Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 August 2018.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

102.A (Stan Dolan)

For some values of m it is possible to find numbers which:

- have m digits;
- are divisible by m;
- have no subsequence divisible by m.

Prove that the sum of the digits of such a number is divisible by m.

[A subsequence of a number is formed by deleting some, but not all, of its digits, with leading zeros not being allowed. Examples of numbers satisfying the above properties are 252, 8000006 and 20111111111111111.]

102.B (Prithwijit De)

Evaluate the following integrals:

$$(a) \qquad \int_0^{\pi/2} \frac{dx}{(\sin^3 x + \cos^3 x)^2};$$

$$(b) \qquad \int_0^{\pi/2} \frac{x}{\sin^3 x + \cos^3 x}\, dx;$$

$$(c) \qquad \int_0^{\pi/2} \cos x \ln\left(\sin^3 x + \cos^3 x\right) dx.$$

102.C (Peter Shiu)

Let $0 < \alpha < 1$ be an irrational number. Show that there are infinitely many Pythagorean triples (a, b, c) with $a^2 + b^2 = c^2$ such that

$$0 < \frac{a}{b} - \alpha < \frac{7}{c}.$$

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102.D (Michael Fox)

This problem is about spheres with collinear centres and a common tangent line. The line ℓ passes through given points $(0, 0, 1)$ and $(1, m, 1)$ and it is the locus $(t, mt, 1)$. The centre of sphere S_0 is the origin. Its radius is 1, and it touches ℓ at the point where $t = 0$. For all natural numbers n, the centre of sphere S_n is $(c_n, 0, 0)$, its radius is r_n and it touches ℓ at $(t_n, mt_n, 1)$. Each S_n touches S_{n-1} externally, with $c_n > c_{n-1}$.

In any order, show that:

(a) if $2m^2$ is an integer, then so are all the r_n, t_n and c_n;

(b) the r_n, t_n, c_n are integer polynomials in m^2;

(c) if $m = \sinh u$, then $r_n = \cosh 2nu$.

Finally, in (c), express t_n and c_n in terms of hyperbolic functions.

Solutions and comments on **101.E**, **101.F**, **101.G**, **101.H** (July 2017).

101.E (Marcel Chiriţă)

The triangle ABC has inradius r and circumradius R. The excircle touching side BC has centre I_a and radius r_a with I_b, r_b and I_c, r_c similarly defined. Prove that

$$\frac{I_aB\cos\frac{1}{2}B}{\sqrt{r_b + r_c}} + \frac{I_bC\cos\frac{1}{2}C}{\sqrt{r_a + r_c}} + \frac{I_cA\cos\frac{1}{2}A}{\sqrt{r_b + r_a}} \leqslant \frac{3}{2}\sqrt{\frac{R(2R - r)}{r}}.$$

There was a variety of interesting approaches to this intriguing geometrical inequality: the one that follows was given by Peter Nüesch. It employs the following identities with the standard triangle notation, including s for semiperimeter and Δ for area.

(1) $r_a = I_aB\cos\frac{1}{2}B$ (A standard piece of trigonometry.)

(2) $\sum\frac{1}{s-a} = \frac{r+4R}{rs}$

(This follows from the relation $\sum ab = s^2 + r^2 + 4rR$ proved in the solution to Problem **101.A** in the November 2017 *Gazette*, p. 548. For

$$\sum\frac{1}{s-a} = \frac{1}{(s-a)(s-b)(s-c)}\sum(s-a)(s-b) = \frac{1}{r^2s}\left(\sum ab - s^2\right)$$

$$= \frac{r^2+4rR}{r^2s}, \text{ as required.})$$

(3) $\sum\frac{a^2}{r_b + r_c} = 4R - 2r$

(For $\sum\frac{a^2}{r_b+r_c} = \frac{1}{\Delta}\sum a(s-b)(s-c)$, using $r_b = \frac{\Delta}{s-b}$, etc.

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$$= \frac{(s-a)(s-b)(s-c)}{\Delta} \sum \left(\frac{s}{s-a} - 1 \right)$$

$$= r\left(\frac{r^2 + 4Rr}{r^2} - 3 \right), \text{ using (2)}$$

$$= 4R - 2r.)$$

By (1), the inequality in **101.E** is equivalent to

$$\left(\sum \frac{r_a}{\sqrt{r_b + r_c}} \right)^2 \leqslant \frac{9R(2R - r)}{4r}.$$

Cauchy-Schwarz gives

$$\left(\sum \frac{r_a}{\sqrt{r_b + r_c}} \right)^2 = \left(\sum \frac{r_a}{a} \cdot \sqrt{\frac{a^2}{r_b + r_c}} \right)^2 \leqslant \sum \frac{r_a^2}{a^2} \sum \frac{a^2}{r_b + r_c}$$

$$= (4R - 2r) \sum \frac{r_a^2}{a^2} \quad \ldots (*), \text{ by (3)}.$$

Then, using $r_a = \dfrac{\Delta}{s - a}$, we have

$$\frac{r_a}{a} = \frac{\Delta}{a(s-a)} = \frac{\Delta}{(s-a)(s-b+s-c)}$$

$$\leqslant \frac{\Delta}{2(s-a)\sqrt{(s-b)(s-c)}} = \frac{1}{2}\sqrt{\frac{s}{s-a}},$$

by the AM-GM inequality and Heron's formula.

Finally, from (*) and using (2)

$$\left(\sum \frac{r_a}{\sqrt{r_b + r_c}} \right)^2 \leqslant (4R - 2r)\frac{s}{4} \sum \frac{1}{s-a} = \frac{(2R - r)(r + 4R)}{2r}$$

$$\leqslant \frac{9R(2R - r)}{4r}, \text{ by Euler's inequality } r \leqslant \frac{R}{2}.$$

As several solvers observed, there is equality throughout if, and only if, triangle *ABC* is equilateral.

Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, J. A. Mundie, P. Nüesch, V. Schindler and the proposer Marcel Chiriţă.

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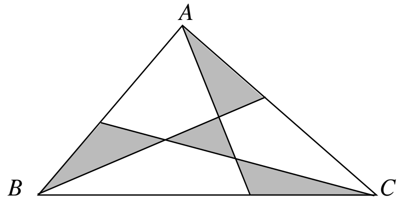
101.F (Stan Dolan)

Given any triangle ABC, show how to construct three Cevians such that the four shaded triangles in the diagram have equal areas.



Answer: The three Cevians divide each side in the ratio $\phi : 1$, where ϕ is the golden ratio.

The most popular solutions to this attractive problem assumed symmetry (with the three Cevians dividing the sides in the same ratio). Michael Fox's solution below caught my eye for the care with which he showed that this has to be the case.

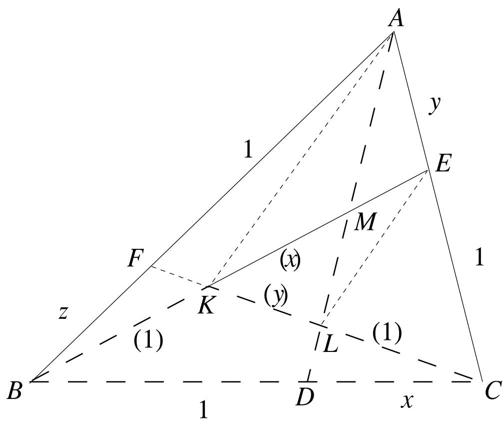


FIGURE 1

The Cevians AD, BE, CF of ΔABC meet to form ΔKLM, where K is not on AD, and so on, cyclically, as in Figure 1. Given that the areas AME, BKF, CLD and KLM are equal, we are to construct D, E and F.

Let $BD : DC = 1 : x$, $CE : EA = 1 : y$ and $AF : FB = 1 : z$, and consider Δs AME, LMK. Since AML and EMK are straight lines, the angles at M in the triangles are equal. And since the areas of the triangles are equal, it follows that $AM.ME = KM.ML$, i.e. $\dfrac{AM}{KM} = \dfrac{EM}{LM}$. Hence ΔAMK is similar to ΔEML, implying $\angle MAK = \angle MLE$, whence $AK \parallel EL$, giving $CL : LK = CE : EA = 1 : y$. Cyclic interchanges then give $AM : ML = 1 : z$, and $BK : KM = 1 : x$.

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We can link the values of x, y and z by applying Menelaus' theorem to $\triangle BKC$ and transversal DLM. We find that $\frac{BM}{MK}.\frac{KL}{LC}.\frac{CD}{DB} = -1$, i.e. $\frac{1+x}{x}.\frac{y}{1}.\frac{x}{1} = 1$. Thus $y(1+x) = 1$, so $y = \frac{1}{1+x}$. It follows similarly that $z = \frac{1}{1+y}$ and $x = \frac{1}{1+z}$. Eliminating z between the last two expressions gives $x = \frac{1+y}{2+y}$, and removing y gives $x = \frac{2+x}{3+2x}$, whence $x^2 + x - 1 = 0$. The positive root of this is equal to $\frac{1}{2}\left(\sqrt{5} - 1\right) \approx 0.618$. Cyclic interchange, which is easier than substitution, gives $x = y = z$. We can also verify that $\frac{BD}{BC} = \frac{DC}{BD}$ $(= \phi)$, i.e. $BD^2 = BC.DC$, so that D divides BC in the golden ratio.

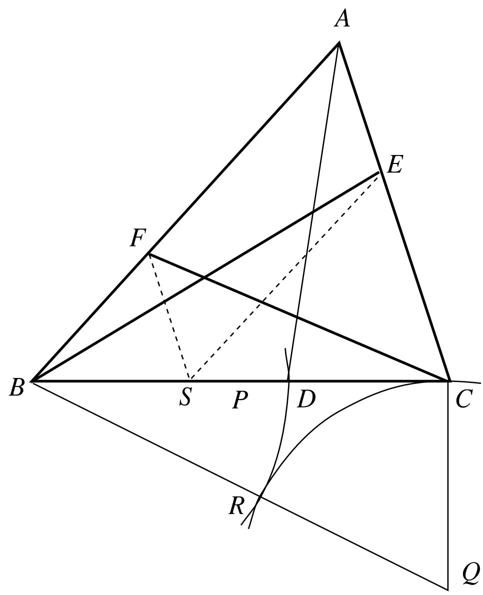

FIGURE 2

There is a simple construction shown in Figure 2.

Let P be the midpoint of BC, and CQ perpendicular to BC with $CQ = CP$. Then R, on BQ, is such that $QR = QC$; and D, on BC, is such that $BD = BR$. If now S is the reflection of D in P, then SE and SF are respectively parallel to BA and CA.

Suppose $BC = 2\ell$, then $CQ = \ell$. Let $BR = x$, then we have $(x + \ell)^2 = (2\ell)^2 + \ell^2$ so $x^2 = (2\ell)^2 - 2\ell x = 2\ell(2\ell - x)$. Thus $BD^2 = BC.DC$, as required. Clearly E and F divide the corresponding sides in the required ratios.

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The quadratic $x^2 + x - 1 = 0$ also has a negative root, for which the Cevians lie outside *ABC*, and the equal sub-triangles *AME*, *BKF*, *CLD* and *KLM* overlap. Still $BD^2 = BC.DC$, but there is an extra property: *K*, *L*, *M* are the respective midpoints of *BE*, *CF*, *AD* (see Figure 3).

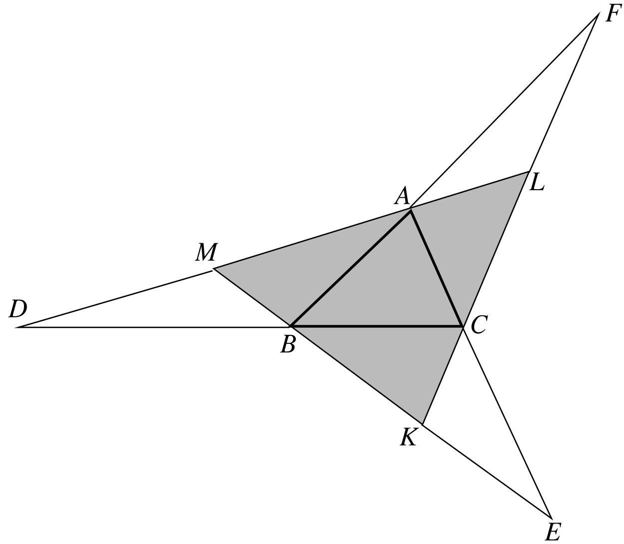

FIGURE 3

As several respondents observed, we can also generate the solution for an arbitrary triangle by an affine transformation of the equilateral case. Here, verifying that three Cevians dividing the sides in the golden ratio $\phi : 1$ satisfies the requirements of **101.F** makes a very nice addition to the collection of problems featuring the golden ratio.

Correct solutions were received from: M. G. Elliott, M. Fox, G. Howlett, T. Kecker, C. Starr (2 solutions), G. Strickland (2 solutions) and the proposer Stan Dolan.

101.G (Dao Thanh Oai)

Let *ABC* be a triangle and let Γ be a circle in the plane of *ABC* such that two circles through *B*, *C* touch Γ at A_1, A_2; two circles through *C*, *A* touch Γ at B_1, B_2 and two circles through *A*, *B* touch Γ at C_1, C_2.

Show that the three lines A_1A_2, B_1B_2, C_1C_2 are concurrent.

Proving this neat geometrical result attracted a wide range of analytical and synthetic methods. Of the latter, Li Zhou's succinct argument (based on the Figure below) follows:

Let Ω be the circumcircle of *ABC*, and *x* be the radical axis of Γ and Ω. Suppose that *x* intersects *BC*, *CA*, *AB* at *D*, *E*, *F*, respectively. Let *O* be the

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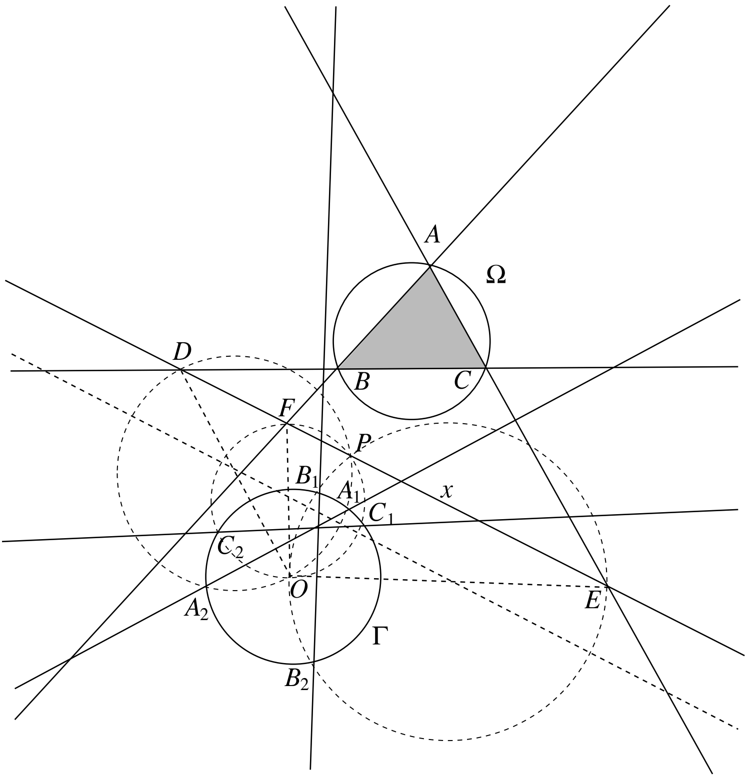
centre of Γ, and Γ_A, Γ_B, Γ_C be the dashed circles with OD, OE, OF as diameters, respectively. Then D is the radical centre of Γ, Ω, and the two circles through B, C and tangent to Γ. Thus A_1A_2 is the radical axis of Γ with Γ_A. Likewise, B_1B_2 and C_1C_2 are the radical axes of Γ with Γ_A and Γ_B respectively. Since the midpoints of OD, OE, OF lie on a line parallel to x, Γ_A, Γ_B, Γ_C concur at a second point P on x. Finally, since A_1A_2, B_1B_2, C_1C_2 are the respective images of Γ_A, Γ_B, Γ_C under the inversion in Γ, they concur at Q, the image of P under the inversion in Γ.



Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, G. Howlett, L. Zhou and the proposer Dao Thanh Oai.

101.H (Mehtaab Sawhney)

Find the smallest positive integer k such that the following inequality holds for all non-negative real numbers a, b, c.

$$(a-b)(a-2b)(ka-c)+(b-c)(b-2c)(kb-a)+(c-a)(c-2a)(kc-b) \geqslant 0.$$

Answer: The smallest positive integer is $k = 12$.

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Solvers clearly relished this unusual cyclic inequality challenge. Appropriate choices of a, b, c establish that $k \geqslant 12$. For example, $a = 7$, $b = 4$, $c = 0$ gives $k \geqslant \frac{504}{43} = 11.72$ (and the choices $a = 5$, $b = 3$, $c = 0$; $a = 9$, $b = 5$, $c = 0$ also suffice).

The main challenge is to establish that the inequality holds when $k = 12$ and solvers displayed a variety of different approaches to accomplish this, including Graham Howlett whose neat solution follows.

When $k = 12$, the left-hand side of the inequality, $f(a, b, c)$, expands to give

$$f(a, b, c) = 12\sum a^3 - 38\sum a^2b + 23\sum ab^2 + 9abc$$

where the summations are cyclic. Since a, b, c may be cyclically permuted, we may suppose $c \leqslant a, b$ in what follows.

If $c = 0$, then

$$\begin{aligned} f(a, b, 0) &= 12a^3 - 38a^2b + 23ab^2 + 12b^3 \\ &= 12a\left(a - \frac{7b}{4}\right)^2 + 4b\left(a - \frac{55b}{32}\right)^2 + \frac{47}{256}b^3 \end{aligned}$$

so that $f(a, b, 0) \geqslant 0$ for non-negative a, b.

If $c > 0$, then $f(a, b, c) = c^3 f(u + 1, v + 1, 1)$ where $u = \frac{a}{c} - 1$, $v = \frac{b}{c} - 1$ with $u, v \geqslant 0$. But

$$\begin{aligned} f(u + 1, v + 1, 1) &= 21\left(u^2 - uv + v^2\right) + \left(12u^3 - 38u^2v + 23uv^2 + 12v^3\right) \\ &= 21\left[(u - v)^2 + uv\right] + f(u, v, 0) \end{aligned}$$

where the first bracketed expression is clearly non-negative and $f(u, v, 0) \geqslant 0$ by the $c = 0$ case above.

Several solvers noted that, if positive *real* values of k are allowed, numerical methods show that (for example) $a = 1.7488$, $b = 1$, $c = 0$ gives the minimum $k \approx 11.721$.

Correct solutions were received from: M. Bataille, S. Dolan, M. G. Elliott, GCHQ Problem Solving Group, G. Howlett, T. Kecker and the proposer Mehtaab Sawhney.

10.1017/mag.2018.35 N.J.L.