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On an unlinking theorem

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1. *Introduction*. An *n-link of multiplicity* $\mu(L^n_\mu)$ is a smooth embedding of the disjoint union of μ copies of S^n in S^{n+2}; $L^n_\mu: S^n_1 \cup \ldots \cup S^n_\mu \to S^{n+2}$. L^n_μ is said to be *trivial* if it extends to a smooth embedding of the disjoint union of μ copies of D^{n+1}. Let $X = S^{n+2} - L^n_\mu$, and $C_{n,\mu}$ denote the wedge product of μ copies of S^1 and $(\mu-1)$ copies of S^{n+1}. Then clearly, if L^n_μ is trivial, then $X \simeq C_{n,\mu}$, where $\simeq$ denotes homotopy equivalence.

At the 1969 Georgia Topology Institute, Gutierrez (6) announced the following result: 'Let L^n_μ be an n-link of multiplicity μ $(n \geqslant 4)$. The condition $\pi_i(X) \cong \pi_i(C_{n,\mu})$ for $i < q$ $(q \leqslant \frac{1}{2}(n+1))$ is equivalent to the existence of μ mutually disjoint, $(q-1)$-connected manifolds $V_i \subset S^{n+2}$ with $\partial V_i = S^n_i$.'

One can produce counter-examples to the above result by a generalized spinning process (2, 5). Gutierrez has since pointed out to the author that the above unlinking theorem is true with one additional hypothesis: 'Let X denote the bounded link complement, and suppose that $\pi_1(X)$ is free on the meridian curves on ∂X.' It is easily seen that the example produced below does not satisfy this extra requirement; in fact, the boundary meridian curves do not generate $\pi_1(X)$ in the example. The author wishes to thank J. J. Andrews for helpful conversations.

2. *The counter-example*.

DEFINITION. *A ball configuration* $K^n_\mu: B^n_1 \cup \ldots \cup B^n_\mu \subset B^{n+2}$ *is a smooth proper embedding of the disjoint union of* μ *copies of* B^n *in* B^{n+2}.

DEFINITION. *One obtains* L^{n+k}_μ *by k-spinning* K^n_μ *as follows*:

$$S^{n+k+2} = (S^k \times B^{n+2}) \cup (D^{k+1} \times \partial B^{n+2})$$
$$\textit{identified along} \quad S^k \times \partial B^{n+2} = \partial D^{k+1} \times \partial B^{n+2}.$$
$$S^{n+k}_i = (S^k \times B^n_i) \cup (D^{k+1} \times \partial B^n_i) \quad (1 \leqslant i \leqslant \mu)$$
$$\textit{identified along} \quad S^k \times \partial B^n_i = \partial D^{k+1} \times \partial B^n_i.$$

If $k = 1 = \mu$, then this is equivalent to the classical spinning technique of Artin (3).

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Lemma 1. *Suppose L_μ^{n+k} is obtained by k-spinning K_μ^n. Let $X = S^{n+k+2} - L_\mu^{n+k}$ and $Y = B^{n+2} - K_\mu^n$. Then $\pi_1(X) \cong \pi_1(Y)$.*

Proof. From the construction, we have $X = (S^k \times Y) \cup (D^{k+1} \times \partial Y)$ identified along $S^k \times \partial Y = \partial D^{k+1} \times \partial Y$, so Van Kampen's Theorem immediately gives us the desired result.

Now consider the ball configuration K_2^1 of Fig. 1. The counterexample is produced by k-spinning K_2^1, for $k \geq 3$. If $k = 1$, then Artin (3) showed that L_2^2 was not isotopically splittable, and Andrews–Curtis (1) showed that S_2^2 was not homotopic to zero in the complement of S_1^2.

The surprising thing about K_2^1 is that $\pi_1(Y) = \mathbf{Z} * \mathbf{Z}$, where as before $Y = B^3 - K_2^1$. This becomes evident when one realizes that the cube-with-knotted-holes obtained from boring out the arcs in Fig. 1 is actually homeomorphic to the cube-with-straight-holes! (See (4), pp. 97.)

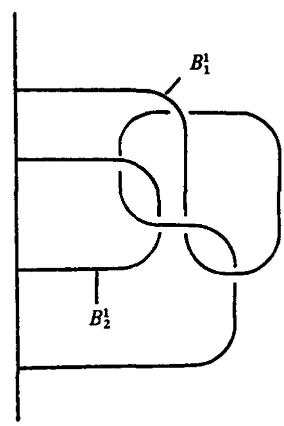

Fig. 1

We now k-spin K_2^1 $(k \geq 3)$ producing L_2^{k+1}, and with $\pi_1(X) = \mathbf{Z} * \mathbf{Z}$ where $X = S^{k+3} - L_2^{k+1}$. So $\pi_1(X) \cong \pi_1(C_{k+1,2})$ and the result of Gutierrez tells us that each of S_1^{k+1} and S_2^{k+1} bounds (simultaneously) a simply connected $(k+2)$-manifold in S^{k+3}. Forget the unknotted component S_2^{k+1}, and concentrate on S_1^{k+1}, the k-spun trefoil. We have $V_1^{k+2} \subset S^{k+3}$, with $\partial V_1^{k+2} = S_1^{k+1}$ and $\pi_1(V) = 1$. Having V simply connected means that $\Delta_1^1 = 1$, where Δ_1^1 is the first Alexander invariant in dimension 1 for the k-spun trefoil (7). This is because, as in Levine (7), if we split S^{k+3} along V_1^{k+2} to obtain W, then $W \simeq S^{k+3} - V_1^{k+2}$ so

$$H_1(W; Q) \cong H^{k+1}(V; Q) \cong H_1(V; Q) = 0$$

where the first isomorphism is Alexander Duality and the second Lefschetz Duality (∂V is a sphere). The infinite cyclic cover of the complement of the k-spun trefoil is built up of copies of W, and the Mayer–Vietoris sequence gives us that H_1 of the infinite cyclic cover $= 0$, hence $\Delta_1^1 = 1$. However, it is well known that $\Delta_1^1 = 1 - t + t^2$, the same as the Alexander polynomial of the trefoil.

3. *Calculating the homotopy groups of X.* We will prove the following theorem, where $X = S^{k+3} - L_2^{k+1}$.

THEOREM 2. *$\pi_i(X) \cong \pi_i(C_{k+1,2})$ for $1 \leqslant i \leqslant k$ and $\pi_{k+1}(X)$ is non-finitely-generated free Abelian.*

Proof. The proof is somewhat analogous to the calculation of Epstein (5). Let $\tilde{X}$ denote the universal cover of X, and $\tilde{Y}$ denote the universal cover of $Y = B^3 - K_2^1$. Now $\partial\tilde{Y}$ = lots of copies of ∂Y^*, where ∂Y^* is a non-simply connected cover of ∂Y. Let $Y_1 = B^3 - B_1^1$ the complement of the trefoil arc. Now the inclusion $\pi_1(Y) \to \pi_1(Y_1)$ is an epimorphism because any loop in Y_1 can be deformed homotopically off B_2^1. Chasing the following diagram with exact rows and vertical maps by inclusion yields that $\pi_1(Y, \partial Y) \to \pi_1(Y_1, \partial Y_1)$ is an epimorphism

$$\begin{array}{ccccc} \to & \pi_1(Y) & \to & \pi_1(Y, \partial Y) & \to 0 \\ & \downarrow & & \downarrow & \\ \to & \pi_1(Y_1) & \to & \pi_1(Y_1, \partial Y_1) & \to 0. \end{array}$$

We have a 1-1 correspondence

$$\pi_1(Y_1, \partial Y_1) \leftrightarrow [\pi_1(Y_1), \pi_1(Y_1)] = \mathbf{Z} * \mathbf{Z},$$

where $[\pi_1(Y_1), \pi_1(Y_1)]$ denotes the commutator subgroup of $\pi_1(Y_1)$, because the following are short exact sequences:

$$\begin{array}{ccccccc} 0 \to & \pi_1(\partial Y_1) & \to & \pi_1(Y_1) & \to & \pi_1(Y_1, \partial Y_1) & \to 0, \\ & \wr\Vert & & & & & \\ & \mathbf{Z} & & & & & \end{array}$$

$$\begin{array}{ccccccc} 0 \to & [\pi_1(Y_1), \pi_1(Y_1)] & \to & \pi_1(Y_1) & \to & H_1(Y_1) & \to 0. \\ & \wr\Vert & & & & \wr\Vert & \\ & \mathbf{Z} * \mathbf{Z} & & & & \mathbf{Z} & \end{array}$$

Now since $\pi_2(Y) = 0$ by asphericity of knots (8), we have from the homotopy exact sequence of the pair $(Y, \partial Y)$ that ∂Y^* is the covering space of ∂Y corresponding to $\pi_2(Y, \partial Y)$.

If $\tilde{X}$ denotes the universal cover of X, then $\tilde{X} = (S^k \times \tilde{Y}) \cup (D^{k+1} \times \partial\tilde{Y})$ identified along $S^k \times \partial\tilde{Y} = \partial D^{k+1} \times \partial\tilde{Y}$. $\tilde{Y}$ is contractible since $Y \simeq S^1 \vee S^1$, and $\partial\tilde{Y}$ is 1-dimensional. Consider now the Mayer–Vietoris sequence for $\tilde{X}$:

Case 1. $k = 1$.

$$\begin{array}{ccccccc} \to H_2(S^1 \times \tilde{Y}) & \oplus & H_2(D^2 \times \partial\tilde{Y}) & \to & H_2(\tilde{X}) & \to & H_1(S^1 \times \partial\tilde{Y}) \\ \Vert & & \Vert & & & & \wr\Vert \\ 0 & & 0 & & & & H_0(\partial\tilde{Y}) \oplus H_1(\partial\tilde{Y}) \\ \to H_1(S^1 \times \tilde{Y}) & \oplus & H_1(D^2 \times \partial\tilde{Y}) & \to & H_1(\tilde{X}). & & \\ \wr\Vert & & \wr\Vert & & \Vert & & \\ \mathbf{Z} & & H_1(\partial\tilde{Y}) & & 0 & & \end{array}$$

The above sequence reduces to

$$0 \to H_2(\tilde{X}) \to H_0(\partial\tilde{Y}) \to \mathbf{Z} \to 0$$

and since $H_0(\partial\tilde{Y})$ is nfg free Abelian, then $\pi_2(X) \cong H_2(\tilde{X})$ is nfg free Abelian.

Case 2. $k > 1$. A similar analysis of the Mayer–Vietoris sequence gives us $H_i(\tilde{X}) = 0$, $1 \leqslant i \leqslant k$ and $0 \to H_{k+1}(\tilde{X}) \to H_0(\partial\tilde{Y}) \to \mathbf{Z} \to 0$ exact. This completes the proof of Theorem 2.

Note that $\pi_{k+1}(X) \ncong \pi_{k+1}(C_{k+1,2}) = 0$ so L_2^{k+1} is non-trivial.

It is clear that the process of Fig. 1 of boring a straight hole in the trefoil complement destroyed one of the crossovers in the trefoil. Hence, one could start with any knot, and by boring a number of straight holes, could destroy all the crossovers. We therefore have the following corollary to the proof of Theorem 2:

COROLLARY 3. *Let K_1^1 denote a knotted ball pair, and $L_1^{k+1} \subset S^{k+3}$ denote the knot obtained by k-spinning K_1^1. Then there exists an integer $\mu_0 \geqslant 1$ such that if $\mu \geqslant \mu_0 + 1$ then L_1^{k+1} can be extended to L_μ^{k+1} where all the components of L_μ^{k+1} (except S_1^{k+1}) are unknotted, and such that $\pi_i(X) \cong \pi_i(C_{k+1,\mu})$, $1 \leqslant i \leqslant k$ and $\pi_{k+1}(X)$ is* nfg *free Abelian. Moreover, the link $L_{\mu-1}^{k+1}$ obtained by deleting S_1^{k+1} is the trivial link.*

μ_0 has been called the 'Gordian' number (**9, 10**) of a classical knot; it is the minimum number of crossovers in the regular projection of the knot which must be changed to produce the unknot.

REFERENCES

(1) ANDREWS, J. J. and CURTIS, M. L. Knotted 2-spheres in the 4-sphere. *Ann. of Math.* **70** (1959), 565–571.

(2) ANDREWS, J. J. and SUMNERS, D. W. On higher-dimensional fibred knots. *Trans Amer. Math. Soc.* **153** (1971), 415–426.

(3) ARTIN, E. Zur Isotopie Zweidimensionaler Flachen im R_4. *Abh. Math. Sem. Univ. Hamburg* **4** (1925), 174–177.

(4) BING, R. H. Mapping a 3-sphere onto a homotopy 3-sphere. *Topology Seminar Wisconsin*, 1965 (*Ann. of Math. Study* no. **60**), 89–99.

(5) EPSTEIN, D. B. A. Linking spheres. *Proc. Cambridge Philos. Soc.* **56** (1960), 215–219.

(6) GUTIERREZ, M. Unlinking spheres in codimension two. Preprint.

(7) LEVINE, J. Polynomial invariants of knots of codimension two. *Ann. of Math.* **84** (1966), 537–544.

(8) PAPKYRIAKOPOULOS, C. D. On Dehn's lemma and the asphericity of knots. *Ann. of Math*, **66** (1957), 1–26.

(9) SMYTHE, N. Trivial knots with arbitrary projection. *J. Austral. Math. Soc.* **7** (1967), 481–489.

(10) WENDT, H. Die gordische Auflösung von Knoten. *Math. Zeit.* **42** (1937), 680–696.