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Hyperbolic metric and membership of conformal maps in the Bergman space

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Abstract. We prove that for $0 < p < +\infty$ and $-1 < \alpha < +\infty$, a conformal map defined on the unit disk belongs to the weighted Bergman space A^p_α if and only if a certain integral involving the hyperbolic distance converges.

1 Introduction

For $0 < p < +\infty$ and $-1 < \alpha < +\infty$, the weighted Bergman space A^p_α is the set of all holomorphic functions f in the unit disk $\mathbb{D}$ such that

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p \left(1-|z|^2\right)^\alpha dA(z) < +\infty,$$

where dA denotes the Lebesgue area measure on $\mathbb{D}$. Closely related to Bergman spaces are the classical Hardy spaces H^p. For $p > 0$, H^p consists of all holomorphic functions in the unit disk such that

$$\|f\|^p_{H^p} := \sup_{0<r<1} \int_0^{2\pi} \left|f(re^{it})\right|^p dt < +\infty.$$

It is well known that $H^p \subset A^p_\alpha$, for all $a \in (-1, +\infty)$, and moreover,

$$\lim_{\alpha \to -1^+} \|f\|_{A^p_\alpha} = \|f\|_{H^p}$$

(see [16]). For the theory of Bergman spaces, see [4, 7]. The problem of characterizing conformal maps which are contained in H^p has been extensively studied in the past with the work, among others, of Prawitz [13], Hardy and Littlewood [6], Pommerenke [12], and Poggi-Corradini [11]. The following characterization of conformal maps in Hardy spaces is due to Prawitz [13], Hardy and Littlewood [6], and Pommerenke [12].

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Theorem A *Let $0 < p < +\infty$ and suppose f is a conformal map on $\mathbb{D}$. Then $f \in H^p$ if and only if*

$$\int_0^1 M(r, f)^p \, dr < +\infty,$$

where $M(r, f) = \max_{|z|=r} |f(z)|$, $0 \le r < 1$, is the maximum modulus of f on the circle of radius r centered at 0.

For a conformal map f on $\mathbb{D}$ and $r > 0$, set $F_r = \{z \in \mathbb{D} : |f(z)| = r\}$. Let $d_{\mathbb{D}}(0, F_r)$ denote the hyperbolic distance in $\mathbb{D}$ between 0 and the set F_r, *i.e.*, $d_{\mathbb{D}}(0, F_r) = \inf_{z\in F_r} d_{\mathbb{D}}(0, z)$, where $d_{\mathbb{D}}(0, z)$ is the hyperbolic distance between $0, z$ in $\mathbb{D}$. Poggi-Corradini in [11] posed the question of whether a conformal map f belongs to H^p if and only if

$$\int_0^{+\infty} r^{p-1} e^{-d_{\mathbb{D}}(0,F_r)} \, dr < +\infty.$$

This question was settled by Karafyllia in [8] providing another characterization for conformal maps in H^p.

Theorem B *Let $0 < p < +\infty$ and suppose f is a conformal map on $\mathbb{D}$. For $r > 0$, let $F_r = \{z \in \mathbb{D} : |f(z)| = r\}$. Then $f \in H^p$ if and only if*

$$\int_0^{+\infty} r^{p-1} e^{-d_{\mathbb{D}}(0,F_r)} \, dr < +\infty.$$

Conformal maps in Bergman spaces have been characterized by Baernstein, Girela, and Peláez [2] and also by Pérez-González and Rättyä [10].

Theorem C *Let $0 < p < +\infty$ and $-1 < \alpha < +\infty$ and suppose f is a conformal map in $\mathbb{D}$. Then $f \in A^p_\alpha$ if and only if*

$$\int_0^1 (1 - r^2)^{\alpha+1} M(r, f)^p \, dr < +\infty.$$

Note that Theorem C is the analogue of Theorem A. It is therefore natural to ask what the counterpart of Theorem B is for Bergman spaces. In this direction we prove the following theorem.

Theorem 1.1 *Let $0 < p < +\infty$ and $-1 < \alpha < +\infty$. Suppose f is a conformal map on $\mathbb{D}$ and for $r > 0$ let $F_r = \{z \in \mathbb{D} : |f(z)| = r\}$. Then $f \in A^p_\alpha$ if and only if*

$$\int_0^{+\infty} r^{p-1} e^{-(\alpha+2) d_{\mathbb{D}}(0,F_r)} \, dr < +\infty.$$

For a bounded function f, the integral that appears in Theorem 1.1 converges trivially. Therefore, we will assume for the rest of this paper that f is an unbounded conformal map on the unit disk.

2 Preliminaries

The hyperbolic distance between $z, w \in \mathbb{D}$ is defined by

$$d_{\mathbb{D}}(z,w) = \log \frac{1 + \frac{|z-w|}{|1-z\overline{w}|}}{1 - \frac{|z-w|}{|1-z\overline{w}|}},$$

and the Green function for the unit disk is

$$g_{\mathbb{D}}(z,w) = \log \frac{|1 - z\overline{w}|}{|z-w|} = \log \frac{e^{d_{\mathbb{D}}(z,w)} + 1}{e^{d_{\mathbb{D}}(z,w)} - 1}.$$

Let D be a simply connected domain in $\mathbb{C}$ and let $f : \mathbb{D} \to D$ be a conformal map from the unit disk onto D. The hyperbolic distance in D between $z, w \in D$ is defined by

$$d_D(z,w) = d_{\mathbb{D}}\big(f^{-1}(z),\ f^{-1}(w)\big).$$

Since $d_{\mathbb{D}}$ is invariant under the group of conformal self maps of the unit disk, it follows that d_D is well defined. Moreover, the function

$$g_D(z,w) = \log \frac{e^{d_D(z,w)} + 1}{e^{d_D(z,w)} - 1} \tag{2.1}$$

is the Green function for the domain D and is also invariant under conformal maps. See, for example, [1, 3, 5, 9].

We will need a few facts about the function $M(r, f)$ that appears in Theorems A and C. Recall that for a (non constant) holomorphic function f on $\mathbb{D}$ and $0 \le r < 1$, the maximum modulus function of f is defined as

$$M(r, f) = \max_{|z|=r} |f(z)|.$$

It is well known that M is a continuous, strictly increasing function of r, and thus its derivative M' exists everywhere in $(0,1)$ except for at most countably many points. Let $0 < a < b < 1$ and let $z_a,\ z_b \in \mathbb{D}$ be points such that $M(a, f) = |f(z_a)|$, $M(b, f) = |f(z_b)|$, $|z_a| = a$, $|z_b| = b$. Also, let z'_a be the point where the segment $[0, z_b]$ meets the circle of radius a. Observe that

$$\begin{aligned} 0 < M(b,f) - M(a,f) = |f(z_b)| - |f(z_a)| &\le |f(z_b)| - |f(z'_a)| \\ \le |f(z_b) - f(z'_a)| &= \Big| \int_{[z'_a, z_b]} f'(w)dw \Big|. \end{aligned}$$

By the triangle inequality,

$$0 < M(b,\ f) - M(a,\ f) \le \sup_{|z|\le b} |f'(z)||b - a|.$$

This shows that M is locally Lipschitz in $[0,1)$ and thus absolutely continuous in $[0, 1 - \varepsilon]$ for any $\varepsilon > 0$ sufficiently small. We can now proceed with a lemma that will be useful in the proof of Theorem 1.1.

Lemma 2.1 *Suppose f is an unbounded conformal map defined on the unit disk such that $f(0) = 0$. Then the maximum modulus function M, defined above, satisfies the* *following change of variable formula:*

$$\int_0^{+\infty} r^{p-1} e^{-(\alpha+2)d_{\mathbb{D}}(0,F_r)}\,dr = \int_0^1 M(s,\ f)^{p-1}\Big(\frac{1-s}{1+s}\Big)^{\alpha+2} M'(s,\ f)\,ds.$$

Proof For $0 < r < +\infty$, recall that $F_r = \{z \in \mathbb{D} :\ |f(z)| = r\}$. Let z_r be a point on F_r such that $d_{\mathbb{D}}(0, F_r) = d_{\mathbb{D}}(0, z_r) = \log\frac{1+\rho}{1-\rho}$, where $\rho = \rho(r) = |z_r|$. The point z_r may not be unique, but it is one of the points of F_r that is closest to the origin. This is true, because of the definition of $d_{\mathbb{D}}$ and the fact that the function $\log\frac{1+x}{1-x}$ is strictly increasing in $[0,1)$. The first integral in the statement of the lemma can therefore be written as

$$\begin{aligned}\int_0^{+\infty} r^{p-1} e^{-(\alpha+2)d_{\mathbb{D}}(0,F_r)}\,dr &= \int_0^{+\infty} r^{p-1} e^{-(\alpha+2)\log\frac{1+\rho(r)}{1-\rho(r)}}\,dr\\ &= \int_0^{+\infty} r^{p-1}\Big(\frac{1-\rho(r)}{1+\rho(r)}\Big)^{\alpha+2} dr.\end{aligned}$$

Let $0 < \varepsilon < 1$. By a standard result in real analysis (see, for example, [15, p. 326]) and the facts about M stated before the lemma, we have that

$$\int_{M(0,\ f)}^{M(1-\varepsilon,\ f)} r^{p-1}\Big(\frac{1-\rho(r)}{1+\rho(r)}\Big)^{\alpha+2} dr = \int_0^{1-\varepsilon} M(s,\ f)^{p-1}\Big(\frac{1-\rho(M(s,\ f))}{1+\rho(M(s,\ f))}\Big)^{\alpha+2} M'(s,\ f)\,ds.$$

By the definition of $\rho(\cdot)$, it is not hard to see that for any $s > 0$, $\rho(M(s,\ f)) = |z_{M(s,\ f)}| = s$. The last equality, therefore, becomes

$$\int_{M(0,\ f)}^{M(1-\varepsilon,\ f)} r^{p-1}\Big(\frac{1-\rho(r)}{1+\rho(r)}\Big)^{\alpha+2} dr = \int_0^{1-\varepsilon} M(s,\ f)^{p-1}\Big(\frac{1-s}{1+s}\Big)^{\alpha+2} M'(s,\ f)\,ds.$$

Since we are assuming that f is unbounded and $f(0) = 0$, letting $\varepsilon \to 0$ and using the Monotone Convergence Theorem yields the required formula. ■

Next, we need to establish a differential inequality for the function M that will also be used in the proof of Theorem 1.1

Lemma 2.2 *Let f be a conformal map on the unit disk satisfying $f(0) = 0$ and $f'(0) = 1$. Then*

$$M'(r,\ f) \le M(r,\ f)\frac{1+r}{r(1-r)},$$

for any $r \in (0,1)$ such that $M'(r,\ f)$ exists. Moreover, equality occurs for some r if and only if f is a Koebe function, i.e., $f(z) = \frac{z}{(1-\lambda z)^2}$, $|\lambda| = 1$.

Proof Let $r \in (0,1)$ be a point such that $M'(r,\ f)$ exists and let h be a small positive number. Write $M(r,\ f) = |f(z_r)|$ and $M(r-h,\ f) = |f(z_{r-h})|$, where $|z_r| =$

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r, $|z_{r-h}| = r - h$. Let z'_{r-h} be the point where the segment $[0, z_r]$ meets the circle of radius $r - h$. Then we have

$$\frac{M(r, f) - M(r-h, f)}{h} = \frac{|f(z_r)| - |f(z_{r-h})|}{h} \le \frac{|f(z_r)| - |f(z'_{r-h})|}{h}.$$

Observe that, as $h \to 0$, the limit of the right-hand side in the last inequality is $\frac{\partial}{\partial r}|f|(z_r)$. Thus letting $h \to 0$ gives

$$M'(r, f) \le \frac{\partial}{\partial r}|f|(z_r).$$

By the Cauchy–Schwarz inequality, $\frac{\partial}{\partial r}|f|(z_r) \le |\nabla|f|(z_r)|$. A quick calculation shows that

$$|\nabla|f|(z_r)| = |f'(z_r)|.$$

It follows that

$$M'(r, f) \le |f'(z_r)|.$$

Since we are assuming that f is a normalized univalent function, *i.e.*, $f \in S$, by a standard estimate for maps in S (see [5, p. 22]), we conclude that

$$|f'(z_r)| \le |f(z_r)| \frac{1 + |z_r|}{|z_r|(1 - |z_r|)} = M(r, f)\frac{1+r}{r(1-r)},$$

and the lemma is proved. Finally, we treat the equality case. If f is a Koebe function, then $M(r, f) = \frac{r}{(1-r)^2}$, and therefore we have equality for all $r \in (0,1)$. Conversely, if equality holds for some r, then we must have equality in the estimate for maps in S that we used above and therefore f is a Koebe function. ∎

Finally, we will make use of the following fact about the norm $\|f\|^p_{A^p_\alpha}$. Let $0 < p < +\infty$ and $-1 < \alpha < +\infty$. The quantity

$$\|f\|^p_{A^p_\alpha} := \int_{\mathbb{D}} |f(z)|^p \left(1 - |z|^2\right)^\alpha dA(z)$$

is comparable (see [14]) to

$$\int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left(\log \frac{1}{|z|}\right)^{\alpha+2} dA(z) + |f(0)|^p.$$

It follows that

$$f \in A^p_\alpha \Longleftrightarrow \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \left(\log \frac{1}{|z|}\right)^{\alpha+2} dA(z) < +\infty. \tag{2.2}$$

3 Proof of Theorem 1.1

The first part of the proof is similar to the proof of [8, Theorem 1.1]. Suppose that for some $0 < p < +\infty$ and some $-1 < \alpha < +\infty$,

$$\int_0^{+\infty} r^{p-1} e^{-(\alpha+2)d_{\mathbb{D}}(0,F_r)} dr < +\infty. \tag{3.1}$$

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Let dA denote the Lebesgue area measure, and let $D = f(\mathbb{D})$. We denote by $g_D(f(0), z)$, $z \in D$ the Green function for D and we set $g_D(f(0), z) = 0$ for $z \notin D$. By a change of variable and the conformal invariance of the Green function, we deduce that

$$
\begin{aligned}
\text{(3.2)} \quad & \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \Big(\log \frac{1}{|z|}\Big)^{\alpha+2} dA(z) \\
&= \int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 g_{\mathbb{D}}(0, z)^{\alpha+2} \, dA(z) \\
&= \int_D |w|^{p-2} g_D(f(0), w)^{\alpha+2} \, dA(w) \\
&= \int_0^{+\infty} r^{p-1} \Big(\int_0^{2\pi} g_D\big(f(0), re^{i\theta}\big)^{\alpha+2} d\theta \Big) dr.
\end{aligned}
$$

By elementary calculus,

$$
\log \frac{e^x + 1}{e^x - 1} \le 3e^{-x}, \tag{3.3}
$$

for all x sufficiently large. Note that for D unbounded and simply connected, $d_D(f(0), f(F_r)) \to +\infty$ as $r \to +\infty$ which also follows from the hypothesis (3.1). Therefore, by (3.3) and (2.1), we deduce that there exists an $r_0 > 0$ and a positive constant C such that for every $r \ge r_0$,

$$
\begin{aligned}
g_D\big(f(0), re^{i\theta}\big)^{\alpha+2} &\le Ce^{-(\alpha+2) d_D\left(f(0), re^{i\theta}\right)} \\
&\le Ce^{-(\alpha+2) d_D\left(f(0), \{w \in D : |w| = r\}\right)} \\
&= Ce^{-(\alpha+2) d_{\mathbb{D}}(0, F_r)}.
\end{aligned}
$$

Integrating with respect to θ, we get

$$
\begin{aligned}
\int_0^{2\pi} g_D\big(f(0), re^{i\theta}\big)^{\alpha+2} d\theta &\le C \int_0^{2\pi} e^{-(\alpha+2) d_{\mathbb{D}}(0, F_r)} d\theta \\
&= 2\pi C e^{-(\alpha+2) d_{\mathbb{D}}(0, F_r)},
\end{aligned} \tag{3.4}
$$

for every $r \ge r_0$. So, by (3.2) and (3.4), we infer that there exist positive constants C_1, C_2 such that

$$
\int_{\mathbb{D}} |f(z)|^{p-2} |f'(z)|^2 \Big(\log \frac{1}{|z|}\Big)^{\alpha+2} dA(z) \le C_1 \int_{r_0}^{+\infty} r^{p-1} e^{-(\alpha+2) d_{\mathbb{D}}(0, F_r)} dr + C_2 < +\infty.
$$

Thus by (2.2), we conclude that $f \in A_\alpha^p$.

For the converse, suppose that $f \in A_\alpha^p$ is conformal. In addition, assume temporarily that $f(0) = 0$ and $f'(0) = 1$. By Lemma 2.1, it suffices to show that

$$
\int_0^1 M(s, f)^{p-1} (1-s)^{\alpha+2} M'(s, f) ds < +\infty.
$$

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Note that the singularity of this integral occurs at 1, and thus it is enough to prove that

$$\int_{\delta}^{1} M(s,\ f)^{p-1}(1-s)^{\alpha+2}M'(s,\ f)ds < +\infty,$$

for some number $\delta \in (0,1)$. By Lemma 2.2,

$$\int_{\delta}^{1} M(s,\ f)^{p-1}(1-s)^{\alpha+2}M'(s,\ f)ds \le 2\int_{\delta}^{1} M(s,\ f)^{p}\frac{(1-s)^{\alpha+1}}{s}ds.$$

Observe that

$$\begin{aligned}2\int_{\delta}^{1} M(s,\ f)^{p}\frac{(1-s)^{\alpha+1}}{s}ds &\le \frac{2}{\delta}\int_{\delta}^{1} M(s,\ f)^{p}(1-s^2)^{\alpha+1}ds\\ &\le \frac{2}{\delta}\int_{0}^{1} M(s,\ f)^{p}(1-s^2)^{\alpha+1}ds.\end{aligned}$$

Since $f \in A^p_\alpha$, the last integral converges by Theorem C. Hence,

$$\int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\mathbb{D}}(0,F_r)}dr < +\infty.$$

We will now remove the extra assumptions $f(0)=0$ and $f'(0)=1$. For $f \in A^p_\alpha$, let $h(z)=\frac{f(z)}{f'(0)}$ and $g(z)=h(z)-h(0)$. For $r>0$, let $H_r=\{z\in\mathbb{D}: |h(z)|=r\}$ and $G_r=\{z\in\mathbb{D}: |g(z)|=r\}$. Set $\Omega=h(\mathbb{D})$. Note that $g\in A^p_\alpha$. Since $g(0)=0$ and $g'(0)=1$, it follows from what we have proved that

$$\int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\mathbb{D}}(0,G_r)}dr < +\infty.$$

By the conformal invariance of the hyperbolic distance,

$$\begin{aligned}\int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\mathbb{D}}(0,G_r)}dr &= \int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\Omega-h(0)}(0,r\partial\mathbb{D}\cap(\Omega-h(0)))}dr\\ &= \int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\Omega}(h(0),(h(0)+r\partial\mathbb{D})\cap\Omega)}dr\\ &= \int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\Omega}(h(0),\tilde{w}_r)}dr,\end{aligned}$$

where $\tilde{w}_r\in\{w:|w-h(0)|=r\}\cap\Omega$. Let Γ be the hyperbolic geodesic in Ω joining $h(0)$ to w_{2r}, where $|w_{2r}|=r$ and $d_\Omega(h(0),2r\partial\mathbb{D}\cap\Omega)=d_\Omega(h(0),w_{2r})$. If r is sufficiently large, then $\{w:|w-h(0)|\le r\}\subset 2r\mathbb{D}$, and thus we can find a point w_r on $\Gamma\cap\{w:|w-h(0)|=r\}$. Then

$$d_\Omega(h(0),\tilde{w}_r)\le d_\Omega(h(0),w_r)<d_\Omega(h(0),w_{2r}).$$

It follows that

$$\begin{aligned}+\infty > \int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\mathbb{D}}(0,G_r)}dr &\ge \int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\Omega}(h(0),w_{2r})}dr\\ &= \int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\Omega}(h(0),2r\partial\mathbb{D}\cap\Omega)}dr\\ &= \int_{0}^{+\infty} r^{p-1}e^{-(\alpha+2)d_{\mathbb{D}}(0,H_{2r})}dr.\end{aligned}$$

By a change of variable, we conclude that

$$\int_0^{+\infty} r^{p-1} e^{-(\alpha+2)d_{\mathbb{D}}(0,H_r)}\,dr < +\infty.$$

Finally, observing that $H_r = F_{r|f'(0)|}$ and using another change of variable gives

$$\int_0^{+\infty} r^{p-1} e^{-(\alpha+2)d_{\mathbb{D}}(0,F_r)}\,dr < +\infty,$$

and the proof is complete. ∎

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