The Journal of Symbolic Logic
Volume 84, Number 1, March 2019

# BASIS THEOREMS FOR $\Sigma^1_2$-SETS

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**Abstract.** We prove the following two basis theorems for $\Sigma^1_2$-sets of reals:

(1) Every nonthin $\Sigma^1_2$-set has a perfect $\Delta^1_2$-subset if and only if it has a nonthin $\Delta^1_2$-subset, and this is equivalent to the statement that there is a nonconstructible real.

(2) Every uncountable $\Sigma^1_2$-set has an uncountable $\Delta^1_2$-subset if and only if either every real is constructible or $\omega_1^L$ is countable.

We also apply the method that proves (2) to show that if there is a nonconstructible real, then there is a perfect $\Pi^1_2$-set with no nonempty $\Pi^1_2$-thin subset, strengthening a result of Harrington [4].

## §1. Introduction and preliminaries.

**1.1. Introduction.** The main theme of this paper concerns basis theorems for $\Sigma^1_2$-sets of reals. Let $\leq_r$ be a reducibility relation. We say that a perfect set $P$ is *r-pointed* if there is a perfect tree $T \subseteq 2^{<\omega}$ such that $[T] = P$ (where $[T]$ denotes the set of infinite paths in $T$) and for all $x \in [T]$, $T \leq_r x$. Martin [8] proved that under the axiom of determinacy (AD), every $A \subseteq 2^\omega$ whose corresponding Turing degrees are cofinal contains a $\Delta^0_1$-pointed perfect subset. Woodin (unpublished) has shown that Turing determinacy implies AD in $L(\mathbb{R})$. It follows that the assumption of AD is necessary for Martin's theorem to hold. If Turing reducibility is replaced with a coarser reducibility notion, then the set-theoretic assumption may be considerably weakened. For example, Martin [9] showed that it is a theorem of ZFC that every uncountable $\Delta^1_1$-set contains a $\Delta^1_1$-pointed perfect subset (this is false in general for uncountable $\Pi^1_1$-sets by a theorem of Mansfield [7] and Solovay [12], see Sacks [10] for a proof). The natural question is how far can Martin's theorem be generalized within ZFC, in particular whether it holds with regard to $\Delta^1_2$-reducibility for $\Sigma^1_2$-sets.

A set is *thin* if it contains no perfect subset. It is known (applying a Cantor-Bendixson–type construction of Mansfield [7] and Solovay [12]) that if $A$ is $\Sigma^1_2$ and contains a nonconstructible real, then it is not thin and in fact contains a perfect subset whose elements range over an upper cone of $L$-degrees. It was asked at the Dagstuhl workshop in February 2017 whether the perfect set could range over an upper cone of $\Delta^1_2$-degrees. The point here is that a Cantor-Bendixson–style construction over a Suslin representation of a nonthin $\Sigma^1_2$-set may involve stages beyond $\omega_1^L$ which is greater than $\delta^1_2$, the least ordinal which is not the order type of

Received March 4, 2018.

2010 *Mathematics Subject Classification.* 03D30, 03D60, 03D65, 03E15, 03E35.

*Key words and phrases*. basis theorem, Shoenfield absoluteness, constructibility, pointed trees, perfect set.


0022-4812/19/8401-0017
DOI:10.1017/jsl.2018.81

a $\Delta^1_2$-well ordering of $\omega$. Nevertheless, the answer to the question is affirmative by a different argument (see Corollary 1.4). We show in Section 2 that every nonthin $\Sigma^1_2$-set has a perfect $\Delta^1_2$-pointed subset if and only if there is a nonconstructible real. Furthermore, the $\Delta^1_2$-pointed perfect subset, if it exists, can be chosen to be the set of paths of a perfect $\Delta^1_2$-tree.

A more general and perhaps basic question regarding $\Sigma^1_2$-sets is whether every uncountable $\Sigma^1_2$-set has an uncountable $\Delta^1_2$-subset. Since $\Sigma^1_2$-subsets of $L$ are $\Sigma_1(L)$-definable, the situation is reminiscent of that in $\omega$ where every infinite recursively enumerable set has an infinite recursive subset. Generalizations of this fundamental theorem to ordinals have been studied in higher recursion theory (see Sacks [10]), most of which stayed within the realm of the constructible universe. Our interest here is to investigate this problem beyond $L$. A characterization of conditions guaranteeing every uncountable $\Sigma^1_2$-set to have an uncountable $\Delta^1_2$-subset is given in Section 3, i.e., the statement is true if and only if $2^\omega \subset L$ or $\omega_1^L < \omega_1$.

Harrington [4] showed that if there is a nonconstructible real, then there is a perfect $\Pi^1_2$-set with no $\Pi^1_2$-singleton. In the concluding section of the paper, we apply the method in Section 3 to strengthen this result by proving that under the same hypothesis, there is a $\Pi^1_2$-set with no thin $\Pi^1_2$-subset.

**1.2. Preliminaries.** We follow the standard terminologies and notations (cf. [2], [5], and [10]). Given a tree $T$, let $[T]$ denote the set of infinite paths in $T$. If $\sigma \in 2^{<\omega}$, then $[\sigma]$ denotes the collection of binary strings which extend $\sigma$. If $\alpha$ is a limit ordinal and $\{x_\beta\}_{\beta<\alpha}$ is a sequence of length $\alpha$, let $x = \lim_{\beta\to\alpha} x_\beta$ denote $\exists\gamma < \alpha\forall\beta \geq \gamma(x_\beta = x)$. If $x$ is a real, then $\omega_1^{L[x]}$ denotes the first uncountable ordinal in the structure $(L[x], \in)$.

The following Spector-Gandy–type characterization of $\Sigma^1_2$-sets (see [2]) will be used throughout the paper.

THEOREM 1.1 (Shoenfield). *Given a set $A \subseteq 2^\omega$, the following are equivalent*:

- *$A$ is $\Sigma^1_2$.*
- *There is a $\Sigma_1$-formula $\varphi$ such that for all reals $x$,*

$$x \in A \Leftrightarrow L_{\omega_1^{L[x]}}[x] \models \varphi(x).$$

- *There is a $\Sigma_1$-formula $\varphi$ such that for all reals $x$,*

$$x \in A \Leftrightarrow L_{\delta^1_2(x)}[x] \models \varphi(x),$$

*where $\delta^1_2(x)$ is the least ordinal not the order type of a $\Delta^1_2(x)$-well ordering of $\omega$.*

Theorem 1.1 enables one to use recursion-theoretic arguments to study $\Sigma^1_2$-sets. It follows that there is a version of "Church-Turing Thesis" for $\Sigma^1_2$-sets that we can appeal to in the construction of such sets.

The following theorem, which shows the pervasive presence of nonconstructible reals, will be used as a basic result in this paper.

THEOREM 1.2 (Groszek and Slaman [3]). *Suppose that there is a nonconstructible real. Then every perfect set contains a nonconstructible real.*

PROPOSITION 1.3. *Suppose that there is a nonconstructible real. If $A$ is $\Sigma^1_2$ and not thin, then $A$ contains a $\Delta^1_2$-perfect subset.*

Proof. Since $A$ is $\Sigma^1_2$, there is a $\Pi^1_1$ set $B \subseteq \omega^\omega \times \omega^\omega$ such that

$$\forall x(x \in A \leftrightarrow \exists y((x, y) \in B)).$$

By $\Pi^1_1$-uniformization, we may assume that $\forall x(\exists y(x, y) \in B \to \exists! y(x, y) \in B)$. By Theorem 1.2, $A$ contains a nonconstructible real and so does $B$. It follows from the Mansfield-Solovay theorem that $B$ is not thin either. Let

$$C = \{T \subseteq \omega^{<\omega} \times \omega^{<\omega} \mid [T] \subseteq B \wedge \forall(\sigma, \tau) \in T \exists(\sigma_0, \tau_0) \in T \exists(\sigma_1, \tau_1) \in T \\ (\sigma_0 \succ \sigma \wedge \sigma_1 \succ \sigma \wedge \sigma_0 | \sigma_1)\}.$$

Since $B$ was replaced by its uniformization, it is not difficult to see that $C$ is a nonempty $\Pi^1_1$-set and hence contains an element $T \in \Delta^1_2$. Let $D = \{x \mid \exists y(x, y) \in [T]\}$ be a $\Sigma^1_1(T)$-set. Since $B$ is uniformized and $[T] \subseteq B$, we have that $D$ must be uncountable and so contains a perfect subset. Again since $B$ is uniformized, we have that $x \in D \Leftrightarrow \exists y \in L_{\omega_1^x}[x](x, y) \in T$. So, by Spector-Gandy's theorem, $D$ must be $\Delta^1_1(T)$. Then by Shoenfield absoluteness relative to $T$, there is a perfect tree $S \in \Delta^1_2(T)$ such that $[T] \subseteq D$.[1] Thus $S$ is $\Delta^1_2$. ⊣

Corollary 1.4. *Every nonthin $\Sigma^1_2$-set has a $\Delta^1_2$-pointed subset.*

Proof. Let $A$ be an uncountable $\Sigma^1_2$-set. We consider two cases:

Case 1. There is a nonconstructible real. By Proposition 1.3, $A$ has a $\Delta^1_2$-perfect, and hence $\Delta^1_2$-pointed subset.

Case 2. Otherwise. Let $B = \{T \mid [T] \subseteq A \wedge T \text{ is perfect}\}$. Let $T_0 \in B$. Then $T_0 \in L$ since every real is constructible. Let $\alpha_0$ be such that $T_0 \in L_{\alpha_0+1} \setminus L_{\alpha_0}$. Then by [1] and [6], there is a master code $z \in L_{\alpha_0+1} \setminus L_{\alpha_0}$. Note that $T_0 \geq_{\Delta^1_2} z$ and $T_0 \leq_T z$. Let $C = [T_0] \setminus \{x \mid x \in L_{\alpha_0}\}$. Then $C$ is $\Delta^1_1(z)$.[2] Since $C$ is uncountable, there is a $\Delta^1_2(z)$-perfect tree $T_1$ such that $[T_1] \subseteq C$. Since for every real $x \in [T_1]$, $x \notin L_{\alpha_0}$, we have $z \leq_{\Delta^1_2} x$. It follows that $[T_1] \subseteq A$ and $T_1$ is a $\Delta^1_2$-pointed tree. ⊣

**§2. Perfect subsets of $2^\omega$.** First note that the argument in Proposition 1.3 does not go through without the assumption $2^\omega \not\subset L$. For example, since $x \in L \cap 2^\omega$ if and only if there is a real $y \in L_{\omega_1^y}$ such that $x \leq_T y$, let $B = \{(x, y) \mid x \leq_T y \wedge y \in L_{\omega_1^y}\}$. Then $B$ is $\Pi^1_1$ and thin. We have $x \in L \cap 2^\omega \leftrightarrow \exists y(x, y) \in B$, so that, assuming that every real is constructible, the $\Sigma^1_2$-set of all reals is the projection of a thin $\Pi^1_1$-set. Then $C$ in the proof of the proposition is not defined. This failure leads us to the following which implies the necessity of the hypothesis in Proposition 1.3.

Lemma 2.1. *If every real is constructible, then there is a co-countable $\Delta^1_2$-set $A$ with no $\Delta^1_2$-perfect subset.*

[1] In fact one can construct a perfect set $S \leq_T T$ such that $[S] \subseteq A$.

[2] This follows from Jensen's fine structure theory of $L$ [6]: Since $T_0 \in L_{\alpha_0+1} \setminus L_{\alpha_0}$, there is an $n$ such that $T_0$ is a new $\Sigma_n(L_{\alpha_0})$ subset of $\omega$. Hence there exists a $\Sigma_n$-master code $z$ in the sense of Jensen in [6]. Then $T_0$ is $\Sigma_0((L_\omega, z))$ and so $T_0 \leq_T z$. Also by the results in [6], there is a $\Sigma_n(L_{\alpha_0})$ partial function $p$ mapping $\omega$ onto $\alpha_0$. Thus the set $E = \{2^i 3^j \mid \text{Both } p(i) \text{ and } p(j) \text{ are defined and } p(i) < p(j)\}$ is a $\Sigma_n(L_{\alpha_0})$ subset of $\omega$ coding a well ordering of length $\alpha_0$. Since $z$ is a $\Sigma_n$-master code, we have $E \leq_T z$ and so $\omega_1^z > \alpha_0$. Now by a simple calculation, one can see that $C$ is $\Delta^1_1(z)$.

PROOF. We $L_{\omega_1}$-recursively build two sets $A$ and $B$ such that $A = 2^\omega \setminus B$ as follows:

By $\Pi^1_1$-uniformization, fix a partial $\Pi^1_1$-function $p : \omega \to 2^\omega$ such that $\mathsf{Range}(p)$ contains all $\Pi^1_1$-singletons. Let $q : \omega \times \omega \to 2^\omega$ be a partial function such that

$$q(e, i) = x \Leftrightarrow p(i)\downarrow \wedge \Phi_e^{p(i)} \text{ is total} \wedge x = \Phi_e^{p(i)} \wedge x \text{ codes a perfect tree } T_{e,i},$$

where $\Phi_e$ is the $e$-th oracle Turing machine. Then $\mathsf{Range}(q)$ contains exactly the $\Delta^1_2$-perfect trees. Since every real is constructible, $q$ is $\Sigma_1$ over $L_{\omega_1}$. We now proceed with the construction.

Let $A_0 = B_0 = \emptyset$. At stage $\gamma < \omega_1$, select the least $(e, i)$ such that $q(e, i)\downarrow$ at stage $\gamma$ and $\bigcup_{\gamma'<\gamma} B_{\gamma'} \cap [T_{e,i}] = \emptyset$. If there is no such $(e, i)$, let $A_\gamma = \bigcup_{\gamma'<\gamma} A_{\gamma'}$, $B_\gamma = \bigcup_{\gamma'<\gamma} B_{\gamma'}$, and go to the next stage. Otherwise, choose the $<_L$-least real $x \in [T_{e,i}] \setminus L_\gamma$ and let $B_\gamma = \bigcup_{\gamma'<\gamma} B_{\gamma'} \cup \{x\}$. Define $A_\gamma = (L_\gamma \cap 2^\omega) \setminus B_\gamma$. Then both $A = \bigcup_{\gamma<\omega_1} A_\gamma$ and $B = \bigcup_{\gamma<\omega_1} B_\gamma$ are $\Sigma_1(L_{\omega_1})$. By Theorem 1.1, A and $B$ are $\Sigma^1_2$. Since they are complementary to each other, both are $\Delta^1_2$, and the construction ensures that $B$ is countable. Furthermore, if $T$ is a $\Delta^1_2$-perfect tree, then there is an $(e, i)$ and a stage $\gamma$ where $q(e, i)\downarrow$, $T_{e,i} = T$, and $[T_{e,i}] \cap B \neq \emptyset$. Hence $A$ contains no $\Delta^1_2$-perfect subset. ⊣

The above Lemma and Proposition 1.3 imply the following:

THEOREM 2.2. *Every nonthin $\Sigma^1_2$-set has a $\Delta^1_2$-perfect subset if and only if there is a nonconstructible real.*

While the set $A$ in Lemma 2.1 was constructed to answer the question under the hypothesis $2^\omega \subset L$, it also presents an "extreme case" as the next observation shows. We say that a set $A$ is ZFC-provably $\Delta^1_2$, or $\Delta^{1,\mathsf{ZFC}}_2$ for short, if there are two $\Sigma^1_2$-formulas $\varphi$ and $\psi$ such that

$$\mathsf{ZFC} \vdash \forall x(x \in A \leftrightarrow \varphi(x) \leftrightarrow \neg\psi(x)).$$

PROPOSITION 2.3. *If $A$ is a nonthin $\Delta^{1,\mathsf{ZFC}}_2$-set, then it contains a $\Delta^1_2$-perfect subset.*

PROOF. By Proposition 1.3, it is sufficient to assume that every real is constructible. Hence there is a perfect tree $T \in L$ such that $[T] \subseteq A$. Adding a Cohen generic real $g$ to $V$, by the Shoenfield absoluteness lemma, $V[g] \models [T] \subseteq A$ since $A$ is $\Delta^1_2$. Then $V[g] \models A$ contains a perfect subset. By Proposition 1.3, there is a $\tilde{T}$ such that

$$V[g] \models A \text{ contains a } \Delta^1_2\text{-perfect subset } [\tilde{T}].$$

Then $\tilde{T} \in V$ and $V \models [\tilde{T}] \subseteq A \wedge \tilde{T}$ is $\Delta^1_2$ also by Shoenfield absolutness. By Shoenfield absoluteness again,

$$V \models (\exists \tilde{T})(A \text{ contains a } \Delta^1_2\text{-perfect subset } [\tilde{T}]).$$ ⊣

A more general problem is to characterize the conditions under which every nonthin $\Sigma^1_2$-set has a nonthin $\Delta^1_2$-subset. This is provided by the following theorem.

THEOREM 2.4. *Every nonthin $\Sigma^1_2$-set has a nonthin $\Delta^1_2$-subset if and only if there is a nonconstructible real.*

PROOF. The direction from right to left follows from Theorem 2.2 immediately. We prove the other direction by an "infinitary" priority argument.

https://doi.org/10.1017/jsl.2018.81 Published online by Cambridge University Press

Fix a recursive enumeration of all $\Sigma^1_2$-pairs $\{(B_i, C_i)\}_{i\in\omega}$ and an $L_{\omega_1}$-effective enumeration $\{T_\gamma\}_{\gamma<\omega_1}$ of perfect trees. We shall construct a co-countable $\Sigma^1_2$-set $A$ satisfying the following requirements:

$$P_i : \exists\gamma([T_\gamma] \subseteq C_i \wedge B_i = 2^\omega \setminus C_i \to \exists x \in C_i \setminus A).$$

*Construction*:

*Stage* 0. Let $A = \emptyset$ and define $\gamma^i_0 = 0$. Let $x_{i,0}$ be the leftmost infinite path through $T_{\gamma^i_0}$ for each $i$.

*Stage* $\alpha =$ *limit ordinal.* For each $i$, let $\gamma^i_\alpha = \min\{\gamma \mid \forall\beta < \alpha(\gamma^i_\beta \leq \gamma)\}$ and let $x_{i,\alpha}$ be the $<_L$-least infinite path through $T_{\gamma^i_\alpha}$ not in $A$. Go to the next stage.

*Stage* $\alpha + i + 1$ *where* $\alpha < \omega_1$ *is a limit ordinal.* If $[T_{\gamma^i_\alpha}] \cap B_i \cap L_\alpha \neq \emptyset$, then let $\gamma^i_{\alpha+i+1} = \gamma^i_{\alpha+i} + 1$ and let $x_{i,\alpha+i+1}$ be the $<_L$-least infinite path through $T_{\gamma^i_{\alpha+i+1}}$ not in $A$. For $j \neq i$, let $x_{j,\alpha+i+1} = x_{j,\alpha}$ and $\gamma^j_{\alpha+i+1} = \gamma^j_\alpha$. Put all reals in $L_{\alpha+i}$ other than the $x_{j,\alpha+i+1}$'s, where $j \in \omega$, into $A$ and proceed to the next stage. If $[T_{\gamma^i_\alpha}] \cap B_i \cap L_\alpha = \emptyset$, let $x_{j,\alpha+i+1} = x_{j,\alpha}$ and $\gamma^j_{\alpha+i+1} = \gamma^j_\alpha$ for all $j \in \omega$, and proceed to the next stage.

By the construction, $A$ is clearly $\Sigma^1_2$. The construction ensures that a real $x$ is not in $A$ if and only if there is an $i$ such that $x = \lim_{\alpha\to\omega_1} x_{i,\alpha}$. Hence $A$ is co-countable.

Now suppose that $C$ is $\Delta^1_2$ and has a perfect subset. Let $i$ be such that $C = C_i$, $B_i = 2^\omega \setminus C_i$, and $[T_{\gamma_i}] \subseteq C_i$ for the least $\gamma_i$. Then $\lim_{\alpha\to\omega_1} \gamma^i_\alpha = \gamma_i$. Let $\alpha$ be the least stage $\alpha$ such that $\gamma^i_\alpha = \gamma_i$. Then an infinite path $x_{i,\alpha}$ in $[T_{\gamma^i_\alpha}]$ was selected to be kept out of $A$ at stage $\alpha$. Since $B_i = 2^\omega \setminus C_i$ and $[T_i] \subseteq C_i$, by the construction, $\forall\beta \geq \alpha(x_{i,\alpha} = x_{i,\beta})$. Then $x_{i,\alpha} \notin A$ and hence $C_i \setminus A \supseteq \{x_{i,\alpha}\} \neq \emptyset$. ⊣

## §3. Uncountable subsets of $\Sigma^1_2$-sets.

Since $\Sigma^1_2$-sets of reals in $L$ are $\Sigma_1$-definable set-theoretically, the following proposition is an immediate consequence of the general theory of recursively enumerable sets in $\alpha$-recursion theory (see [10]).

PROPOSITION 3.1. *Suppose that every real is constructible. Then every uncountable $\Sigma^1_2$-set has an uncountable $\Delta^1_2$-subset. In fact, every uncountable $\Sigma^1_2$-set contains a pair of disjoint uncountable $\Sigma^1_2$-subsets.*

COROLLARY 3.2. *Every nonthin $\Sigma^1_2$-set $A$ has an uncountable $\Delta^1_2$-subset.*

PROOF. This is an immediate consequence of Proposition 3.1 if every real is constructible, and a consequence of Proposition 1.3 otherwise. ⊣

The rest of the section studies conditions for the converse of Corollary 3.2 to hold.

LEMMA 3.3. *If $\omega^L_1 < \omega_1$, then every uncountable $\Sigma^1_2$-set has a $\Delta^1_2$-perfect subset.*

PROOF. Since $\omega^L_1$ is countable, every uncountable $\Sigma^1_2$-set has a nonconstructible member and hence not thin by the Mansfield-Solovay Theorem. By Proposition 1.3, every uncountable $\Sigma^1_2$-set has a $\Delta^1_2$-perfect subset. ⊣

In subsequent proofs, we will need to decide if "$[T] \subseteq A$" holds for a perfect tree $T$ and a $\Sigma^1_2$-set $A$. However, the statement is $\Pi^1_3(T)$ which is not absolute. To overcome this difficulty, we require an analysis of the set of reals finer than what the Shoenfield absoluteness lemma provides, in the sense of the following Lemma.

LEMMA 3.4. *Suppose that there is a nonconstructible real. Given a perfect tree $T \in L$ and a $\Sigma^1_2$-set $A$ such that $[T] \cap A$ contains a perfect subset, there is a perfect tree $\tilde{T} \in L$ such that $[\tilde{T}] \subseteq [T] \cap A$. Furthermore, $\tilde{T}$ may be computed uniformly and effectively from $T$ and $A$. In other words, there is a partial function $f : \mathcal{P}(\omega^{<\omega}) \times \omega \to \mathcal{P}(\omega^{<\omega})$ $\Sigma_1$ definable over $L_{(\omega_1)^L}$ such that for any perfect tree $T \subseteq \omega^{<\omega}$ in $L$ and any index $e$ of a $\Sigma^1_2$-set $A$, if $[T] \cap A_e$ has a perfect subset, then $f(T, e)$ is a perfect subset of $[T] \cap A_e$ in $L$.*[3]

PROOF. The proof follows essentially that of either Lemma 9.3 in [10] or Lemma 4.3.2 in [2].

We assume that $T = 2^{<\omega}$ first. As in the proof of Proposition 1.3, one can effectively compute (an index of) a $\Pi^1_1$-set $B$ such that

$$x \in A \Leftrightarrow \exists! y \in L_{\omega_1^{x\oplus y}}[x]((x, y) \in B).$$

By Theorem 1.2, there is a nonconstructible real $x_0 \in A$. Fix the corresponding unique real $y_0 \in L[x_0]$ so that $(x_0, y_0) \in B$. Note that $x_0 \oplus y_0 \notin L_{\omega_1^{x_0\oplus y_0}}$. So the $\Sigma^1_2$ set $\{(x, y) \mid (x, y) \in B \wedge x \oplus y \notin L_{\omega_1^{x\oplus y}}\}$ is not empty. Thus there exists a pair $(x, y) \in B$ such that $(x, y) \in L$ but $x \oplus y \notin L_{\omega_1^{x\oplus y}}$. Let $(x_1, y_1)$ be such a pair.

We may also effectively obtain a tree $T_0 \subseteq 2^{<\omega} \times 2^{<\omega} \times \omega_1^{<\omega} \in L$ such that

$$(x, y) \in B \Leftrightarrow \exists f \in L_{\omega_1^{x\oplus y}}[x \oplus y]((x, y, f) \in [T_0]).$$

Since the set $\{(x, y) \mid x \oplus y \notin L_{\omega_1^{x\oplus y}}\}$ is $\Sigma^1_1$, there is a recursive tree $S \subseteq \omega^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ such that

$$\exists z \in L[x \oplus y](x, y, z) \in [S] \Leftrightarrow x \oplus y \notin L_{\omega_1^{x\oplus y}}.$$

Note that, since $(x_1, y_1) \in B \cap L$ and $(x_1, y_1) \notin L_{\omega_1^{x_1\oplus y_1}}$, we may fix $\alpha_0$ to be the least admissible ordinal $\alpha$ for which there is a pair $(x_2, y_2) \in B \cap L$ such that $(x_2, y_2) \notin L_\alpha$ and $\omega_1^{x_2\oplus y_2} = \alpha$. Let $T_0^{\alpha_0} = T_0 \cap L_{\alpha_0}$ and

$$T_1 = \{(\sigma, \tau, v, u) \mid (\sigma, \tau, v) \in T_0^{\alpha_0} \wedge (\sigma, \tau, u) \in S\}.$$

By the assumption on $\alpha_0$, there is an infinite path through $T_1$. Moreover for any infinite path $(x, y, f, z)$ though $T_1$, $\omega_1^{x\oplus y} \geq \alpha_0$. Note that for any $(\sigma, \tau, v, u) \in T_1$, if there is an infinite path through $T_1$ extending $(\sigma, \tau, v, u)$, then there must be incompatible pairs of strings $(\sigma_1, \tau_1)$ and $(\sigma_2, \tau_2)$ extending $(\sigma, \tau)$ and pairs $(v_1, u_1)$ and $(v_2, u_2)$ for which there exist infinite paths through $T_1$ extending $(\sigma_1, \tau_1, v_1, u_1)$ and $(\sigma_2, \tau_2, v_2, u_2)$ respectively (otherwise, there will be an infinite path $(x, y, f, z)$ through $T_1$ such that $(x, y, f, z) \in L_{\omega_1^{x\oplus y}}$). Then $(x, y, z) \in [S]$ but $x \oplus y \in L_{\omega_1^{x\oplus y}}$, a contradiction. More details can be found in the proof of either Lemma 9.3 in [10] or Lemma 4.3.2 in [2].

Now, by the property of $T_1$, it is not difficult to see that there is a perfect tree $T_2$ in $L$ such that $[T_2] \subseteq B$. For our purpose, we fix an algorithm to construct $T_2$ as following. Let $T_1'$ be the outcome of Cantor-Bendixson construction with the input

[3]Note that we do not claim it is a theorem of ZFC that we may find such a tree $\tilde{T}$. The point is that if $\tilde{T}$ exists, then we may find it effectively. To do this, one needs the assumption that there is a nonconstructible real.

https://doi.org/10.1017/jsl.2018.81 Published online by Cambridge University Press

being the countable tree $T_1$. Then $T_1' \subseteq \omega^{<\omega} \times \alpha_0^{<\omega} \times \omega^{<\omega} \times \omega^{<\omega}$ has the property that for any $(\sigma, \tau, v, u) \in T_1'$, there are nodes $(\sigma_1, \tau_1, v_1, u_1)$ and $(\sigma_2, \tau_2, v_2, u_2)$ in $T_1'$ so that $(\sigma_1, \tau_1)$ and $(\sigma_2, \tau_2)$ are distinct nodes extending $(\sigma, \tau)$. Now let $T_2$ be the projection of $T_1'$ into first two coordinate. Since $B$ is already uniformized, we conclude that there is a perfect tree $\tilde{T}$ derived from $T_2$ so that $[\tilde{T}] \subseteq A$.

For an arbitrary perfect tree $T \in L$, one relativizes the proof above to $T$ to obtain the corresponding perfect tree $\tilde{T}$.

By the uniformity of the construction, we see that the function $(T, A_e) \mapsto \tilde{T}_e$ is a $\Sigma_1(L_{(\omega_1)^L})$ partial function. ⊣

Corollary 3.5. *Suppose that there is a nonconstructible real. If $A$ is $\Sigma^1_2$ and $2^\omega \setminus A$ is thin, then for any perfect tree $T \in L$, there is a perfect tree $T' \in L$ such that $[T'] \subseteq [T] \cap A$. Moreover, the partial function $p : (T, A) \mapsto \tilde{T}$ is $\Sigma_1(L_{(\omega_1)^L})$.*

Proof. Let $A$ and $T \in L$ be as given. It is clear that there is a perfect tree $S \subseteq T$ such that $[S] \cap L = \emptyset$. Hence $A \cap [T]$ contains a nonconstructible real and therefore a perfect subset. Now by Lemma 3.4, there is a perfect tree $T' \in L$ such that $[T'] \subseteq [T] \cap A$. The definability also follows from the Lemma. ⊣

Let $\mathscr{B} = \{B \mid B \text{ is } \Sigma^1_2 \wedge 2^\omega \setminus B \text{ is thin}\}$.

Lemma 3.6. *Suppose that $\omega_1^L = \omega_1$ and there is a nonconstructible real. Then there is an uncountable $\Sigma^1_2$-set $A$ with no uncountable $\Delta^1_2$-subset. In fact there is an uncountable $\Sigma^1_2$-set $A \subset L$ such that $|A \setminus \bigcap_{B \in \mathscr{B}} B| \leq \aleph_0$.*

Proof. The proof of Lemma 3.6 is a finite injury priority argument over $L_{\omega_1}$. The idea is as follows: Every $\Sigma^1_2$-set $B \in \mathscr{B}$ is nonthin by assumption, and hence by the proof of Proposition 1.3 contains a $\Delta^1_2$-perfect tree in $L$. By Corollary 3.5, one may wait for a stage where a perfect tree $\tilde{T}$ with $[\tilde{T}] \subset B$ is enumerated in $L$. The aim then is to make $A$ a subset of $[\tilde{T}]$ except for possibly countably many elements. To satisfy this for each $B \in \mathscr{B}$ clearly creates conflicts. These are resolved by the use of a fusion argument.

We fix a $\Sigma_1(L_{(\omega_1)^L})$ partial function $p$ as in Corollary 3.5.

*Construction:* Fix a recursive enumeration $\{B_i\}_{i \in \omega}$ of all $\Sigma^1_2$-sets. We construct a $\Sigma^1_2$-set $A$ such that

$$|A \setminus \bigcap_{B_i \in \mathscr{B}} B_i| \leq \aleph_0.$$

Let $P_i$ be the requirement stating that $A \setminus B_i$ is countable if $B_i \in \mathscr{B}$.

*Stage* 0. We may assume that $B_0 = 2^\omega$. Let $T_0 = 2^{<\omega}$ and $f_0 = id : 2^{<\omega} \to 2^{<\omega}$ be the identity map. By definition, $P_0$ receives attention at stage 0 and no $P_i$ receives attention at stage 0 for $i > 0$.

*Stage* $\alpha + 1$. Let $f_\alpha : 2^{<\omega} \cong T_\alpha$ be the canonical homeomorphism. We say that $P_i$ *requires attention at stage* $\alpha + 1$ if it has not received attention thus far, or is injured at stage $\alpha$, and for each $\sigma \in 2^i$, there is a perfect tree

$$T^\sigma_\alpha \subseteq \{\tau \mid \tau \in T_\alpha \wedge \tau \text{ is compatible with } f_\alpha(\sigma)\}$$

enumerated in $L_{\alpha+1} \setminus L_\alpha$ where $[T^\sigma_\alpha] \subseteq B_i$. In other words, $p(\{\tau \mid \tau \in T_\alpha \wedge \tau \text{ is compatible with } f_\alpha(\sigma)\}, i)$ is defined at stage $\alpha + 1$; and we let $T^\sigma_\alpha = p(\{\tau \mid \tau \in T_\alpha \wedge \tau \text{ is compatible with } f_\alpha(\sigma)\}, i)$.

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If no $i$ requires attention, let $f_{\alpha+1} = f_\alpha, T_{\alpha+1} = T_\alpha$. Declare all the requirements injured at stage $\alpha$ to remain injured at stage $\alpha+1$ and go to the next stage. Otherwise, let $i_{\alpha+1}$ be the least such $i$. Define $T_{\alpha+1} = \bigcup_{\sigma\in 2^{i_{\alpha+1}}} T^\sigma_\alpha$ for $\sigma \in 2^{i_{\alpha+1}}$, $f_{\alpha+1} : 2^{<\omega} \cong T_{\alpha+1}$ the canonical homeomorphism, and declare $P_{i_{\alpha+1}}$ to have received attention. Every $P_j$ for $j > i_{\alpha+1}$ is injured at stage $\alpha+1$.

*Stage $\alpha$ a limit ordinal*. By Claim 1(ii) below, for each $i \in \omega$ there is a $\beta_i < \alpha$ such that $\forall\beta\forall\sigma \in 2^i(\beta_i < \beta < \alpha \to f_\beta(\sigma) = f_{\beta_i}(\sigma))$. Hence $\lim_{\beta\to\alpha} f_\beta(\sigma)$ exists for each $\sigma \in 2^{<\omega}$. Let $T_\alpha$ be the perfect tree generated by $\{\lim_{\beta\to\alpha} f_\beta(\sigma) \mid \sigma \in 2^{<\omega}\}$. Choose the least stage $\beta \le \alpha$ such that $T_\beta = T_\alpha$. For each real $x \in [T_\alpha]\cap(L_\alpha \setminus L_\beta)$, put it into $A$.

This completes the construction of $A$ at stage $\alpha$. It is clear that $A \subseteq L$ is a $\Sigma^1_2$-set.

Claim 1.

(i) Each requirement receives attention at most finitely many times.
(ii) For each $\sigma$, $\{\beta \mid f_\beta(\sigma) \neq f_{\beta+1}(\sigma)\}$ is finite. In particular, $\lim_{\alpha\to\omega_1} f_\alpha(\sigma)$ exists for each $\sigma$.
(iii) $\exists\alpha\forall\beta \ge \alpha(T_\beta = T_\alpha)$.

Proof of Claim 1. We prove (i) and (ii) by induction on $i$. The construction ensures that if $P_i$ receives attention at stage $\alpha$, then it receives attention again at some $\beta > \alpha$ only if some $P_j$, $j < i$, receives attention at $\beta'$ where $\alpha < \beta' < \beta$.

Now $P_0$ clearly receives attention at most once (we identify $\sigma \in 2^0$ with the empty string). Let $n$ be given and assume that each $P_i$, $i \le n$, receives attention finitely many times. Let $\beta_n$ be a stage such that for all $i \le n$ and all $\beta \ge \beta_n$, $P_i$ does not receive attention at stage $\beta$. Then $\forall\beta \ge \beta_n\forall\tau(|\tau| \le n \to f_\beta(\tau) = f_{\beta_n}(\tau))$. If $P_{n+1}$ never receives attention after stage $\beta_n$, then it will have received attention at most finitely many times, and so (i) holds for $P_{n+1}$.

Suppose now $P_{n+1}$ receives attention at some stage $\beta > \beta_n$ and let $\beta_{n+1}$ be the least such $\beta$. Then by the construction, $\beta_{n+1} = \alpha+1$ for some $\alpha$ and $T_{\beta_{n+1}} \subset T_\alpha$. But this implies that for all $\sigma \in 2^{n+1}$ and $\beta > \beta_{n+1}$, $f_{\beta_{n+1}}(\sigma) = f_\beta(\sigma)$, proving (ii) for all strings $\sigma$ of length $n+1$. This also implies that $P_{n+1}$ does not require attention after stage $\beta_{n+1}$. Hence $P_{n+1}$ satisfies (i).

Note that (iii) follows immediately from (i) and (ii).

Claim 2.

(i) $A$ is $\Sigma^1_2 \setminus \Delta^1_2$ and uncountable.
(ii) $A \setminus \bigcap_{B_i\in\mathscr{B}} B_i$ is countable.

Proof of Claim 2. (i). To show that $A$ is not $\Delta^1_2$, note that $A \subset L$ and hence thin by Theorem 1.2. If $A$ is $\Delta^1_2$, then $2^\omega \setminus A = B_i$ for some $B_i \in \mathscr{B}$, and this contradicts the fact that $A \cap B_i \neq \emptyset$ by the construction.

By Claim 1, let $\alpha_\infty < \omega_1$ be such that for all $\alpha > \alpha_\infty$, $i \in \omega$ and all strings $\sigma$, $P_i$ does not receive attention at stage $\alpha$, and $f_{\alpha_\infty}(\sigma) = f_\alpha(\sigma)$. Then $T_\alpha = T_{\alpha_\infty}$. This implies that $[T_{\alpha_\infty}] \cap (L_{\omega_1} \setminus L_{\alpha_\infty}) \subseteq A$. So $A$ is uncountable.

(ii). Let $B_i \in \mathscr{B}$. We claim that for each $\sigma$ of length $i$ and $\tilde{T}^\sigma_{\alpha_\infty} = \{\tau \mid \tau \in T_{\alpha_\infty} \wedge \tau$ is compatible with $f_{\alpha_\infty}(\sigma)\}$, $[\tilde{T}^\sigma_{\alpha_\infty}] \subseteq B_i$. Otherwise, since $B_i \in \mathscr{B}$, by Corollary 3.5, there is a stage $\alpha \ge \alpha_\infty$ such that for any $\sigma$ of length $i$, there is a perfect tree $T^\sigma_\alpha \subseteq \tilde{T}^\sigma_{\alpha_\infty}$ such that $[T^\sigma_\alpha] \subseteq B_i$ and $T^\sigma_\alpha \in L_{\alpha+1} \setminus L_\alpha$. Then $P_i$ receives attention at stage $\alpha$, a contradiction. Thus $[T_{\alpha_\infty}] \subseteq B_i$ and so $[T_{\alpha_\infty}] \subseteq \bigcap_{B_i\in\mathscr{B}} B_i$.

Thus $A \setminus \bigcap_{B_i \in \mathscr{B}} B_i \subseteq L_{\alpha_\infty}$ and hence countable.

By Claim 2, $A$ is an uncountable $\Sigma^1_2$-set without any uncountable $\Delta^1_2$-subset, completing the proof of Lemma 3.6. ⊣

Corollary 3.7. *Assume that $\omega_1^L = \omega_1$ and there is a nonconstructible real. Then there is a countable $\Sigma^1_2$-set $A \subset L$ such that $A \cap \bigcap_{B \in \mathscr{B}} B \neq \emptyset$.*

Proof. We repeat the construction of the trees $T_\alpha$ in Lemma 3.6 but only put countably many elements in $T_{\alpha_\infty}$ into $A$ at stage $\alpha_\infty$ and none after that stage. Then it is immediate that $A$ is a countable $\Sigma^1_2$-set satisfying the lemma. ⊣

In summary, we have the following theorem.

Theorem 3.8. *Every uncountable $\Sigma^1_2$-set has an uncountable $\Delta^1_2$-set if and only if either $2^\omega \subset L$ or $\omega_1^L < \omega_1$.*

We end this section with a strengthened version of Lemma 3.6.

Theorem 3.9. *Assume $\omega_1^L = \omega_1$ and that there is a nonconstructible real. Then there is an uncountable $\Sigma^1_2$-set $A$ with no uncountable $\Delta^1_2(x)$-subset for any $x \in L$. In fact, there is an uncountable $\Sigma^1_2$-set $A \subset L$ such that for any $x \in L$ and any $\Sigma^1_2(x)$-set $B$ with $2^\omega \setminus B \subset L$,[4] $|A \setminus B| \leq \aleph_0$.*

Proof. To see that the last statement in the theorem implies that $A$ has no uncountable $\Delta^1_2(x)$-subset for any $x \in L$, assume for the sake of contradiction that $C \subset A$ is a counterexample. Then $B = 2^\omega \setminus C$ is $\Delta^1_2(x)$ and $2^\omega \setminus B = C \subset L$. But then $C = A \setminus B$ is at most countable.

Now note that the proof of Lemma 3.6 can be carried out uniformly within a perfect tree $T \in L$ relative to an oracle $x$. In other words, given a perfect tree $T \in L$ and a real $x$, we may $T \oplus x$-effectively perform the construction in the proof of Lemma 3.6 by replacing $2^{<\omega}$ with $T$ and working within $[T]$ to define $A$, so that $A$ is contained in $\bigcap_{B \in \mathscr{B}(x)} B$ except at most countably many points, where $\mathscr{B}(x) = \{B \mid B \text{ is } \Sigma^1_2(x) \wedge 2^\omega \setminus B \text{ is thin}\}$. We will use this as the blueprint of the construction below as $x$ ranges all constructible reals.

Fix a $\Sigma_1(L_{\omega_1})$-sequence of reals $\vec{x} = \{x_\beta\}_{\beta \in \omega_1}$ such that $\vec{x}$ is a cofinal increasing chain of Turing degrees in $L$. For $\beta < \omega_1$, let

$$\mathscr{B}(x_\beta) = \{B \mid B \text{ is } \Sigma^1_2(x_\beta) \wedge 2^\omega \setminus B \text{ is thin}\}.$$

Then for any $x \in L$, every $\Sigma^1_2(x)$ set is $\Sigma^1_2(x_\beta)$ for some $\beta$. We use an idea in Simpson [11] as presented in Chong and Yu [2] to construct $A$. For any perfect tree $T$, we use $f_T : 2^{<\omega} \to T$ to denote the canonical homeomorphism from $2^\omega$ to $[T]$.

Define by induction a $\Sigma_1(L_{\omega_1})$-collection of sets $\{\mathscr{I}_\beta\}_{\beta<\omega_1}$ of perfect trees such that for $\gamma < \beta < \omega_1$,

(i) $\forall S \in \mathscr{I}_\gamma \exists S_0, S_1 \in \mathscr{I}_\beta (S_0 \subset S \cap [f_S(0)] \wedge S_1 \subset S \cap [f_S(1)])$, where $f_S : 2^{<\omega} \cong S$ is the canonical homeomorphism;
(ii) $\forall S \in \mathscr{I}_\beta \exists n \exists \{S_i\}_{i \leq n} \subseteq \mathscr{I}_\gamma ([S] \subseteq \bigcup_{i \leq n} [S_i])$, and
(iii) $\forall S \in \mathscr{I}_\beta ([S] \subseteq \bigcap_{B \in \mathscr{B}(x_\beta)} B)$.

[4] Actually, as the referee pointed out, $2^\omega \setminus B \subset L$ can be replaced with "$2^\omega \setminus B$ is thin".

https://doi.org/10.1017/jsl.2018.81 Published online by Cambridge University Press

The requirements to satisfy are

$$P_\beta : \forall S \in \mathscr{I}_\beta (S \subseteq \bigcap_{B \in \mathscr{B}(x_\beta)} B).$$

*Construction*: Let $\mathscr{I}_0 = \{2^{<\omega}\}$. At stage $\alpha$, let $\gamma_\alpha$ be the least $\gamma \leq \alpha$ such that $\mathscr{I}_\gamma$ is undefined.

CASE 1. $\gamma_\alpha = \beta + 1$ for some $\beta$. Then for each $S \in \mathscr{I}_\beta$, let $S_0, S_1$ be the $<_L$-least pair of perfect trees such that $S_0 \subset S \cap [f_S(0)]$ and $S_1 \subset S \cap [f_S(1)]$. Let $S_0, S_1 \in \mathscr{I}_{\beta+1}$.

CASE 2. $\gamma_\alpha$ is a limit ordinal. Let $\{\beta_i\}_{i<\omega}$ be the $<_L$-least increasing $\omega$-sequence of ordinals with limit $\gamma_\alpha$. For $\beta < \gamma_\alpha$, let $i_\beta$ be the least $i$ such that $\beta \leq \beta_i$. For any $S \in \mathscr{I}_\beta$, by inductive hypothesis, it is not difficult to see that there is a $<_L$-least fusion sequence $\{S_\sigma\}_{\sigma \in 2^{<\omega}}$ such that

- $S_\emptyset = S$;
- $(\forall n)(\forall \sigma \in 2^{n+1})(S_\sigma \in \mathscr{I}_{\beta_{i_\beta + n}})$;
- $(\forall n)(\forall \sigma \in 2^n)(S_{\sigma^\frown 0} \subset S_\sigma \cap [f_S(\sigma^\frown 0)] \wedge S_{\sigma^\frown 1} \subset S \cap [f_S(\sigma^\frown 1)])$.

Let $T = \bigcap_{n<\omega} \bigcup_{\sigma \in 2^n} S_\sigma$. Put $T \cap [f_T(0)]$ and $T \cap [f_T(1)]$ into $\mathscr{I}_{\gamma_\alpha}$.

Now for any $\beta \leq \gamma_\alpha$ and each $T \in \mathscr{I}_\beta$, we perform the construction as in Lemma 3.6 up to stage $\alpha$. More precisely, we attempt to trim $T$ to a tree $S$ where $[S] \subseteq B$ for all $B \in \mathscr{B}(x_\beta)$. During the process, once an action is taken on the least $\beta \leq \gamma_\alpha$ with a $T \in \mathscr{I}_\beta$ to meet requirement $P_\beta$, we put each real in $[S] \cap L_\alpha$ into $A$, and initialize all the parameters associated with each $\beta' > \beta$ and go to the next stage.

This completes the construction at stage $\alpha$.

The $\Sigma_1(L)$-definability of $A$ in the construction ensures that it is an uncountable $\Sigma^1_2$-set. As in the proof of Claim 1 in Lemma 3.6, for any $\beta$, $\mathscr{I}_\beta$ will stabilize after a countable ordinal stage $\alpha_\beta$, i.e., no new trees are added to $\mathscr{I}_\beta$ and no more trimming of trees in $\mathscr{I}_\beta$ takes place. By induction on $\beta < \omega_1$, it follows that by stage $\omega_1$ properties (i)–(iii) and $P_\beta$, for $\beta < \omega_1$, are satisfied. Since (by construction) $\mathscr{I}_\beta$ is countable for each $\beta$ at any stage, it means that at most countably many reals are put into $A$ at stage $\beta$. Then by (ii), (iii), and the discussion above, after stage $\alpha_\beta$, we only enumerate reals in $\bigcap_{B \in \mathscr{B}(x_\beta)} B$ into $A$. Hence $A \setminus \bigcap_{B \in \mathscr{B}(x_\beta)} B$ is countable. ⊣

**§4. An anti-basis theorem for $\Pi^1_2$-sets.** H. Friedman observed that if every real is constructible, then every nonempty $\Pi^1_2$-set contains a $\Pi^1_2$-singleton. Harrington [4] showed that the conclusion fails if $2^\omega \not\subset L$: If there is a nonconstructible real, then there is a perfect $\Pi^1_2$-set containing no $\Pi^1_2$-singleton. We apply the method of the previous section to obtain the following stronger result.

THEOREM 4.1. *If there is a nonconstructible real, then there is a perfect $\Pi^1_2$-set $A$ with no thin $\Pi^1_2$-subset.*

PROOF. Let $\mathscr{B}$ be as defined in Section 3.

*Construction*: Fix a recursive enumeration $\{B_i\}_{i \in \omega}$ of $\Sigma^1_2$-sets. We construct a $\Sigma^1_2$-set $A$ such that

$$2^\omega \setminus \bigcap_{B \in \mathscr{B}} B \subseteq A.$$

Assume that $B_0 = 2^\omega$. At stage 0, let $T_0 = 2^{<\omega}$ and $f_0 = id : 2^{<\omega} \to 2^{<\omega}$. We declare that requirement 0, and no other requirement, receives attention at stage 0.

Stage $\alpha + 1$. Let $f_\alpha : 2^{<\omega} \cong T_\alpha$. Search for an $i$ such that requirement $i$ requires attention, i.e., either it has not received attention previously, or was injured at stage $\alpha$, and such that for each $\sigma \in 2^i$, there is a perfect tree

$$T_\alpha^\sigma \subseteq \{\tau \mid \tau \in T_\alpha \wedge \tau \text{ is compatible with } f_\alpha(\sigma)\}$$

in $L_\alpha$ and $[T_\alpha^\sigma] \subseteq B_i$. Let $i_{\alpha+1}$ be the least such $i$. Define $T_{\alpha+1} = \bigcup_{\sigma\in 2^i} T_\alpha^\sigma$ for $\sigma \in 2^{i_{\alpha+1}}$. Enumerate all the reals *not* in $[T_{\alpha+1}]$ into $A$ and declare $i_{\alpha+1}$ to have received attention. Let $f_{i_{\alpha+1}} : 2^{<\omega} \cong T_{\alpha+1}$ be the canonical homeomorphism. For $i > i_{\alpha+1}$, requirement $i$ is declared to be injured. If $i_{\alpha+1}$ does not exist, let $T_{\alpha+1} = T_\alpha$ and $f_{\alpha+1} = f_\alpha$ and go to the next stage.

If $\alpha$ is a limit ordinal, then as in Lemma 3.6, each requirement is injured at most finitely many times before stage $\alpha$. For $i \in \omega$ and $\sigma \in 2^i$, let $f_\alpha(\sigma) = \lim_{\beta\to\alpha} f_\beta(\sigma)$. Let $T_\alpha$ be the image of $2^{<\omega}$ under $f_\alpha$ and proceed to the next stage.

This completes the construction of $A$. By the construction, $A$ is a $\Sigma^1_2$-set. As in the proof of Lemma 3.6, there is a stage $\alpha_0$ such that for all $\alpha \geq \alpha_0$, $T_\alpha = T_{\alpha_0}$. Thus $[T_{\alpha_0}] = 2^\omega \setminus A$ is a perfect $\Pi^1_2$-set. By the construction, it is clear that $[T_{\alpha_0}] \subseteq \bigcap_{B\in\mathscr{B}} B$ and so $2^\omega \setminus \bigcap_{B\in\mathscr{B}} B \subseteq 2^\omega \setminus [T_{\alpha_0}] = A$. ⊣

We end this paper with two questions.

QUESTION 4.2. *Suppose that there is a nonconstructible real and $\omega_1^L = \omega_1$. Is there a nonempty $\Pi^1_2$-set $A$ with no thin $\Pi^1_2(x)$-subset for any $x \in L$?*

QUESTION 4.3. *Let $n > 1$. When does a nonthin $\Sigma^1_{2n}$-set contain a $\Delta^1_{2n}$-perfect subset? When does an uncountable $\Sigma^1_{2n}$-set contain an uncountable $\Delta^1_{2n}$-subset?*

**Acknowledgment.** Chong's research was partially supported by NUS grants C-146-000-042-001 and WBS: R389-000-040-101. Wu's research was partially supported by NSFC grants 11871464 and No. 11621061. Yu was partially supported by the National Natural Science Fund of China through grants No. 11671196 and 11322112.

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