1 and 2 we obtain our improved Corollaries 1.2 and 1.3.

§4. *Proof of Theorem* 1.6 *and its Corollaries.* Let $S(t) = 1/\pi \arg \zeta(\frac{1}{2}+it)$ and $N(t)$ denote the number of zeros of $\zeta(s)$ with ordinates between 0 and t. Then

$$\frac{\zeta^r(\frac{1}{2}+it)}{\zeta^r(\frac{1}{2}-it)} = e^{2\pi i r S(t)} = e^{2\pi i r N(t)} \chi^r(\tfrac{1}{2}+it).$$

It follows that

$$|J_4| \leqslant \left| \sum_{2T \geqslant \gamma_l \geqslant T} e^{2\pi i r l} \int_{\gamma_l}^{\gamma_{l+1}} |\zeta(\tfrac{1}{2}+it)|^2 \chi^r(\tfrac{1}{2}+it) A_r(\tfrac{1}{2}-it, P)^2\,dt \right|$$

where $0 < \gamma_1 \leqslant \gamma_2 \leqslant \ldots$ are the ordinates of the zeros of the ζ-function in the upper half-plane.

Let

$$J_4(\gamma_l, \gamma_{l+1}) = \int_{\gamma_l}^{\gamma_{l+1}} |\zeta(\tfrac{1}{2}+it)|^2 \chi^r(\tfrac{1}{2}+it) A_r(\tfrac{1}{2}-it, P)^2\,dt.$$

Let $\mu > \frac{1}{2}$ be some fixed constant and put

$$J_{41}(x, y) = \int_x^y \zeta(\mu+it)\zeta(1-\mu-it)\chi^r(\mu+it) A(1-\mu-it, P)^2\,dt$$

and

$$J_{42}(x) = \int_{1/2}^{\mu} \zeta(\sigma+ix)\zeta(1-\sigma-ix)\chi^r(\sigma+ix) A_r(1-\sigma-ix, P)^2\,d\sigma.$$

Using Cauchy's theorem it is easily seen that

$$iJ_4(\gamma_l, \gamma_{l+1}) = iJ_{41}(\gamma_l, \gamma_{l+1}) + J_{41}(\gamma_l) - J_{41}(\gamma_{l+1}).$$

Thus

$$\begin{aligned}|J_4| &\leqslant \left|\sum_{2T \geqslant \gamma_l \geqslant T} e^{2\pi i r l} J_4(\gamma_l, \gamma_{l+1})\right| \\ &\leqslant \left|\sum_{2T \geqslant \gamma_l \geqslant T} e^{2\pi i r l} J_{41}(\gamma_l, \gamma_{l+1})\right| + |e^{2\pi i r} - 1| \left|\sum_{2T \geqslant \gamma_l \geqslant T} e^{2\pi i r l} J_{42}(\gamma_l)\right| \\ &\leqslant \int_{\mu+iT}^{\mu+2iT} |\chi^r(s) A_r(1-s, P)^2 \zeta(s)\zeta(1-s) ds| + |e^{2\pi i r} - 1| \\ &\quad \times \int_{1/2}^{\mu} \sum_{2T \geqslant \gamma_l \geqslant T} |\zeta(\sigma + i\gamma_l)\zeta(1-\sigma-i\gamma_l)\chi^r(\sigma+i\gamma_l) A_r(1-\sigma-i\gamma_l, P)^2| d\sigma \\ &= L_1 + |e^{2\pi i r} - 1| L_2,\end{aligned}$$

say.

Clearly

$$\begin{aligned}L_1 &\ll T^{(1/2-\mu)(r-1)+\varepsilon} \int_T^{2T} |A_r(1-\mu-it, P)|^2 dt \\ &\ll T^{(1/2-\mu)(r-1)+\varepsilon} \times T \times T^{(2\mu-1)\theta} \ll T^{1-\varepsilon},\end{aligned}$$

if $\theta \leqslant \frac{1}{2}(r-1)$. Next,

$$\begin{aligned}L_2 &\leqslant \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} \left(\sum_{2T \geqslant \gamma_l \geqslant T} |A_r(1-\sigma-i\gamma_l, P)\zeta(\sigma+i\gamma_l)|^2\right) d\sigma \\ &= \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)} X(\sigma) d\sigma,\end{aligned}$$

say. By the approximate functional equation (see [**13**])

$$\zeta(\sigma+it) = \sum_{n \leqslant \sqrt{t/2\pi}} \frac{1}{n^{\sigma+it}} + \chi(\sigma+it) \sum_{n \leqslant \sqrt{t/2\pi}} \frac{1}{n^{1-\sigma-it}} + O(t^{-\sigma/2}).$$

Hence

$$\begin{aligned}X(\sigma) &= \sum_{\gamma_l} |A_r(1-\sigma-i\gamma_l, P)\zeta(\sigma+i\gamma_l)|^2 \\ &= \sum_{\gamma_l} \left| A_r(1-\sigma-i\gamma_l, P) \left(\sum_{n \leqslant \sqrt{T/2\pi}} \left(\frac{1}{n^{\sigma+i\gamma_l}} + \frac{\chi(\sigma+i\gamma_l)}{n^{1-\sigma-i\gamma_l}} \right) \right) \right|^2 \\ &\leqslant 2(X_1(\sigma) + X_2(\sigma)),\end{aligned}$$

where

$$X_1(\sigma)=\sum_{\gamma_l}\left|A_r(1-\sigma-i\gamma_l, P)\sum_{n\leqslant\sqrt{T/2\pi}}\frac{1}{n^{\sigma+i\gamma_l}}\right|^2$$

and

$$X_2(\sigma)=\sum_{\gamma_l}\left|A_r(1-\sigma-i\gamma_l, P)\chi(\sigma+i\gamma_l)\sum_{n\leqslant\sqrt{T/2\pi}}\frac{1}{n^{1-\sigma-i\gamma_l}}\right|^2.$$

For brevity, write

$$A_r(s, P)=\sum_{n\leqslant T^\theta} b_n n^{-s}$$

and

$$A_r(1-\sigma-it, P)\sum_{n\leqslant\sqrt{T/2\pi}} n^{-\sigma+it}=\sum_{n\leqslant T^{\theta+1/2}/\sqrt{2\pi}} c_n(\sigma)n^{it}$$

where

$$c_n(\sigma)=\sum_{\substack{uv=n\\ u\leqslant T^\theta,\, v\leqslant\sqrt{T/2\pi}}} b_u u^{\sigma-1}v^{-\sigma}=n^{-\sigma}\sum_{\substack{u|n\\ n\sqrt{2\pi/T}\leqslant u\leqslant T^\theta}} b_u u^{2\sigma-1}.$$

Thus,

$$\frac{2\pi X_1(\sigma)}{T\log T}=\frac{2\pi}{T\log T}\sum_{m,n\leqslant T^{\theta+1/2}/\sqrt{2\pi}} c_m(\sigma)c_n(\sigma)\sum_{T\leqslant\gamma\leqslant 2T}\left(\frac{m}{n}\right)^{i\gamma}$$

$$\leqslant(1+o(1))\sum_{n\leqslant T^{\theta+1/2}/\sqrt{2\pi}} c_n(\sigma)^2+2\left|\sum_{m>n} c_m(\sigma)c_n(\sigma)\sum_{T\leqslant\gamma\leqslant 2T}\left(\frac{m}{n}\right)^{i\gamma}\right|.$$

From Lemma 2.5 and the Riemann Hypothesis we see that

$$\sum_{T\leqslant\gamma\leqslant 2T}\left(\frac{m}{n}\right)^{i\gamma}=-\frac{T}{2\pi}\sqrt{\frac{n}{m}}\Lambda\left(\frac{m}{n}\right)$$

$$+O\left(\sqrt{\frac{m}{n}}\log^2 T+\frac{\sqrt{m}\log T}{\sqrt{n}\langle m/n\rangle}+\frac{\sqrt{n}\log T}{\sqrt{m}\log(m/n)}\right).$$

Observe that

$$\int_{1/2}^{\mu} c_m(\sigma)c_n(\sigma)T^{(r-1)(1/2-\sigma)}d\sigma$$

$$=\sum_{\substack{u|n\\ n\sqrt{2\pi/T}\leqslant u\leqslant T^\theta}}\ \sum_{\substack{v|m\\ m\sqrt{2\pi/T}\leqslant v\leqslant T^\theta}}\frac{b_u b_v}{\sqrt{nm}}\int_{1/2}^{\mu}\left(\frac{T^{(r-1)}nm}{(uv)^2}\right)^{1/2-\sigma}d\sigma$$

$$=\sum_u\sum_v\frac{(1+o(1))b_u b_v}{\sqrt{nm}((r-1)\log T+\log(nm)-2\log(uv))},$$

since $\theta < \frac{1}{2}(r-1)$. In particular

$$\int_{1/2}^{\mu} c_m(\sigma)c_n(\sigma)T^{(r-1)(1/2-\sigma)}d\sigma = O\left(\frac{d_k(n)d_k(m)}{\sqrt{nm}}\right).$$

From this estimate and our earlier estimate for $\sum_{T\leqslant\gamma\leqslant 2T}(m/n)^{i\gamma}$, it easily follows that

$$\int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)}X_1(\sigma)d\sigma$$

$$\leqslant \frac{T}{2\pi}\int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)}\Bigg(\log T \sum_{n\leqslant T^{\theta+1/2}/\sqrt{2\pi}} c_n(\sigma)^2$$

$$+2\sum_{T^{\theta+1/2}/\sqrt{2\pi}>m>n} c_m(\sigma)c_n(\sigma)\sqrt{\frac{n}{m}}\,\Lambda\left(\frac{m}{n}\right)\Bigg)d\sigma.$$

Now

$$\int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)}\sum_{n\leqslant T^{\theta+1/2}/\sqrt{2\pi}} c_n(\sigma)^2 d\sigma$$

$$\leqslant \sum_{\substack{u,v\leqslant T^{\theta} \\ n\leqslant\sqrt{Tuv/2\pi},[u,v]|n}} \frac{(1+o(1))b_u b_v}{(r-1)\log T+2\log n-2\log(uv)}$$

$$\leqslant \sum_{u,v\leqslant T^{\theta}} \frac{b_u b_v (u,v)}{2uv}\log\left(\frac{r\log T-\log(uv)}{(r-1)\log T-2\log(u,v)}\right).$$

Next,

$$\int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)}\sum_{n\leqslant m} c_n(\sigma)c_m(\sigma)\left(\frac{n}{m}\right)^{1/2}\Lambda\left(\frac{m}{n}\right)d\sigma$$

$$=\sum_{p}\frac{\log p}{p}\sum_{u,v\leqslant T^{\theta}} b_u b_v \sum_{\substack{u|n,\ v|np \\ n\leqslant\sqrt{T}\min(u,v/p)}} \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)}\frac{(uv)^{2\sigma-1}}{(n^2p)^{\sigma}}d\sigma$$

$$=\sum_{p}\frac{\log p}{p}\sum_{u,v\leqslant T^{\theta}} b_u b_v \sum_{\substack{u|n,\ v|np \\ n\leqslant\sqrt{T}\min(u,v/p)}} \int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)}\frac{(uv)^{2\sigma-1}}{(n^2p)^{\sigma}}d\sigma$$

$$=\sum_{p}\frac{\log p}{p}\sum_{u,v\leqslant T^{\theta}} b_u b_v \sum_{n}\frac{(1+o(1))}{n((r-1)\log T+\log(n^2p)-2\log(uv))}$$

$$=Y_1+Y_2$$

where Y_1 corresponds to those terms for which $p\nmid v$ and Y_2 to those for which $p|v$.

Putting $h=h(u, v, p)=\min(u, v/p)\leqslant\sqrt{uv/p}$, we see that,

$$Y_1\leqslant\sum_p\frac{\log p}{p}\sum_{u,v\leqslant T^\theta}b_ub_v\sum_{\substack{[u,v]|n\\ n\leqslant h\sqrt{T}}}\frac{1+o(1)}{n((r-1)\log T+\log(n^2p)-2\log(uv))}$$

$$\leqslant\sum_p\frac{\log p}{p}\sum_{u,v}\frac{b_ub_v}{[u, v]}\sum_{n\leqslant\sqrt{Th/[u,v]}}\frac{1+o(1)}{n((r-1)\log T-2\log(u, v)+\log(pn^2))}$$

$$\leqslant(1+o(1))\sum_{p\leqslant T^\theta}\frac{\log p}{p}\sum_{u,v}\frac{b_ub_v}{2[u, v]}\log\left(\frac{r\log T+\log(h^2p)-2\log(uv)}{(r-1)\log T-2\log(u, v)+\log p}\right)$$

$$\leqslant(\theta+o(1))\log T\sum_{u,v}\frac{b_ub_v}{2[u, v]}\log\left(\frac{r\log T-\log(uv)}{(r-1)\log T-2\log(u, v)}\right).$$

Further

$$Y_2\leqslant\sum_p\frac{\log p}{p}\sum_{u\leqslant T^\theta, v\leqslant T^\theta/p}\sum_{\substack{[u,v]|n\\ n\leqslant\min(u,v)\sqrt{T}}}\frac{(1+o(1))b_ub_{vp}}{n((r-1)\log T+\log(n^2p)-2\log(uvp))}$$

$$\leqslant\sum_p\frac{\log p}{p}\sum_{u,v}\frac{b_ub_{pv}}{[u, v]}\sum_{n\leqslant v\sqrt{T}/[u,v]}\frac{1+o(1)}{n((r-1)\log T+\log(n^2)-2\log(u, v)-\log p)}$$

$$\leqslant\sum_p\frac{\log p}{p}\sum_{u,v}\frac{b_ub_{pv}}{2[u, v]}\log\left(\frac{r\log T-2\log v-\log p}{(r-1)\log T-2\log(u, v)-\log p}\right)$$

$$\leqslant r\sum_{u,v}\frac{b_ub_v}{2[u, v]}(\theta\log T-\log v)\log\left(\frac{(r-\theta)\log T-\log v}{(r-1-\theta)\log T-\log(u, v)}\right).$$

Piecing these results together and using $\theta<\min(\frac{1}{2}, \frac{1}{2}(r-1))$ we see that

$$\int_{1/2}^{\mu}T^{(r-1)(1/2-\sigma)}X_1(\sigma)d\sigma\leqslant\frac{T\log T}{2\pi}\sum_{u,v}\frac{b_ub_v}{[u, v]}\left(\log\left(\frac{r\log T-\log(uv)}{(r-1)\log T-2\log(u, v)}\right)\right.$$

$$\left.+r\left(\theta-\frac{\log v}{\log T}\right)\log\left(\frac{(r-\theta)\log T-\log v}{(r-1-\theta)\log T-\log(u, v)}\right)\right).$$

Similarly, we see that,

$$\int_{1/2}^{\mu}T^{(r-1)(1/2-\sigma)}X_2(\sigma)d\sigma\leqslant\frac{T\log T}{2\pi}\sum_{u,v}\frac{b_ub_v}{[u, v]}\left(\log\left(\frac{(r+1)\log T-2\log[uv]}{r\log T-\log(u, v)}\right)\right.$$

$$\left.+r\left(\theta-\frac{\log v}{\log T}\right)\log\left(\frac{(r-\theta)\log T+2\log(u, v)-\log u}{(r-\theta)\log T-\log v}\right)\right).$$

Thus

$$\frac{\pi L_2}{T\log T} \leqslant \frac{2\pi}{T\log T}\int_{1/2}^{\mu} T^{(r-1)(1/2-\sigma)}(X_1(\sigma)+X_2(\sigma))\,d\sigma$$

$$\leqslant \sum_{u,v} \frac{b_u b_v}{[u,v]}\Bigg(\log\left(\frac{(r+1)\log T - 2\log[u,v]}{(r-1)\log T - 2\log(u,v)}\right)$$

$$+ r\left(\theta - \frac{\log v}{\log T}\right)\log\left(\frac{(r-\theta)\log T + 2\log(u,v) - \log u}{(r-1-\theta)\log T - \log(u,v)}\right)\Bigg)$$

$$\leqslant \sum_{u,v} \frac{b_u b_v}{[u,v]} H(u, v, (u,v)).$$

This proves Theorem 1.6. To obtain Corollary 1.7, note that if $u, v \leqslant T^{\theta}$, then

$$H(u, v, (u,v)) \leqslant \log\left(\frac{r+1-2\theta}{r-1-2\theta}\right) + r\left(\theta - \frac{\log v}{\log T}\right)\log\left(\frac{r}{r-1-2\theta}\right).$$

Hence

$$\frac{\pi L_2}{T\log T} \leqslant \sum_{u,v\leqslant T^{\theta}} \frac{b_u b_v}{[u,v]}\left(\log\left(\frac{r+1-2\theta}{r-1-2\theta}\right) + r\left(\theta - \frac{\log v}{\log T}\right)\log\left(\frac{r}{r-1-2\theta}\right)\right)$$

$$= \sum_{w\leqslant T^{\theta}} \sum_{\substack{u,v\leqslant T^{\theta}/w \\ (u,v)=1}} \frac{b_{uw} b_{vw}}{uvw}$$

$$\times\left(\log\left(\frac{r+1-2\theta}{r-1-2\theta}\right) + r\left(\theta - \frac{\log wv}{\log T}\right)\log\left(\frac{r}{r-1-2\theta}\right)\right)$$

$$= \sum_{w\leqslant T^{\theta}} \sum_{u,v\leqslant T^{\theta}/w} \left(\sum_{d|u, d|v} \mu(d)\right)\frac{b_{uw} b_{vw}}{uvw}$$

$$\times\left(\log\left(\frac{r+1-2\theta}{r-1-2\theta}\right) + r\left(\theta - \frac{\log wv}{\log T}\right)\log\left(\frac{r}{r-1-2\theta}\right)\right)$$

$$= \sum_{w\leqslant T^{\theta}} \sum_{d\leqslant T^{\theta}/w} \frac{\mu(d)}{d^2 w} \sum_{u,v\leqslant T^{\theta}/wd} \frac{b_{udw} b_{vdw}}{uv}$$

$$\times\left(\log\left(\frac{r+1-2\theta}{r-1-2\theta}\right) + r\left(\theta - \frac{\log wdv}{\log T}\right)\log\left(\frac{r}{r-1-2\theta}\right)\right).$$

Using partial summation in conjunction with Lemma 2.6, we see

$$\sum_{u,v\leqslant T^\theta/wd}\frac{b_{udw}b_{vdw}}{uv}\left(\log\left(\frac{r+1-2\theta}{r-1-2\theta}\right)+r\left(\theta-\frac{\log wdv}{\log T}\right)\log\left(\frac{r}{r-1-2\theta}\right)\right)$$

$$\leqslant\frac{D_r(wd,1)^2}{\Gamma^2(r+1)}\left(\int_1^{T^\theta/wd}\frac{r(\log y)^{r-1}}{y}P\left(\frac{\log(ydw)}{\theta\log T}\right)dy\right)$$

$$\times\left(\int_1^{T^\theta/wd}\frac{r(\log y)^{r-1}}{y}Q\left(\frac{\log(ydw)}{\theta\log T}\right)dy\right).$$

By an obvious change of variables this is

$$=\frac{D_r(wd,1)^2}{\Gamma^2(r+1)}(\theta\log T)^{2r}\left(\int_{\log(dw)/(\theta\log T)}^1 r\left(z-\frac{\log(dw)}{\theta\log T}\right)^{r-1}P(z)dz\right)$$

$$\times\left(\int_{\log(dw)/(\theta\log T)}^1 r\left(z-\frac{\log(dw)}{\theta\log T}\right)^{r-1}Q(z)dz\right)$$

$$=G\left(\frac{\log(wd)}{\theta\log T}\right)\frac{D_r(wd,1)^2}{\Gamma^2(r+1)}(\theta\log T)^{2r}.$$

Thus

$$\frac{\pi L_2}{T\log T}\leqslant\frac{(\theta\log T)^{2r}}{\Gamma^2(r+1)}\sum_{w\leqslant T^\theta}\sum_{d\leqslant T^\theta/w}\frac{\mu(d)D_r(wd,1)^2}{d^2w}G\left(\frac{\log(dw)}{\theta\log T}\right)$$

$$=\frac{(\theta\log T)^{2r}}{\Gamma^2(r+1)}\sum_{n\leqslant T^\theta}\frac{D_r(n,1)^2\varphi(n)}{n^2}G\left(\frac{\log n}{\theta\log T}\right).$$

Corollary 1.7 follows immediately by partial summation and Lemma 2.6. Corollary 1.8 is a simple consequence of Corollary 1.7 and Corollary 2 of Conrey and Ghosh, [4].

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