APPROXIMATION TO TRANSIENTS BY MEANS OF LAGUERRE SERIES

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ABSTRACT. Ward (1) has discussed a method, introduced by Tricomi (6), of calculating transients by means of series involving Laguerre functions which in some cases makes it unnecessary to determine poles of the relevant characteristic function. This method is here investigated with special reference to conditions for convergence and adjustments for improving convergence; some of the examples discussed by Ward are reconsidered.

Both in Ward's paper and here the location of poles of the characteristic function is assumed to be approximately known. In some cases the determination of poles of outstandingly small or large modulus and their separation from the remainder may be the most satisfactory procedure. Lin's method (2) for determining a quadratic factor of a polynomial is more widely applicable than has previously been supposed, and this is discussed in bare outline, without proof, here, but in detail, with adequate numerical examples, elsewhere (3).

1. *Introduction.* Instead of determining the motion of some element of a dynamical system by building up the Heaviside or Laplace operational equivalent of the motion and expressing this in partial fractions, Ward (1) transformed his operational variable so that the operational equivalent became an infinite series, each term of which was equivalent to a Laguerre function of some multiple of the time. The series involved was always convergent provided that the motion was stable, but convergence was sometimes slow. Here we first show, in § 2, that rapidity of convergence is related to the geometry of the 'operational world'. In § 3 geometrical considerations are applied to determining the most appropriate scale factors for the Laguerre series appropriate to some of the examples discussed by Ward. In some cases series more rapidly convergent than those used by Ward are obtainable by systematic methods; in other difficult cases Ward has obtained by inspection a scale factor almost as good as any systematically obtainable. In Ward's last example difficulty was experienced because the poles of the characteristic function were very widely separated. We suggest that in some such cases widely separated poles should be determined first and their contribution calculated in exponential form, while for the remainder of the poles a reasonably rapidly convergent Laguerre series could be found. Up to this point we have assumed, like Ward, that the location of poles of the characteristic function was known. In § 4 we consider in outline procedure for finding these poles, that is to say, normally, for solving algebraic equations. Lin's method (2) for determining a quadratic factor of a polynomial can be extended so that any pole of interest (not necessarily one having large or small modulus) can often be found easily. Full details of this work will be published elsewhere (3).

2. *A Laguerre-function series for $e^{-\alpha t}$. Conditions of rapid convergence.* Suppose we wish to express $e^{-\alpha t}$ for any α, real or complex, as a series of Laguerre functions

$$\sum_{r=0}^{\infty} a_r e^{-\beta t} L_r(2\beta t) = \sum_{r=0}^{\infty} a_r \lambda_r(2\beta t), \tag{1}$$

where β is any real positive number, and $L_n(x)$ means the Laguerre polynomial

$$L_n(x) = 1 - \binom{n}{1}\frac{x}{1!} + \binom{n}{2}\frac{x^2}{2!} - \dots + (-1)^n \frac{x^n}{n!}. \tag{2}$$

In (2) $\binom{n}{r}$ is to be interpreted as the binomial coefficient $n!/\{(n-r)!\,r!\}$. Then the operational equivalent of $e^{-\alpha t}$, starting suddenly when $t = 0$, is

$$\begin{aligned}\frac{p}{p+\alpha} &= \frac{p}{p+\beta}\frac{p+\beta}{p+\alpha} \\ &= \frac{p}{p+\beta}\left[\frac{\alpha+\beta}{2\beta} - \frac{\alpha-\beta}{2\beta}x\right]^{-1},\end{aligned}$$

where $x = (p-\beta)/(p+\beta)$. It follows that $p = \beta(1+x)/(1-x)$ and

$$\frac{p}{p+\alpha} = \frac{2\beta}{\alpha+\beta}\frac{p}{p+\beta}\sum_{r=0}^{\infty}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^r x^r, \tag{3}$$

so that, returning to the time world,

$$e^{-\alpha t} = \frac{2\beta}{\alpha+\beta}\sum_{r=0}^{\infty}\left(\frac{\alpha-\beta}{\alpha+\beta}\right)^r e^{-\beta t} L_r(2\beta t), \tag{4}$$

where the Laguerre-function terms, like the original exponential, are assumed to start at $t = 0$. Equations (3) and (4) are valid provided that the geometric series involved in the former is convergent, and the condition for this is

$$X = \left|\frac{\alpha-\beta}{\alpha+\beta}\right| < 1, \tag{5}$$

which implies that α has a positive real part and that $e^{-\alpha t}$ is associated with exponential decay (with or without oscillation) and not with growth.

If now β is fixed, the locus of the point α so that X remains constant is a circle of the coaxal system having $(\beta, 0)$ and $(-\beta, 0)$ as limiting points. These circles do not intersect, and the small circles of the system are those for which X is small. Their centres are all on the real axis Ox, and every circle of the system cuts at right angles any circle through the limiting points $(\pm\beta, 0)$. For fixed β, there is one circle of the system passing through the point $\alpha_0 = \gamma_0 + i\omega_0$, represented on Fig. 1 by A_0 or (γ_0, ω_0), and the value of X at all points on the circumference of this circle is

$$\left\{\frac{(\gamma_0-\beta)^2+\omega_0^2}{(\gamma_0+\beta)^2+\omega_0^2}\right\}^{\frac{1}{2}}, \tag{6}$$

while the value of X is less at all interior points. If now we have only a single exponential term (or single conjugate pair of exponential terms) to consider, we are at liberty to vary β so as to make (6) as small as possible, and it can be shown that under these conditions X has its least value

$$\omega_0/\{\gamma_0 + (\gamma_0^2+\omega_0^2)^{\frac{1}{2}}\} \tag{7}$$

when $\beta = (\gamma_0^2 + \omega_0^2)^{\frac{1}{2}}$, so that A_0 is on the circle having the limiting points $(\pm\beta, 0)$ at the ends of a diameter. If, however, we have several exponential terms we are no longer at liberty to choose β. There will be a smallest circle C with its centre on the real axis which has all the points like A_0 associated with the exponential terms in the time world (or poles of the characteristic function in the operational world) either on its circumference or inside it; at least two of the points A_0 will be on the circumference of C. If this circle has centre $(g, 0)$ and radius r, the system of coaxal circles whose equations are

$$x^2 + y^2 - 2(g+k)x + g^2 - r^2 = 0, \tag{8}$$

where k varies, will include C as a member and be such that the convergence associated with all the exponential terms is not slower than that associated with points on the circumference of C. The limiting points of the system (8) are $(\pm(g^2 - r^2)^{\frac{1}{2}}, 0)$ and the value of the constant ratio X for the circle C is then

$$r/\{g + (g^2 - r^2)^{\frac{1}{2}}\} = \{g - (g^2 - r^2)^{\frac{1}{2}}\}/r, \tag{9}$$

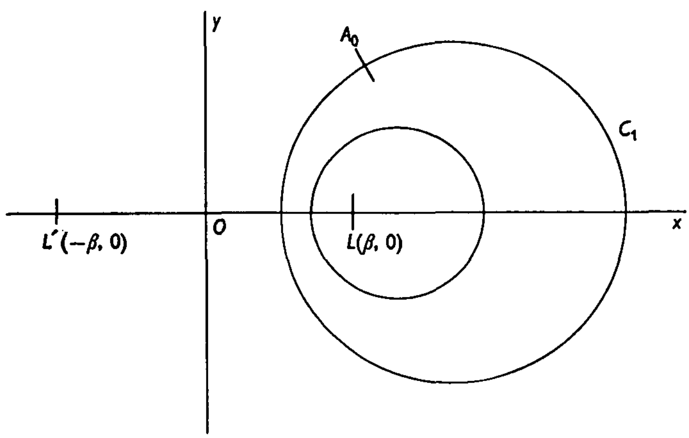

Fig. 1. Coaxal circles.

which is a function only of the ratio g/r. X is nearly unity when $(g/r) - 1$, which must be positive, is small and X is small when (g/r) is large.

In cases where (9) gives too high a value of X, we can improve convergence as follows: If $e^{-\alpha_r t}$ is a typical term in the transient response found by conventional methods, rewrite it as

$$e^{Gt}\, e^{-(\alpha_r + G)t} \quad (G > 0), \tag{10}$$

keeping the e^{Gt} as a factor outside, and applying the above to $e^{-(\alpha_r+G)t}$. In effect we have to replace g by $(g+G)$ in (9) so that X is reduced. The effective exponential multiplier of the Laguerre polynomial in (4) now becomes e^{-Kt}, where, allowing for the factor e^{Gt} in (10),

$$K = \{(G+g)^2 - r^2\}^{\frac{1}{2}} - G. \tag{11}$$

This increases as G increases (for fixed g, r) from $(g^2 - r^2)^{\frac{1}{2}}$ when $G = 0$ to g when $G + g \gg r$. For large G it increases slowly; for $G = r$ we have, expanding (11) in powers of (r/g),

$$K_r = g - (r^2/2g) + (r^3/2g^2) - \dots, \tag{12}$$

and the appropriate value of X is found to be less than $(r/2g)^{\frac{1}{2}}$ so that there is probably little advantage in using values of G above r.

So far we have assumed that the exponential terms have coefficients of the same order of magnitude. These coefficients are associated with the residues at poles of the characteristic function in the operational world. We may in some cases find that a pole associated with an unusually small residue can be safely omitted from consideration in determining the circle C so that a more favourable value of (g/r) is available for the poles with high residues.

3. *Application to some of the examples discussed by Ward.* In Ward's first example, the operational equivalent of the characteristic function is*

$$F_1[p] = p/\{(p+1)(p+2)\}. \tag{13}$$

Here the poles are such that in our notation the circle C is that on the points $(1, 0)$ and $(2, 0)$ as diameter. It follows that $g = 1{\cdot}5$ and $r = 0{\cdot}5$ so that β, which is Ward's scale factor b, is $\sqrt{2}$, and the ratio X is $3-2\sqrt{2} = 0{\cdot}171572$. Ward, by taking his scale factor as unity, in effect reduced his series for the factor $(p+1)$ to a single term and then had a series for the factor $(p+2)$ in which the ratio X associated with convergence was $\frac{1}{3}$, so that in this case he has the advantage of simplicity. We, on the other hand, have infinite series associated with both factors, but a much more rapid convergence, and in more complicated cases rapidity of convergence will be what matters most.

If in (13) we make the substitution

$$x = \frac{p-\sqrt{2}}{p+\sqrt{2}} \quad \text{or} \quad p = \sqrt{2}(1+x)/(1-x), \tag{14}$$

so that (13) contains a factor $p/(p+\sqrt{2})$ but everything else is in terms of x, we have

$$F_1[p] = \frac{2\sqrt{2}\,p}{p+\sqrt{2}}\left\{\frac{1-x}{(4+3\sqrt{2})-(3\sqrt{2}-4)\,x^2}\right\} \tag{15}$$

$$= (6-4\sqrt{2})\,\frac{p}{p+\sqrt{2}}\,(1-x)\,(1+kx^2+k^2x^4+\dots), \tag{16}$$

where $$k = (3\sqrt{2}-4)/(3\sqrt{2}+4) = 17-12\sqrt{2} = 0{\cdot}0294372. \tag{17}$$

The series (16) is particularly simple because the denominator of (15) happens not to contain any term linear in x, but in the general case the required series in ascending powers of x could have been obtained by dividing the denominator of (15) into the numerator and cancelling the lowest power of x left at each stage of the division. Table 1 below gives a comparison between true and Laguerre-approximation values obtained from the first four terms of (16) only. The good agreement for small values of t is mainly due to the fact that the $(2r-1)$th term of (16) and the $2r$th have equal and opposite coefficients while the Laguerre functions $\lambda_{2r-2}(2\sqrt{2}t)$ and $\lambda_{2r-1}(2\sqrt{2}t)$ associated are nearly equal. The values of t were chosen to be as nearly as possible comparable with Ward's, but $(2\sqrt{2}t)$ was chosen to have tabular arguments in the Laguerre-

* As I prefer the Heaviside to the Laplace notation, I have replaced Ward's s by p throughout, and added the appropriate factor p.

function tables used, giving four decimal places, or two significant figures when the tabular entry was numerically less than 10^{-3}.

For Ward's second example, the transient in our notation would be represented operationally by

$$F_2[p] = \frac{p(p+0\cdot8)}{p^2+0\cdot8p+1}. \tag{18}$$

Here there is only one pair of complex conjugate poles $p = -0\cdot4 \pm 0\cdot91652i$ of $F_2(p)$, and the modulus of each pole is unity, so that, according to (7), the best value of the scale factor is unity and the value of the ratio X is then 0·65466. In this case, therefore, Ward has used the best available scale factor, but the convergence is rather slow, unless we introduce a multiplying factor as in (10). This in effect means that we have to replace (18) by

$$f_2[p] = \frac{p(p+G+0\cdot8)}{(p+G)^2+0\cdot8(p+G)+1}, \tag{19}$$

Table 1. *Transient represented by* (13)

t (to 2 places)	$2t\sqrt{2}$	True values	Laguerre series (16) (4 terms)	Discrepancy $10^{-5}\times$
0	0	0	0	0
0·07	0·2	0·06358	0·06358	0
0·14	0·4	0·11451	0·11445	−6
0·21	0·6	0·15461	0·15460	−1
0·28	0·8	0·18568	0·18565	−3
0·49	1·4	0·23799	0·23803	+4
0·64	1·8	0·24914	0·24923	+9
0·71	2·0	0·24995	0·25001	+6
0·85	2·4	0·24483	0·24485	+2
0·99	2·8	0·23352	0·23350	−2
1·59	4·5	0·16222	0·16215	−7
1·94	5·5	0·12259	0·12257	+2
2·47	7·0	0·07709	0·07717	+8
3·18	9·0	0·03978	0·03986	+8
3·89	11·0	0·02005	0·02006	+1

so that the poles are now $p = -(0\cdot4+G) \pm 0\cdot91652i$ and their modulus is

$$\rho \equiv \{G^2+0\cdot8G+1\}^{\frac{1}{2}}$$

instead of 1. The best scale factor according to (7) is now also ρ and the value of X is, from (7),

$$(0\cdot84)^{\frac{1}{2}}/\{(G+0\cdot4)+\rho\}. \tag{20}$$

A convenient value of G is 1·6, since ρ is then 2·2, and the value of X is then 0·21822, one-third of the value already obtained for $G = 0$. Since $(0\cdot21822)^2$ is approximately equal to $(0\cdot65466)^7$, we should expect to obtain roughly the same degree of accuracy with three terms using the multiplier $e^{+1\cdot6t}$ as Ward did with seven terms and no multiplier. Theoretically there is no reason why a larger multiplier should not be used if desired; there are, however, two practical difficulties about using short Laguerre-function series for large values of the argument, namely (a) the Laguerre functions are then sparsely tabulated, (*b*) the ratio $|\lambda_{n+1}(x)/\lambda_n(x)|$ is large for any large value of x until n approaches the value N at which $|\lambda_n(x)|$ reaches its first maximum for increasing n and fixed x. The critical values of x at which N changes are very approximately as follows:

Critical x	2	5·3	9	12·5	16	20	24	27	31	35	40
Change in N	0–1	1–2	2–3	3–4	4–5	5–6	6–7	7–8	8–9	9–10	10–11

As the Laguerre-function series obtained has to be multiplied by e^{Gt}, which is large for large t, we find that it is necessary to evaluate it to more decimal places for large values of t than for small ones. Thus in the case under discussion, the true solution of (18) is

$$\phi_2(t) = 1{\cdot}09109\, e^{-0{\cdot}4t} \cos\{0{\cdot}915615t - \sin^{-1} 0{\cdot}4\}. \qquad (21)$$

If in (19) we put $G = 1{\cdot}6$ and $x = (p-2{\cdot}2)/(p+2{\cdot}2)$ we can write

$$f_2[p] = \frac{4{\cdot}4p}{p+2{\cdot}2}\,\frac{4{\cdot}6-0{\cdot}2x}{18{\cdot}48+0{\cdot}88x^2}$$

$$= 1{\cdot}09524\frac{p}{p+2{\cdot}2}[1-0{\cdot}043478x]\,[1+0{\cdot}047619x^2]^{-1}, \qquad (22)$$

so that the Laguerre series required is

$$\phi_2^*(t) = 1{\cdot}09524\, e^{1{\cdot}6t}[\lambda_0(4{\cdot}4t) - 0{\cdot}04378\lambda_1(4{\cdot}4t) - 0{\cdot}047619\lambda_2(4{\cdot}4t) + 0{\cdot}0020738\lambda_3(4{\cdot}4t) + 0{\cdot}00226757\lambda_4(4{\cdot}4t) + \dots]. \qquad (23)$$

Expressions (21) and (22) are compared for various values of t in Table 2. For small values of t up to say 3 sec, the 3-term approximation (column G of Table 2) is quite adequate, but for larger values of t it rapidly degenerates. If the approximation is

Table 2

$A = 1{\cdot}09109\, e^{-0{\cdot}4t}$
$B = \cos\{0{\cdot}91652t - 0{\cdot}41151\}$
$C = \lambda_0(4{\cdot}4t) - 0{\cdot}043478\lambda_1(4{\cdot}4t) - 0{\cdot}047619\lambda_2(4{\cdot}4t)$
$D = e^{1{\cdot}6t}$

$E = 1{\cdot}09524D$
$F = 10^{-3}\{2{\cdot}07038\,\lambda_3(4{\cdot}4t) + 2{\cdot}26757\,\lambda_4(4{\cdot}4t)\}$
$G = EC$
$H = EF$
$I = E(C+F) = G+H$

t	$4{\cdot}4t$	AB (true)	C	$F \times 10^3$	3-term Approx. G	H	5-term Approx. I	$10^4 \times$ Discrepancy G	$10^4 \times$ Discrepancy I
0·00	0	1	0·908903	4·3380	0·9955	0·0048	1·0003	− 45	+ 3
0·50	2·2	0·8923	0·36580	0·3719	0·8916	0·0009	0·8925	− 7	+ 2
1·02	4·5	0·6249(5)	0·110772	0·6072	0·6231(5)	0·0034(2)	0·6266	− 18	+ 16
1·59	7	0·2890	0·021540	−0·8197	0·3008	−0·0114	0·2894	+ 118	+ 4
2·05	9	0·0517	0·002532	−0·7145	0·0732	−0·0206	0·0526	+ 215	+ 9
2·50	11	−0·1221	$-1{\cdot}808 \times 10^{-3}$	−0·2163	−0·1081	−0·0129	−0·1210	+ 140	+ 11
2·73	12	−0·1812(5)	$-2{\cdot}0988 \times 10^{-3}$	−0·0039	−0·1805(5)	−0·0003(4)	−0·1809	+ 7	+ 4
2·95	13	−0·2221	$-1{\cdot}9745 \times 10^{-3}$	+0·1494	−0·2443	+0·0185	−0·2258	− 222	− 37
3·64	16	−0·2486	$-1{\cdot}0002 \times 10^{-3}$	0·2962	−0·3685	0·1091	−0·2594	−1199	−108
4·09	18	−0·2083	$-5{\cdot}363 \times 10^{-4}$	0·2505	−0·4088	0·1910	−0·2178	−2005	− 95
4·55	20	−0·1449	$-2{\cdot}652 \times 10^{-4}$	0·1783	−0·4184	0·2813	−0·1371	−2735	+ 78
5·00	22	−0·0761	$-1{\cdot}249 \times 10^{-4}$	0·1141	−0·4078	0·3725	−0·0353	−3317	+408

improved by taking two more terms of (23), as in column I of Table 2, the approximation is adequate until t is between 4·5 and 5. The values of t chosen have been those as nearly comparable to Ward's values as possible, such that $(4{\cdot}4t)$ is a tabular entry in our Laguerre tables. For values of t above about 5, the terms involving λ_5 and λ_6 will be the most important and so on. It is only by this constant increase with time in the value of n associated with the leading Laguerre function that (23) can follow the oscillation truly described by (21). The penalty we pay for introducing the multiplying factor e^{Gt} will be that we shall be obliged to take rather more terms than we might have hoped were necessary, if we insist on detailed knowledge of the motion at times such that high Laguerre-function arguments occur.

Ward's third example is automatically shown by our approach to require a tenfold increase in the scale factor, and need not be discussed further. For his fourth example, in our notation the operational equivalent of the motion would be

$$F[p] = p^3/\{p^3+31p^2+155p+125\}, \tag{24}$$

and the zeros of the denominator are $p = -1$, -5 and -25. The circle (8) in this case has the points $(1, 0)$ and $(25, 0)$ at opposite ends of a diameter; we thus find $g = 13$ and $r = 12$. If therefore we have $G = 0$, the scale factor will be $(g^2-r^2)^{\frac{1}{2}} = 5$ and the worst value of X involved, from (9), will be $\frac{2}{3}$. If we take $G = 2$, we find that the scale factor becomes 9 and X is reduced to 0·5, while if we take $G = 7$, the scale factor becomes 16 and X is reduced to $\frac{1}{3}$. We have deliberately discarded for the moment the possibility of omitting the point $(1, 0)$, since the corresponding pole happens to be associated with a small residue, because we want to avoid the necessity of determining more poles and residues associated with (24) than we can help. As Ward has only considered values of t up to 0·3, the highest Laguerre argument involved if $G = 7$ is 9·6 and the first maximum of $|\lambda_n(10)|$ occurs for $n = 3$, so we are not likely to be involved in neglecting important terms as in our study of (21) and (23) if we take $G = 7$; this we now do, so that in effect we have to find a Laguerre series of argument $32t$ for

$$f[p] = p(p+7)^2/\{(p+7)^3+31(p+7)^2+155(p+7)+125\}. \tag{25}$$

By putting $x = \dfrac{p-16}{p+16}$ so that $p = \dfrac{16(1+x)}{1-x}$, we find

$$f[p] = \frac{32p}{p+16}\left[\frac{529+414x+81x^2}{32256+4608x-3584x^2-512x^3}\right]$$

$$= \frac{p}{p+16}0{\cdot}524802[1+0{\cdot}639752x+0{\cdot}172837x^2+0{\cdot}062265x^3+0{\cdot}020464x^4 +0{\cdot}006738x^5+0{\cdot}002299x^6+\ldots], \tag{26}$$

so that the required time function is

$$\phi^*(t) = e^{7t}[0{\cdot}524802\lambda_0(32t)+0{\cdot}335743\lambda_1(32t)+0{\cdot}090705\lambda_2(32t) +0{\cdot}032677\lambda_3(32t)+0{\cdot}010740\lambda_4(32t)+0{\cdot}003536\lambda_5(32t) +0{\cdot}001207\lambda_6(32t)+\ldots]. \tag{27}$$

This is compared with the true solution

$$\phi(t) = 0{\cdot}010417e^{-t}-0{\cdot}312500\,e^{-5t}+1{\cdot}302083\,e^{-25t} \tag{28}$$

in Table 3. It thus appears that the discrepancies using six terms of (27) are slightly greater than those using seven terms of Ward's series (in which G was zero and the scale factor was 10). Our approach would have suggested a scale factor of 5 for zero G; the reason why Ward's choice of 10 as scale factor in this case is better is that the residue associated with the pole $p = -1$ of (24) happens to be small. As already explained, we have deliberately refrained from taking any advantage of this fact. With a different numerator in (24), there might be a marked advantage in using an approximation of the form (27); the convergence ratio X associated with the point $(1, 0)$ when the scale factor is 10 and G is zero is $\frac{9}{11}$, whereas in our approximation it would be $\frac{1}{3}$.

Table 3

t	$32t$	True	Approximation (Laguerre series × $0{\cdot}524802\,e^{7t}$)			Discrepancy × 10^5		
			$n = 4$	$n = 5$	$n = 6$	$n = 4$	$n = 5$	$n = 6$
0·000	0	1	0·98393	0·99467	0·99820	−1607	−533	−180
0·009	0·3	0·74216	0·74234	0·74285	0·74255	+ 18	+ 69	+ 39
0·019	0·6	0·54050	0·54697	0·54281	0·54121	647	231	71
0·025	0·8	0·43135	0·43868	0·43350	0·43188	733	215	53
0·038	1·2	0·26087	0·26556	0·26137	0·26070	469	50	− 17
0·050	1·6	0·13959	0·13955	0·13834	0·13891	− 4	−125	− 68
0·075	2·4	−0·00543	−0·01235	−0·00772	−0·00591	− 692	−229	− 58
0·094	3·0	−0·06111	−0·06860	−0·06225	−0·06096	− 749	−114	+ 15
0·125	4·0	−0·10087	−0·10247	−0·09898	−0·09998	− 160	+189	+ 89
0·156	5·0	−0·10797	−0·10128	−0·10468	−0·10744	+ 669	+329	+ 53
0·188	6·0	−0·10175	−0·08989	−0·09982	−0·10231	1186	+193	− 56
0·250	8·0	−0·07891	−0·07236	−0·08330	−0·08034	+ 655	−439	−143
0·313	10·0	−0·05737	−0·07072	−0·06366	−0·05636	−1335	−629	+101

For Ward's last example, the operational equivalent of the transient, in our notation, is

$$F[p] = \frac{p^6 + 42{\cdot}65620dp^5 + 363{\cdot}6445d^2p^4 + 734{\cdot}947265d^3p^3}{(p+1)(p+4)(p+16)(p+64)(p+256)(p+1024)}, \tag{29}$$

where $d = 32$. Here the poles are all real, but are very widely separated. If we do not use a multiplier e^{Gt} as in previous examples, we find that the system of circles (8) must include the circle having $(1, 0)$ and $(1024, 0)$ at opposite ends of a diameter; this gives $g = 512{\cdot}5$, $r = 511{\cdot}5$ and $(g^2 - r^2)^{\frac{1}{2}}$, the scale factor, is 32. The value of X is 0·9393. Now it so happens that the circle having $(4, 0)$ and $(256, 0)$ at the ends of a diameter, and the circle having $(16, 0)$ and $(64, 0)$ at the ends of a diameter, belong to the same coaxal system associated with scale factor 32; the values of X are respectively $\frac{7}{9}$ and $\frac{1}{3}$. With the scale factor 4 chosen by Ward, the corresponding covergence ratios for the six poles are 0·6, 0 (i.e. a single term), 0·6, 0·882, 0·969 and 0·992. As the poles associated with high convergence ratios in (29) happen to have small residues, the Laguerre-series approximation with scale factor 32 will not be markedly better than Ward's, and as the poles of (29) are so widely separated, r is inevitably large and this makes it impossible to gain much from using the multiplier e^{Gt}. On the other hand, when poles are widely separated, it is relatively easy to find those of largest or smallest modulus, and their associated residues. We therefore suggest that for (29) the first step is to find the poles $p = -1$, $p = -1024$ and their associated residues. The contribution from these two poles is then determined in the time world as

$$0{\cdot}032062\, e^{-t} - 0{\cdot}000455\, e^{-1024t}. \tag{30}$$

If we now obtain a Laguerre-function approximation for the remainder of the transient response in the time world, we shall have the scale factor 32 obtained from the poles (4, 0) and (256, 0); the Laguerre series obtained from

$$f[p] = F[p] - \frac{0{\cdot}032062p}{p+1} + \frac{0{\cdot}000455p}{p+1024} \tag{31}$$

will in fact be found to converge with ratio 0·777 instead of 0·939 for (29). As 0·777 is still rather high and the poles of $f[p]$ are still widely separated, it is probably in this sort of case better to find the residues associated with $p = -4$ and $p = -256$ and express the contribution from these poles exponentially, especially as the poles involved are real. In general, we suggest that in complicated cases it will be advantageous to find the isolated poles of relatively very large or small modulus (especially when these poles are real) and their associated residues. Having expressed the contribution from these poles exponentially, we form a remainder equation analogous to (31) to which the isolated poles do not contribute. We then obtain a Laguerre approximation for the time equivalent of this remainder equation. Consider for examples the case

$$F[p] = \frac{p^5 + 4p^4 + 36{\cdot}25p^3 + 34{\cdot}5p^2 + 20{\cdot}25p}{(p+\frac{1}{2})^2 (p^2+p+1)(p+16)} \tag{32}$$

$$= \frac{p}{(p+\frac{1}{2})^2} + \frac{p}{p^2+p+1} + \frac{p}{p+16}. \tag{33}$$

If we first locate the pole at $p = -16$ and determine its contribution exponentially, the appropriate value of the scale factor is 1 (from (7)), with $\gamma_0 = \frac{1}{2}$, $\omega_0 = \frac{1}{2}\sqrt{3}$; the point $(\frac{1}{2}, 0)$ cannot but be inside any circle through the points $(\frac{1}{2}, \pm\frac{1}{2}\sqrt{3})$ and the fact that the associated pole is double is irrelevant; and the corresponding convergence ratio is $1/\sqrt{3}$ for the points $(\frac{1}{2}, \pm\frac{1}{2}\sqrt{3})$ and $\frac{1}{3}$ for the point $(\frac{1}{2}, 0)$. The fact that the pole at $p = -\frac{1}{2}$ is double merely means that its contribution to the Laguerre series is associated with the expansion of $(1+x)^{-2}$ instead of $(1+x)^{-1}$; for a multiple pole this might mean that an extra term or two of the series was required, but it would not affect the rate of convergence in the long run.

4. *Location of poles of the characteristic function.* In Ward's paper the location of poles of the characteristic function in the operational world is given for each of his examples, together with the associated residues, and hitherto we have also regarded these poles and residues as known, at any rate approximately. If the poles are not widely separated, Ward's suggestion of taking the scale factor as numerically equal to the geometric mean of the roots will often give a satisfactorily convergent Laguerre series without the necessity of locating the poles more accurately. For the equation

$$f[p] \equiv p^n + a_{n-1}p^{n-1} + \ldots + a_1 p + a_0 = 0 \tag{34}$$

the geometric mean of the roots is $-(a_0)^{1/n}$. But there remain cases like (29) in which convergence is inevitably slow unless the position of at any rate *some* poles, and their contribution to the total transient, can be determined. Our discussion is therefore incomplete without some reference to the problem of finding roots of (34). Among many possible and well-known methods of doing this, we wish in particular to call attention to that due to Lin (2). Hitherto uncertainty about the convergence of Lin's process has hindered its application, but in fact even if Lin's process is divergent it is often possible to determine with reasonable facility the root of the equation from which it diverges. A full discussion of this matter requires considerable space and adequate numerical examples, and is therefore published elsewhere (3), but it seems appropriate to include here the following brief outline, without proof.

If (34) has a real root $-\alpha$ to which an approximation $-\alpha_0$ is known, and all its roots are well separated, we divide the left-hand side of (34) by $p+\alpha_0$ stopping at the linear term. Let the remainder at this stage be $\lambda_0 p + a_0$. Then the next divisor is $p + a_0/\lambda_0$ or $(p+\alpha_1)$, and the process is repeated to give a third divisor $p+\alpha_2$, etc. Now it is shown in (2) that the successive divisors $\alpha_0, \alpha_1, \alpha_2$, etc., form a sequence whose differences form a geometric progression, provided that the difference between α_0 and the root being sought is sufficiently small. We now assume that $(\alpha_1-\alpha_0)$ and $(\alpha_2-\alpha_1)$ are in fact the first two terms of a geometric progression, and deduce that the root being sought is β, where

$$\beta = \frac{\alpha_1^2-\alpha_0\alpha_2}{2\alpha_1-\alpha_0-\alpha_2}, \tag{35}$$

whatever the relative values of $|\alpha_1-\alpha_0|$ and $|\alpha_2-\alpha_1|$ may be. As a check we now start with the divisor $(p+\beta)$ and obtain the next two divisors $(p+\beta_1)$, $(p+\beta_2)$ as before and we can then use (35) again with β for α_0, β_1 for α_1 and β_2 for α_2. Convergence is usually rapid. If we are seeking a complex root $\gamma+i\omega$ of (34) to which an approximation $\gamma_0+i\omega_0$ is known, in practice in the form of a real quadratic divisor

$$P_0 \equiv p^2+2\gamma_0 p+[\gamma_0^2+\omega_0^2],$$

we divide the left-hand side of (34) by P_0 stopping at the quadratic stage, so that there is a remainder $\lambda_0 p^2+\mu_0 p+a_0$, and divide this remainder by $p+\gamma_0+i\omega_0$ to obtain the next approximation $p+\gamma_1+i\omega_1$ to the divisor as in the real case. We now repeat the process with divisor $P_1 = p^2+2\gamma_1 p+\gamma_1^2+\omega_1^2$ and our next divisor is $p+\gamma_2+i\omega_2$. Applying (35) with α_r replaced by $\gamma_r+i\omega_r$ for $r = 0, 1, 2$, we determine a complex quantity β; if $\bar{\beta}$ is the conjugate of β, then our next divisor is $p^2+(\beta+\bar{\beta})p+\beta\bar{\beta}$ and the process is repeated as in the case of a real root. To obtain a starting factor like P_0 or $(p+\alpha_0)$, the following devices are available:

(*a*) If we seek the root of smallest modulus for (34), try the divisor

$$p^2+(a_1\,|\,a_2)\,p+(a_0\,|\,a_2),$$

or one of its factors if these are real.

(*b*) If we seek the root of largest modulus, put $x = 1/y$, clear of fractions, and apply (*a*) to the y-equation.

(*c*) Seek obvious real roots first by examining the behaviour of (34) for values of p which are simple negative numbers like $-n10^k$ (n an integer less than 10 and k an integer). If the left-hand side of (34) changes sign between $p = -\delta$ and $p = -\Delta$ try the divisor $p + D$ $(\delta < D < \Delta)$.

(*d*) Write (34) in the form

$$(p^2+\lambda_1)(p^2+\lambda_2)\dots(p^2+\lambda_r)+kp(p^2+\mu_1)(p^2+\mu_2)\dots(p^2+\mu_s) = 0. \qquad (36)$$

If (34) is associated with a stable system, the λ's and μ's are all real and positive, and separate each other (4), and they are obtained from equations whose degree in p^2 is not more than half the degree of the original equation. For small k, a useful starting divisor is

$$p^2+\lambda_t, \quad \text{or better} \quad p^2+\lambda_t+kp\frac{(\mu_1-\lambda_t)(\mu_2-\lambda_t)\dots(\mu_r-\lambda_t)}{(\lambda_1-\lambda_t)(\lambda_2-\lambda_t)\dots(\lambda_r-\lambda_t)}, \qquad (37)$$

where t may have any value from 1 to r, and the factor $(\lambda_t-\lambda_t)$ is omitted from the denominator of (37). For large k the corresponding useful starting divisor is

$$p^2+\mu_t, \quad \text{or better} \quad p^2+\mu_t-\frac{p}{k}\frac{(\lambda_1-\mu_t)(\lambda_2-\mu_t)\dots(\lambda_r-\mu_t)}{\mu_t(\mu_1-\mu_t)(\mu_2-\mu_t)\dots(\mu_s-\mu_t)}, \qquad (38)$$

where again the factor $(\mu_t-\mu_t)$ is omitted from the denominator of (38) and t takes any values from 1 to s. If k has intermediate values, some of (37) or (38) may still be useful starting divisors.

(*e*) Finally, if equal roots or clustered roots are suspected, carry out the H.C.F. process for $f(p)$ given by (34), and its derivative. The presence of equal or clustered roots will be indicated by the presence of abnormally small coefficients throughout the remainders $f_2(p), f_3(p), \dots, f_n(p)$ after a certain stage, say from $f_m(p)$ onwards. In this case $f_{m-1}(p)$ has factors which are or are nearly multiple factors of $f(p)$. It is probably necessary to transform the equation in the manner suggested by Olver ((5), § 9) before Lin's or any other iterative process can usefully be applied to such equations.

(*f*) At any stage we may reduce the degree of $f(p)$ by dividing out by the factors already found, but in this process some accuracy may be lost so that any root found must be checked in the original equation.

The above considerations enable us to solve many of the equations associated with practical problems; if we have any *a priori* reason for expecting a root in a particular neighbourhood α_0 (because for example a similar but not identical case gave a root α_0) the trial divisor $p+\alpha_0$ (α_0 real) or $p^2+(\alpha_0+\bar{\alpha}_0)p+\alpha_0\bar{\alpha}_0$ ($\alpha_0, \bar{\alpha}_0$ conjugate) is probably better than any we have already suggested. Moreover, the case where Ward's Laguerre approximation had slow convergence was just the case where the poles of the characteristic function are well separated, so that the Lin process for determining them would work best, and in the case where multiple or clustered poles might make the Lin process most difficult, Ward's Laguerre approximation with the geometric mean of the roots of (34) as the scale factor will be likely to give a rapidly convergent series; the presence of the multiple poles may slightly increase the coefficients of the Laguerre functions involved but the series still contains nothing but a linear combination of Laguerre functions in contrast to the complication of the conventional exponential solution when terms of the form $t^r e^{-\alpha t}$ are introduced by multiple poles.

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