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GV and GW invariants via the enhanced movable cone

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Abstract

Given any smooth germ of a 3-fold flopping contraction, we first give a combinatorial characterisation of which Gopakumar–Vafa (GV) invariants are non-zero, by prescribing multiplicities to the walls in the movable cone. On the Gromov–Witten (GW) side, this allows us to describe, and even draw, the critical locus of the associated quantum potential. We prove that the critical locus is the infinite hyperplane arrangement of Iyama and the second author and, moreover, that the quantum potential can be reconstructed from a finite fundamental domain. We then iterate, obtaining a combinatorial description of the matrix that controls the transformation of the non-zero GV invariants under a flop. There are three main ingredients and applications: (1) a construction of flops from simultaneous resolution via cosets, which describes how the dual graph changes; (2) a closed formula, which describes the change in dimension of the contraction algebra under flop; and (3) a direct and explicit isomorphism between quantum cohomologies of different crepant resolutions, giving a Coxeter-style, visual proof of the Crepant Transformation Conjecture for isolated cDV singularities.

Contents

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1. Introduction

Many key results in algebraic geometry can be established using topological and combinatorial descriptions of a given variety or of its degenerations and deformations. However, even with a clear combinatorial model on such a degeneration or deformation, determining which properties of the original variety can be controlled by combinatorics is still, in general, a difficult question.

This article considers arbitrary, smooth 3-fold flopping contractions, which form a fundamental building block of the minimal model programme. Our main point is that as far as their enumerative geometry is concerned, all such flopping contractions *are* combinatorial, provided we are content with describing only the shape of the enumerative invariants, rather than their precise values. This qualitative perspective allows us to extract, and prove rather easily, many fundamental results. We determine which curve classes give rise to non-zero invariants, and then we control how these invariants transform under flop, in a visually pleasing and satisfyingly combinatorial manner. Along the way it is necessary to enhance existing geometric structures, such as the movable cone.

1.1 Gopakumar–Vafa: finite arrangements

Let $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ be a crepant resolution of a 3-fold isolated cDV singularity, which is, equivalently, a germ of a smooth 3-fold flopping contraction. The morphism f contracts a finite collection $\{\mathrm{C}_i \subseteq \mathcal{X} \mid i \in \mathcal{I}^{\mathrm{c}}\}$ of complete curves to a point, and these freely generate the group of algebraic curve classes

$$\mathrm{A}_1(\mathcal{X}) = \langle \mathrm{C}_i \mid i \in \mathcal{I}^{\mathrm{c}} \rangle_{\mathbb{Z}}.$$

Given $\beta \in \mathrm{A}_1(\mathcal{X})$, Katz defines an associated Gopakumar–Vafa (GV) invariant [Kat08]

$$n_\beta = n_\beta(\mathcal{X}) \in \mathbb{Z}_{\geq 0},$$

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which we recall in Section 3.1. It is an important invariant of flopping contractions, with close connections to other known invariants [DW16, Tod15].

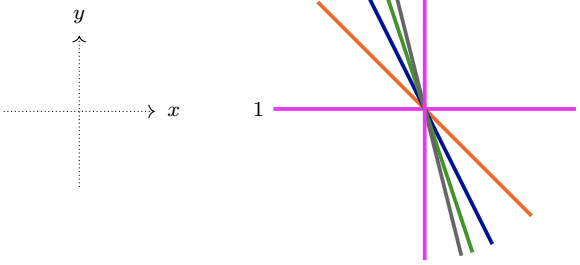
Our first result determines those β for which $n_\beta \neq 0$. The description is direct and combinatorial, encoded in the associated finite hyperplane arrangement $\mathcal{H}_\mathcal{I}$ of Iyama and the second author's work [IW]. The chambers in $\mathcal{H}_\mathcal{I}$ are precisely the ample cones of the different birational models, so we can identify $\mathcal{H}_\mathcal{I}$ with the movable cone. It turns out that the GV invariants are, to first approximation, encoded by the walls of this cone. There is, however, a slight catch: *combinatorially*, the walls carry multiplicities, and this data is not part of the definition of the movable cone. This multiplicity, which is new information (see Remark 5.6), turns out to be the key to determining whether $n_\beta \neq 0$.

As is standard, and as recalled in Section 2.2, slicing $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ by a generic hyperplane section gives rise to a partial crepant resolution of an ADE surface singularity. From this slicing we thus obtain the Dynkin diagram Δ of the ADE surface singularity, together with a subset $\mathcal{I}$ of nodes: the full minimal resolution dominates the partial resolution, and $\mathcal{I}$ are the curves that are contracted by this morphism.

Example 1.1. As the running example, consider a two-curve smooth 3-fold flop for which the corresponding Dynkin data is  , where by convention $\mathcal{I}$ equals the six black nodes. The Dynkin data gives rise to a finite intersection arrangement $\mathcal{H}_\mathcal{I} \subseteq \mathbb{R}^{|\Delta|-|\mathcal{I}|} = \mathbb{R}^2$ [IW, Section 3]. One method of calculating $\mathcal{H}_\mathcal{I}$ is to first restrict all 120 positive roots of E_8 to the subset $\mathcal{I}^c = \Delta \setminus \mathcal{I}$, and thus obtain the set

$$\{01, 11, 21, 42, 31, 41, 10, 20, 30\}.$$

These so-called restricted roots give rise to hyperplanes in the dual space, where, for example, 42 gives rise to the hyperplane $4x + 2y = 0$. The output is thus the following hyperplane arrangement, which we emphasise is constructed entirely from $\mathcal{I} \subseteq \Delta$.



Restricted Root	
01	
11	
21, 42	
31	
41	
10, 20, 30	

(1.A)

Note that the restricted root 42 gives rise to the hyperplane $2(2x + y) = 0$, so the blue diagonal $2x + y = 0$ line carries the list $[1, 2]$ of multiplicities. We write 2 beside the blue line to emphasise this fact. Similarly, the line $x = 0$ carries the list $[1, 2, 3]$ of multiplicities, as a consequence of 20 and 30.

Returning to general flopping contractions $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, since by construction the nodes in $\Delta \setminus \mathcal{I}$ can be identified with the curves in $\mathcal{X}$, after some natural identifications, $\mathcal{H}_\mathcal{I}$ (with multiplicities) can be viewed inside $\operatorname{Pic} \mathcal{X} \otimes \mathbb{R}$. So can the movable cone. After ignoring the multiplicities, $\mathcal{H}_\mathcal{I}$ is equal to the movable cone [Pin83, Wem18].

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The following is our first main result. It describes the non-zero GV invariants in an elementary combinatorial way, and it asserts that it is the hyperplanes of $\mathcal{H}_{\mathcal{I}}$, counted with multiplicity, that control the non-zero GV invariants.

Theorem 1.2 (3.1). *For $\beta \in \mathrm{A}_1(\mathcal{X})$, the GV invariant n_β is non-zero if and only if β is a restricted root.*

The theme of this article is that the shape of the enumerative geometry of $\mathcal{X}$ is controlled, in a very visual way, from this finite amount of initial data.

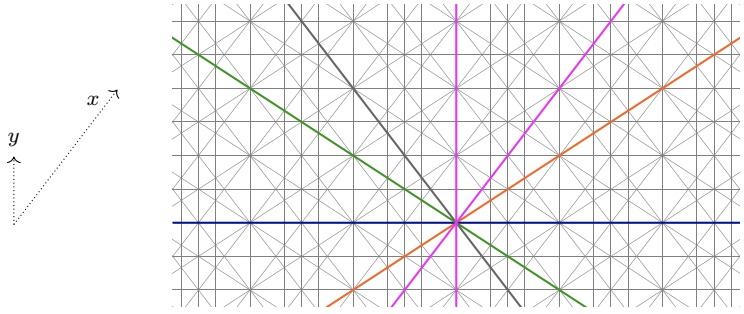
1.2 Gromov–Witten: infinite arrangements

Given any subset $\mathcal{I}$ of a Dynkin diagram, the finite arrangement $\mathcal{H}_{\mathcal{I}}$ of the previous subsection has an infinite cousin $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$. Given a restricted root $\beta = (\beta_i)_{i\in\mathcal{I}^c}$, the hyperplane $\sum_{i\in\mathcal{I}^c}\beta_i x_i = 0$, which appears in the finite arrangement, gets translated over the integers $\mathbb{Z}$, to give an infinite family

$$\sum_{i\in\mathcal{I}^c}\beta_i x_i \in \mathbb{Z}. \tag{1.B}$$

Repeating this over every restricted root results in an infinite arrangement of affine hyperplanes, written $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$. Note that multiplicities on hyperplanes of $\mathcal{H}_{\mathcal{I}}$ result in more translations, as if, say, $2\beta = (2\beta_i)$ is also a restricted root; then its translations give rise to the family $\sum 2\beta_i x_i \in \mathbb{Z}$, i.e. to $\sum \beta_i x_i \in \frac{1}{2}\mathbb{Z}$. This is larger than (1.B).

In the running Example 1.1, taking all the relevant translations of (1.A) results in the following $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$.



(1.C)

Returning to a general flopping contraction $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$, the next main result relates the Gromov–Witten (GW) theory of $\mathcal{X}$ to the associated infinite arrangement $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$. The GW invariants are virtual degrees of moduli spaces of stable maps and provide a system of curve counts equivalent to the more enumerative GV invariants via multiple cover formulae. The GW invariants form the structure constants for the quantum cohomology algebra, but for our purposes it is more convenient to package this information in a generating function, called the *quantum potential* (see Section 3.2 for details). Combining Theorem 1.2 and the multiple cover formulae gives the second main result.

Theorem 1.3 (3.4). *Let $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$ be a smooth 3-fold flopping contraction. The pole locus of the quantum potential is the complexification of the infinite arrangement $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$.*

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By [HW23], the complement of the complexified arrangement $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$ forms the base of the Bridgeland stability covering map, for a natural, compactly supported subcategory of the derived category of $\mathcal{X}$. Theorem 1.3 therefore connects quantum-cohomological Frobenius manifolds and spaces of stability conditions, a phenomenon that has been observed in other contexts [Bri06, IQ23, McA].

1.3 Flops via simultaneous resolution

To track the change of GV/GW invariants under iterated flops requires us to first rework some of the theory of simultaneous resolutions, which may be of independent interest. Our new contribution is to use the wall-crossing formula from [IW], which indexes chambers of the movable cone by certain cosets, to construct iterated flops from simultaneous (partial) resolutions, and to explain how the dual graph changes under flop. This completes the work of Reid [Rei83], Pinkham [Pin83], and Katz–Morrison [KM92] in the 80s and 90s, rounding off a circle of ideas going back to Brieskorn [Bri68].

For any Kleinian singularity $\mathbb{C}^2/G$, consider the corresponding Dynkin diagram Δ, root space $\mathfrak{h}$, and Weyl group W. As is standard, $\mathbb{C}^2/G$ admits a versal deformation $\operatorname{Spec}\mathcal{V}$ over the base $\mathfrak{h}_{\mathbb{C}}/W$. For any subset $\mathcal{I}\subseteq\Delta$, consider the parabolic subgroup $W_{\mathcal{I}} := \langle s_i \mid i\in\mathcal{I}\rangle$, and take the pullback to obtain

$$\begin{array}{ccc} \operatorname{Spec}\mathcal{V}_{\mathcal{I}} & \longrightarrow & \operatorname{Spec}\mathcal{V} \\ \downarrow{\scriptstyle \mathsf{g}_{\mathcal{I}}} & & \downarrow \\ \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}} & \longrightarrow & \mathfrak{h}_{\mathbb{C}}/W. \end{array}$$

When $\mathcal{I}=\emptyset$, the parabolic $W_{\mathcal{I}}=\mathbb{1}$, and in this case, classically $\operatorname{Spec}\mathcal{V}_{\emptyset}$ admits a simultaneous resolution.

As is now standard, to describe smooth 3-fold flops requires singular surface geometry, so the ability to consider $\mathcal{I}\neq\emptyset$ is crucial. By [KM92], for each $\mathcal{I}$ there is a preferred, or a *standard,* simultaneous partial resolution $\mathsf{h}_{\mathcal{I}}\colon \mathcal{Y}_{\mathcal{I}}\to\operatorname{Spec}\mathcal{V}_{\mathcal{I}}$ (see Section 2.4). Further, by *loc. cit.* all smooth flops can be constructed via appropriate classifying maps $\mu\colon \mathcal{D}\mathrm{isc}\to\mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}}$ from the formal disc to $\mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}}$ for some $\mathcal{I}$, giving the following cartesian diagram

$$\begin{array}{ccccc} \mathcal{X} & \longrightarrow & \mathcal{Y}_{\mathcal{I}} & & \\ \downarrow & & \downarrow{\scriptstyle \mathsf{h}_{\mathcal{I}}} & & \\ \operatorname{Spec}\mathcal{R} & \longrightarrow & \operatorname{Spec}\mathcal{V}_{\mathcal{I}} & \longrightarrow & \operatorname{Spec}\mathcal{V} \\ \downarrow & & \downarrow{\scriptstyle \mathsf{g}_{\mathcal{I}}} & & \downarrow \\ \mathcal{D}\mathrm{isc} & \xrightarrow{\mu} & \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}} & \longrightarrow & \mathfrak{h}_{\mathbb{C}}/W. \end{array}$$

Given $\mathcal{X}\to\operatorname{Spec}\mathcal{R}$, the question is how to construct the flop at a given curve from the classifying map μ. This was solved in the case $\mathcal{I}=\emptyset$ by Reid [Rei83], but the general case is harder, since the subset $\mathcal{I}$ changes under flop. Pinkham [Pin83] counts only the number of simultaneous resolutions.

We solve this problem by appealing to the wall-crossing combinatorics of [IW]. The key point is that when $\mathcal{I}\neq\emptyset$, chambers in the movable cone are indexed by cosets, not by elements of the Weyl group. For any subset $\Gamma\subseteq\Delta$, let $W_{\Gamma}\subseteq W$, denote the parabolic subgroup generated by

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reflections dual to the elements of Γ, and let $\ell_\Gamma \in W_\Gamma$ denote the longest element. For any curve $C_i \subseteq \mathcal{X}$, set $w_i = \ell_\mathcal{I} \ell_{\mathcal{I}\cup\{i\}} \in W$. Then there is a unique subset $\omega_i(\mathcal{I}) \subseteq \Delta$, described explicitly in Section 4, for which $W_\mathcal{I} w_i = w_i W_{\omega_i(\mathcal{I})}$.

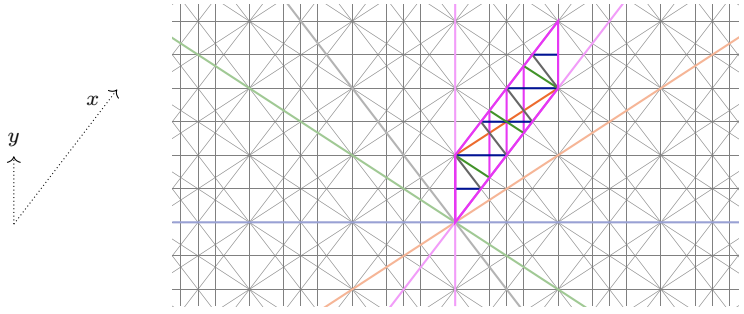
Theorem 1.4 (4.4). *Post-composing* μ *with* $w_i^{-1}\colon \mathfrak{h}_\mathbb{C}/W_\mathcal{I} \to \mathfrak{h}_\mathbb{C}/W_{\omega_i(\mathcal{I})}$ *and taking the pullback constructs the flop* $\mathcal{X}_i^+ \to \operatorname{Spec} \mathcal{R}$ *of the curve* $C_i \subset \mathcal{X}$. *In particular*

1. $\omega_i(\mathcal{I})$ *is the dual graph of the exceptional locus of* $\mathcal{X}_i^+ \to \operatorname{Spec} \mathcal{R}$.
2. *All other crepant resolutions can be obtained from the fixed* μ *by post-composing with* $x^{-1}\colon \mathfrak{h}_\mathbb{C}/W_\mathcal{I} \to \mathfrak{h}_\mathbb{C}/W_\mathcal{K}$ *and pulling back along* $\mathcal{Y}_\mathcal{K}$, *as the pair* $(x, \mathcal{K})$ *ranges over the (finite) indexing set* $\mathsf{Cham}(\Delta, \mathcal{I})$ *described in Notation 2.2.*

We describe the above in detail in Section 4, but we emphasise here that everything is formed intrinsically from the Dynkin data, once $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, and thus μ is fixed.

1.4 Tracking fundamental regions

With the above in mind, tracking the change in GV/GW invariants under all possible flops becomes easy, and satisfyingly visual. In our running Example 1.1, each of the 12 crepant resolutions $\mathcal{X}_i \to \operatorname{Spec} \mathcal{R}$ admits a fundamental region in (1.C). As a calibration, the fixed $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ corresponds to the unit box in the purple axes



(1.D)

The other 12 chambers in the movable cone generate similar fundamental regions as in (1.D), and this is illustrated in Figure 1.

Reassuringly, flopping a single curve turns out to correspond to the neighbouring region. Although Figure 1 illustrates only the two-curve flop in the running Example 1.1, similar things happen in full generality (see Section 5). The following is our third main result, which controls how GV invariants transform under iterated flops. As for the notation, set $\mathcal{X}_i^+$ to be the scheme obtained from $\mathcal{X}$ after flopping only a single curve C_i, and, further, write $n_{\beta,\mathcal{X}}$ for the GV invariant of curve class β in $\mathcal{X}$. In what follows M_i is an explicit invertible matrix, defined in (5.A), that can be easily built using Dynkin combinatorics.

Theorem 1.5 (5.4). *With the notation as above,*

$$n_{\beta,\mathcal{X}_i^+} = \begin{cases} n_{\beta,\mathcal{X}} & \text{if } \beta \in \mathbb{Z}C_i \\ n_{\mathsf{M}_i\beta,\mathcal{X}} & \text{else.} \end{cases}$$

In other words, the set of GV invariants is exactly the same before and after the flop; they are just re-indexed by the change of basis M_i. Example 5.5 illustrates this re-indexing in the case of the running Example 1.1.

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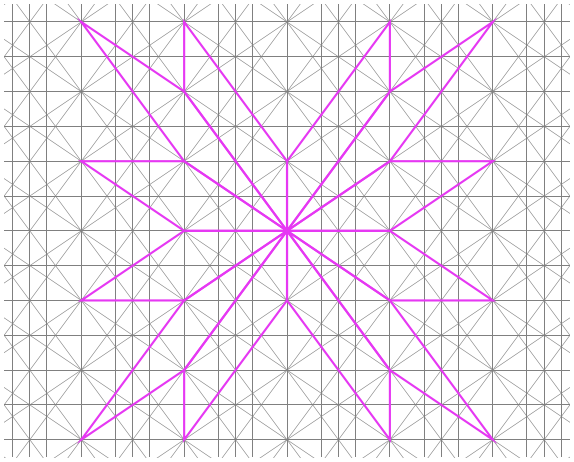

FIGURE 1. The 12 fundamental regions corresponding to the 12 crepant resolutions.

1.5 Applications

The above results have a series of corollaries. The first is a direct and explicit proof of the Crepant Transformation Conjecture for germs of 3-fold flopping contractions. Indeed, combining Theorems 1.3 and 1.5 allows us to easily extract the following, which recovers the main result of [LR01]. We remark here that our simplified approach also gives more refined information in the form of the explicit matrix $\mathsf{N}_i = (\mathsf{M}_i^{-1})^\star$, which identifies the quantum potentials. Our proof also avoids the use of symplectic cuts, thus side-stepping the associated technical difficulties.

COROLLARY 1.6 (5.13). *Under the identification of the Novikov parameters given by the explicit matrix* M_i *of Theorem 1.5, the quantum potentials of* $\mathcal{X}$ *and* $\mathcal{X}_i^+$ *coincide, up to a correction term that does not depend on the Novikov parameters, namely*

$$\Phi_{\mathsf{r}}^{\mathcal{X}_i^+}(\gamma_1,\gamma_2,\gamma_3) - \Phi_{\mathsf{r}}^{\mathcal{X}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3) = -(\gamma_1\cdot \mathrm{C}_i^+)(\gamma_2\cdot \mathrm{C}_i^+)(\gamma_3\cdot \mathrm{C}_i^+)\sum_{k\geq 1} k^3 n_{k\mathrm{C}_i,\mathcal{X}}. \tag{1.E}$$

Here, $\mathsf{N}_i\colon \mathrm{H}^2(\mathcal{X}_i^+;\mathbb{C}) \to \mathrm{H}^2(\mathcal{X};\mathbb{C})$ *is the dual of* M_i^{-1}. *The above identification holds after a specific analytic continuation in the quantum parameters.*

The correction terms on the right-hand side arise due to the non-compactness of $\mathcal{X}$ (see Remarks 3.7 and 5.14). The key point is that the quantum potential of $\mathcal{X}_i^+$ can be effectively reconstructed from the quantum potential of $\mathcal{X}$. Thus, whilst the GW invariants themselves are not combinatorial, their transformation across the flop *is* combinatorial, which is why we obtain such an elementary proof; compare [LR01] and [McL20].

The explicit matrix N_i turns out to have many different incarnations: it arises naturally as the image in K-theory of Bridgeland's flop functor, but, more interestingly, it can be calculated using very simple Dynkin-style combinatorics (see Remark 5.12). However, simply by iterating and multiplying matrices, it is possible to obtain a direct isomorphism between the generating functions of any two crepant resolutions of $\operatorname{Spec}\mathcal{R}$.

The second corollary is algebraic. The flopping contraction $\mathcal{X}\to\operatorname{Spec}\mathcal{R}$ has an associated contraction, algebra $\mathrm{A}_{\mathrm{con}}$ [DW16, DW19], and it is known by Hua–Toda [HT18] for single curves

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and by Toda [Tod18] in general (see A.4), the dimension of the contraction algebra is determined as the weighted sum of GV invariants

$$\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{conv}} = \sum_{\beta \in \mathrm{A}_1(\mathcal{X})} n_\beta (\beta \cdot \mathbb{1})^2$$

where $\beta \cdot \mathbb{1}$ is the sum of the entries of β. For any curve $\mathrm{C}_i \subseteq \mathcal{X}$; the contraction algebra can be intrinsically mutated to obtain $\nu_i \mathrm{A}_{\mathrm{con}}$; and this is the contraction algebra for the flop $\mathcal{X}_i^+ \to \operatorname{Spec} \mathcal{R}$.

It is known [Dug15, Aug20b] that $\mathrm{A}_{\mathrm{con}}$ and $\nu_i \mathrm{A}_{\mathrm{con}}$ are derived equivalent via a two-term tilting complex, but what is surprising here is that their dimension transforms in a very elementary manner, dictated by the K-theory of that derived equivalence.

Corollary 1.7 (5.16). *Under mutation at vertex i,*

$$\dim_{\mathbb{C}} \nu_i \mathrm{A}_{\mathrm{con}} = \sum_{\beta \in \mathrm{A}_1(\mathcal{X})} n_\beta \big((\mathsf{M}_i^{-1} \beta) \cdot \mathbb{1} \big)^2$$

where M_i is the explicit matrix in Corollary 1.5.

The above is remarkable: it says that not only are there just finitely many algebras in the derived equivalence class of the finite dimensional algebra $\mathrm{A}_{\mathrm{con}}$ (by [Aug20a]), but that furthermore, the dimensions of all the other algebras can be easily obtained combinatorially from the first. The proof of Corollary 1.7 is slightly subtle, since it is not a priori clear that the GV invariants defined by Toda are the same as the GV invariants defined here, but this is all discussed in Appendix A.

Conventions.
All cDV singularities and related algebraic geometry take place over $\mathbb{C}$. Vector spaces will be over $\mathbb{R}$, unless stated otherwise, and the complexification of a vector space V will be written $V_{\mathbb{C}}$.

2. Root theory, deformations and perturbations

Fix an isolated 3-fold cDV singularity $\operatorname{Spec} \mathcal{R}$ with a crepant resolution $f \colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$. There is a finite collection of exceptional complete curves in $\mathcal{X}$, contracted to a point $p \in \operatorname{Spec} \mathcal{R}$, such that f restricts to an isomorphism on the complement. In other words, $f \colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is a germ of a smooth 3-fold flopping contraction, and, conversely, every such germ arises in this way [Rei83].

2.1 Elephants

The pullback along f of a general hyperplane section through $p \in \operatorname{Spec} \mathcal{R}$ is a *partial* crepant resolution of an ADE surface singularity, the so-called general elephant [Rei83, (1.14)]

$$\begin{array}{ccc} Y & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathbb{C}^2/G \cong \operatorname{Spec} \mathcal{R}/g & \to & \operatorname{Spec} \mathcal{R}. \end{array} \tag{2.A}$$

Let Δ be the Dynkin diagram associated to $\mathbb{C}^2/G$, and let the composition

$$Z \to Y \to \mathbb{C}^2/G$$

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be the full minimal resolution. By the McKay correspondence, the exceptional curves $\mathrm{C}_i \subseteq Z$ are indexed by the nodes $i \in \Delta$. We write

$$\mathcal{I} \subseteq \Delta$$

for the subset indexing those curves $\mathrm{C}_i \subseteq Z$, which are contracted by the morphism $Z \to Y$, so the complement $\mathcal{I}^{\mathrm{c}} = \Delta \setminus \mathcal{I}$ indexes the curves that survive. In particular,

$$\{\mathrm{C}_i \mid i \in \mathcal{I}^{\mathrm{c}}\}$$

forms the set of exceptional curves in both Y and $\mathcal{X}$, and the group $\mathrm{A}_1(Y) = \mathrm{A}_1(\mathcal{X})$ is freely generated by their cycle classes.

Notation 2.1. Write $Y = Y_{\mathcal{I}}$ for the partial resolution of $\mathbb{C}^2/G$ obtained from the full minimal resolution Z by blowing down the curves in $\mathcal{I}$.

The geometry of $\mathcal{X}$ will be studied by viewing it as the total space of a one-parameter deformation of $Y_{\mathcal{I}}$. This requires detailed control over the associated root theory, which we establish in the following subsections.

2.2 Root theory

For any Dynkin diagram Δ, let $\mathfrak{h}$ be the $\mathbb{R}$-vector space based on the set of simple roots $\{\alpha_i \mid i \in \Delta\}$ so that

$$\mathfrak{h} = \bigoplus_{i \in \Delta} \mathbb{R}\alpha_i,$$

and write $\Theta = \mathfrak{h}^\star$ for the dual. The Weyl group W acts naturally on both $\mathfrak{h}$ and Θ. For every positive root $\alpha \in \mathfrak{h}$, write $\mathcal{D}_\alpha \subseteq \mathfrak{h}$ for the perpendicular hyperplane, and write $\mathsf{H}_\alpha \subseteq \Theta$ for the dual hyperplane.

Notation 2.2. For any subset $\mathcal{I} \subseteq \Delta$, consider the following data:

1. The complement $\mathcal{I}^{\mathrm{c}} = \Delta \setminus \mathcal{I}$.
2. The parabolic subgroup $W_{\mathcal{I}} := \langle s_i \mid i \in \mathcal{I} \rangle \subseteq W$.
3. The $\mathbb{R}$-vector space $\mathfrak{h}_{\mathcal{I}}$ obtained as the quotient of $\mathfrak{h}$ by the $\mathbb{R}$-subspace spanned by $\{\alpha_i \mid i \in \mathcal{I}\}$. The associated quotient map will be written
$$\pi_{\mathcal{I}} \colon \mathfrak{h} \to \mathfrak{h}_{\mathcal{I}}.$$
Note that $\mathfrak{h}_{\mathcal{I}}$ has basis $\{\pi_{\mathcal{I}}(\alpha_i) \mid i \in \mathcal{I}^{\mathrm{c}}\}$ and may be identified with the subspace of $\mathfrak{h}$ based on $\{\alpha_i \mid i \in \mathcal{I}^{\mathrm{c}}\}$.
4. The restricted positive roots are represented by $\mathfrak{h}_{\mathcal{I}}$, which are precisely the non-zero images of positive roots under $\pi_{\mathcal{I}}$.
5. For $\vartheta_i \in \mathbb{R}$ with $i \in \Delta$, write $(\vartheta_i) = \sum_{i \in \Delta} \vartheta_i \alpha_i^\star$, and consider
$$\Theta_{\mathcal{I}} := \{(\vartheta_i) \in \Theta \mid \vartheta_i = 0 \text{ for all } i \in \mathcal{I}\} \subseteq \Theta.$$
The reflecting hyperplanes in Θ intersect $\Theta_{\mathcal{I}}$, and in this way, $\Theta_{\mathcal{I}}$ inherits the structure of a finite hyperplane arrangement. Note that $\Theta_{\mathcal{I}}$ has basis $\{\alpha_i^\star \mid i \in \mathcal{I}^{\mathrm{c}}\}$. Of course, $\mathfrak{h}_{\mathcal{I}}$ and $\Theta_{\mathcal{I}}$ are dual, and both have dimension $|\mathcal{I}^{\mathrm{c}}|$.
6. The set $\mathsf{Cham}(\Delta, \mathcal{I})$ indexes chambers of $\Theta_{\mathcal{I}}$ [IW, 1.8]. Combinatorially, $\mathsf{Cham}(\Delta, \mathcal{I})$ can be defined as the set of all pairs $(x, \mathcal{K})$ with $x \in W$ and $\mathcal{K} \subseteq \Delta$ for which $W_{\mathcal{I}} x = x W_{\mathcal{K}}$ and $\mathrm{length}(x) = \min\{\mathrm{length}(y) \mid y \in x W_{\mathcal{K}}\}$.

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For a given restricted positive root $\beta \in \mathfrak{h}_{\mathcal{I}}$, there are in general many different positive roots $\alpha \in \mathfrak{h}$ such that $\pi_{\mathcal{I}}(\alpha) = \beta$. The following result controls the possible lifts and is a very mild generalisation of [BKL01, Lemma 2.4]. It is used in the proof of Theorem 2.6, which in turn is used to relate enumerative invariants to hyperplane arrangements in Section 3.

Lemma 2.3. *For any ADE Dynkin diagram Δ and any subset $\mathcal{I} \subseteq \Delta$, let $\alpha, \alpha' \in \mathfrak{h}$ be positive roots such that $\pi_{\mathcal{I}}(\alpha), \pi_{\mathcal{I}}(\alpha') \in \mathfrak{h}_{\mathcal{I}}$ are non-zero. Then the following are equivalent:*

1. $\pi_{\mathcal{I}}(\alpha) = \pi_{\mathcal{I}}(\alpha')$.
2. *α and α' are identified under the action of $W_{\mathcal{I}}$ on $\mathfrak{h}$.*
3. *$\mathcal{D}_\alpha$ and $\mathcal{D}_{\alpha'}$ are identified under the action of $W_{\mathcal{I}}$ on $\mathfrak{h}$.*

Proof. Since $\mathcal{I}$ is fixed, to ease notation, set $\pi = \pi_{\mathcal{I}}$.

(1)⇒(2) This is the only difficult part, and it proceeds by case analysis. Consider first the A_n root system, where the positive roots are precisely the connected chains of 1s on the Dynkin graph

$$\alpha_{ij} := \underset{i \quad j}{0 \ldots 0\, 1 \ldots 1\, 0 \ldots 0} \quad (1 \leq i \leq j \leq n).$$

By definition of the action of W on $\mathfrak{h}$, the reflection s_i acts by

$$s_i(\alpha_{ij}) = \alpha_{i+1\, j} \qquad \text{for } i < j, \tag{2.B}$$

which has the effect of replacing the leftmost 1 with a 0. Similarly $s_j(\alpha_{ij}) = \alpha_{i\, j-1}$ for $i < j$, which has the effect of replacing the rightmost 1 with a 0.

Suppose now that we are given positive roots α_{ij} and α_{kl}, and assume without loss of generality that $i \leq k$. Since $\pi(\alpha) = \pi(\alpha')$ is non-zero, the two chains of 1s must overlap, and any position at which they do not overlap must be indexed by an element of $\mathcal{I}$. By iteratively applying (2.B), we can shorten both α_{ij} and α_{kl} using only elements of $W_{\mathcal{I}}$ and force them to line up on the left, at the leftmost element belonging to the overlap. They can similarly be forced to line up on the right as well, proving the claim.

The case D_n for $n \geq 4$ is similar in spirit, albeit more involved. As usual, write $\alpha_1, \ldots, \alpha_n$ for the simple roots; then the positive roots are given by the following linear combinations. Note that the support of each is a connected subset of the Dynkin graph.

$$\alpha_{ij}^{pq} := \underset{i \quad j}{0 \ldots 0\, 1 \ldots 1\, 0 \ldots 0\, \overset{q}{p}} \qquad \begin{array}{l} 1 \leq i \leq j \leq n-2, \\ p, q \in \{0, 1\}, \\ (p, q) \neq (0, 0) \Rightarrow j = n-2 \end{array}$$

$$\beta_{ij} := \underset{i \quad j}{0 \ldots 0\, 1 \ldots 1\, 2 \ldots 2\, \overset{1}{1}} \qquad 1 \leq i < j \leq n-2$$

In addition, the collection α_{ij}^{pq} includes two special cases α_{n-1} and α_n, which we interpret as $(p, q) = (1, 0)$ and $(p, q) = (0, 1)$, with the string of 1s between i and j being empty.

For every $1 \leq i \leq n$, the reflection s_i acts on the coefficient in the ith position by negating it and then by adding the sum of the coefficients in adjacent positions. The coefficients in all other positions are left unchanged.

Consider now positive roots α, α', with $\pi(\alpha) = \pi(\alpha') \neq 0$. We work through the different cases. The first case is $\alpha = \alpha_{ij}^{pq}$ and $\alpha' = \alpha_{kl}^{rs}$. If $(p, q) \neq (r, s)$, then we can use the reflections s_{n-1}, s_n to transform α, α' until $(p, q) = (r, s)$. For instance, if $(p, q) = (0, 1)$ and $(r, s) = (0, 0)$,

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then, since $\pi(\alpha)=\pi(\alpha')$, we must have $n\in\mathcal{I}$. Applying $s_n\in W_{\mathcal{I}}$ to α replaces $(p,q)=(0,1)$, with $(p,q)=(0,0)$. The other cases are similar, so we may assume that $(p,q)=(r,s)$. Given this, we follow the strategy in the A_n case. The only difference is when $j=n-2$ and $l<n-2$, in which case $(p,q)=(r,s)=(0,0)$ since the chain of 1s is connected. Then $s_{n-2}\in W_{\mathcal{I}}$ and $s_{n-2}(\alpha^{00}_{i\,n-2})=\alpha^{00}_{i\,n-3}$. This, and all other situations, then simply mirror the A_n proof.

The next case is $\alpha=\alpha^{pq}_{ij}$ and $\alpha'=\beta_{kl}$. If $i\leq k$ then necessarily $j\geq k$ since otherwise the supports do not overlap. In particular $s_i,s_{i+1},\ldots,s_{k-1}\in W_{\mathcal{I}}$ which gives

$$(s_{k-1}\circ\cdots\circ s_i)(\alpha^{pq}_{ij})=\alpha^{pq}_{kj}.$$

Similarly if $i\geq k$ we use $s_k\circ\cdots\circ s_{i-1}$ to achieve the same transformation. For the next step, notice that $s_{j+1},\ldots,s_{n-2}\in W_{\mathcal{I}}$, and applying these left-to-right gives

$$(s_{n-2}\circ\cdots\circ s_{j+1})(\alpha^{pq}_{kj})=\alpha^{pq}_{k\,n-2}.$$

As in the previous case, if $(p,q)\neq(1,1)$ then we may use the reflections s_{n-1},s_n to transform $\alpha^{pq}_{k\,n-2}$ into $\alpha^{11}_{k\,n-2}$. Now, going back from right-to-left gives the required

$$(s_l\circ\cdots\circ s_{n-2})(\alpha^{11}_{k\,n-2})=\beta_{kl}.$$

The next case is $\alpha=\beta_{ij}$ and $\alpha'=\beta_{kl}$, where without loss of generality $i\geq k$. But then $s_{i-1},\ldots,s_k\in W_{\mathcal{I}}$ which is applied directly to give $(s_k\circ\cdots\circ s_{i-1})(\beta_{ij})=\beta_{kj}$. If $j\geq l$, then $s_{j-1},\ldots,s_l\in W_{\mathcal{I}}$, and $(s_l\circ\cdots\circ s_{j-1})(\beta_{kj})=\beta_{kl}$. The case $j\leq l$ is similar.

This completes the proof for D_n, except for some special cases involving α_{n-1} and α_n. These are more elementary, and so are left to the reader.

The remaining cases E_6,E_7,E_8 encompass finitely many possibilities. It is possible to verify these by hand, but it is also possible to employ computer algebra [BCP97]. Source code is available from the authors upon request. This completes the proof of (1)⇒(2).

(2)⇒(1) This holds since applying elements of $W_{\mathcal{I}}$ to a given positive root cannot change the coefficients associated to elements of $\mathcal{I}^c$.

(2)⇒(3) If $s\in W$ and $v\in\mathfrak{h}$ then $s(\mathcal{D}_v)=\mathcal{D}_{s(v)}$ since s preserves the Cartan pairing. In particular this applies when $v=\alpha$ is a positive root and $s(v)=\alpha'$ is its image.

(3)⇒(2) Suppose there is an element $s\in W_{\mathcal{I}}$ such that $s(\mathcal{D}_\alpha)=\mathcal{D}_{\alpha'}$. We then have $\mathcal{D}_{\alpha'}=\mathcal{D}_{s(\alpha)}$ and since both α' and $s(\alpha)$ are roots it follows that $s(\alpha)$ equals either α' or $-\alpha'$. We claim that $s(\alpha)$ must be a positive root, thus ruling out the latter possibility. If $s=s_i$ for some $i\in\mathcal{I}$, then s_i permutes the set of positive roots excluding α_i. But certainly $\alpha\neq\alpha_i$ since $\pi(\alpha)\neq 0$, so $s_i(\alpha)$ must be a positive root. The argument for general s follows by induction, since $\pi(\alpha)=\pi(s_i(\alpha))$ for any α and any s_i with $i\in\mathcal{I}$. □

2.3 Hyperplane arrangements: finite and infinite

To the above data of a Dynkin diagram Δ and a subset $\mathcal{I}\subseteq\Delta$ it is possible to associate two hyperplane arrangements encoding the set of restricted positive roots: one finite and one infinite [IW]. Recalling Notation 2.2, both arrangements live inside $\Theta_{\mathcal{I}}\cong\mathbb{R}^{|\mathcal{I}^c|}$.

The real vector space $\mathfrak{h}_{\mathcal{I}}$ is dual to $\Theta_{\mathcal{I}}$, and so for each restricted root $0\neq\beta=\pi_{\mathcal{I}}(\alpha)\in\mathfrak{h}_{\mathcal{I}}$ we may consider the dual hyperplane

$$\mathsf{H}_\beta:=\{(\vartheta_i)\mid\sum\beta_i\vartheta_i=0\}\subseteq\Theta_{\mathcal{I}}.$$

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Since there are only finitely many restricted roots, the collection of H_β forms a finite hyperplane arrangement in $\Theta_{\mathcal{I}}$, which we refer to as the *finite linear arrangement*, namely

$$\mathcal{H}_{\mathcal{I}} := \{\mathsf{H}_\beta \mid \beta \text{ is a restricted positive root}\}\,. \tag{2.C}$$

The above list includes repetitions, whenever two restricted roots are proportional. As in Example 1.1, we remember these repetitions by attaching a finite list of multiplicities to each hyperplane. We refer to this data as the *enhanced finite arrangement*.

Remark 2.4. In the setting of Section 2.1, $\mathcal{H}_{\mathcal{I}}$ is the set of walls in the movable cone of $\mathcal{X}$ [Pin83, Wem18]. However, the movable cone does not remember multiplicities.

To define the infinite arrangement, for each restricted positive root β, consider the infinite disjoint union of affine hyperplanes in $\Theta_{\mathcal{I}}$ defined by

$$\mathsf{H}_\beta^{\mathsf{aff}} = \left\{ (\vartheta_i) \mid \sum_{i\in\mathcal{I}^c} \beta_i\vartheta_i = z \text{ for some } z\in\mathbb{Z} \right\}.$$

The infinite affine arrangement $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$ is then defined to be

$$\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}} := \bigcup_\beta \mathsf{H}_\beta^{\mathsf{aff}}. \tag{2.D}$$

There is an inclusion $\mathcal{H}_{\mathcal{I}} \subseteq \mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$, as $\mathcal{H}_{\mathcal{I}}$ can be recovered from $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$ as the subset of hyperplanes which pass through the origin. On the other hand to construct $\mathcal{H}_{\mathcal{I}}^{\mathsf{aff}}$ from the finite arrangement $\mathcal{H}_{\mathcal{I}}$ it is necessary to remember the multiplicities, since $\mathsf{H}_{2\beta}^{\mathsf{aff}} \neq \mathsf{H}_\beta^{\mathsf{aff}}$.

2.4 Simultaneous partial resolution

The enumerative geometry of the 3-fold $\mathcal{X}$ will be studied by replacing $\mathcal{X}$ with a generic perturbation, a strategy employed by many authors [Mor96, Wil99, BKL01]. In the first instance this will allow us to qualitatively characterise the GV invariants, and extract the poles of the quantum product. However the real strength in this approach, and indeed our new contribution, is to use the wall crossing formula from [IW] to construct iterated flops via simultaneous (partial) resolution. This iteration step is harder, and so will be delayed until Section 4.

In this subsection we simply recall the necessary background and set notation, largely following [BKL01, Section 2], together with [Bri68, Pin83, Fri86, KM92].

2.4.1 *Simultaneous resolution.* With notation as in Section 2.2, given any Kleinian singularity $\mathbb{C}^2/G$ with associated Dynkin diagram Δ, consider the complex vector space $\mathfrak{h}_{\mathbb{C}} = \bigoplus_{i\in\Delta}\mathbb{C}\alpha_i$, based by the simple roots. Write $\operatorname{Spec}\mathcal{V}$ for a versal deformation of $\mathbb{C}^2/G$, then as is very well known, base changing with respect to the Weyl group

$$\begin{array}{ccc} \operatorname{Spec}\mathcal{W} & \dashrightarrow & \operatorname{Spec}\mathcal{V} \\ \downarrow & & \downarrow \\ \mathfrak{h}_{\mathbb{C}} & \longrightarrow & \mathfrak{h}_{\mathbb{C}}/W \end{array}$$

gives $\operatorname{Spec}\mathcal{W}\to\mathfrak{h}_{\mathbb{C}}$, which admits a simultaneous resolution. Since $\dim\mathcal{W}\geq 3$, there are in fact many such simultaneous resolutions, since minimal models are not unique.

In [KM92, Theorem 1] Katz–Morrison construct a particular simultaneous resolution, from a particular $\operatorname{Spec}\mathcal{V}\to\mathfrak{h}_{\mathbb{C}}/W$, for which positive roots and their hyperplanes control those curve

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classes that survive under deformation [KM92, Theorem 1(c)]. We will recap this result in greater generality in Theorem 2.6 below, but for now write $\mathcal{Z} \to \operatorname{Spec} \mathcal{W}$ for this preferred resolution. Katz–Morrison refer to their particular choice of $\mathcal{Z}$ as the standard simultaneous resolution [KM92, Section 6].

2.4.2 *Simultaneous partial resolution.* Given any subset $\mathcal{I} \subseteq \Delta$, consider the standard simultaneous resolution $\mathcal{Z} \to \operatorname{Spec} \mathcal{W}$ from Section 2.4.1. As explained in [KM92, above Theorem 3] following [Pin83] it is possible to blow down $\mathcal{Z}$ at the curves in $\mathcal{I}$ and take the quotient by $W_{\mathcal{I}}$ to obtain $\mathcal{Y}_{\mathcal{I}}$, which sits in the following commutative diagram

$$\begin{array}{ccccc} \mathcal{Z} & & & & \\ \downarrow & & & & \\ \mathcal{Y}_{\mathcal{I}}^{\dagger} & \longrightarrow & \mathcal{Y}_{\mathcal{I}} & & \\ \downarrow & & \downarrow{\scriptstyle \mathsf{g}_{\mathcal{I}}} & & \\ \operatorname{Spec} \mathcal{W} & \longrightarrow & \operatorname{Spec} \mathcal{V}_{\mathcal{I}} & \longrightarrow & \operatorname{Spec} \mathcal{V} \\ \downarrow & & \downarrow{\scriptstyle \mathsf{h}_{\mathcal{I}}} & & \downarrow \\ \mathfrak{h}_{\mathbb{C}} & \longrightarrow & \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}} & \longrightarrow & \mathfrak{h}_{\mathbb{C}}/W \end{array} \tag{2.E}$$

with all squares cartesian. By construction, the fibre $(\mathsf{h}_{\mathcal{I}} \circ \mathsf{g}_{\mathcal{I}})^{-1}(0)$ is the partial resolution of $\mathbb{C}^2/G$ obtained from the full minimal resolution by blowing down the curves in $\mathcal{I}$. Namely, recalling Notation 2.1, $(\mathsf{h}_{\mathcal{I}} \circ \mathsf{g}_{\mathcal{I}})^{-1}(0) = Y_{\mathcal{I}}$.

In a similar way as in Section 2.4.1, the middle morphism $\mathsf{h}_{\mathcal{I}} \colon \operatorname{Spec} \mathcal{V}_{\mathcal{I}} \to \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}}$ admits simultaneous *partial* resolutions. Again these are not unique, however we will refer to the choice $\mathcal{Y}_{\mathcal{I}} \to \operatorname{Spec} \mathcal{V}_{\mathcal{I}}$ constructed above as the standard simultaneous partial resolution associated to $\mathcal{I}$. From our perspective, the point is that $\mathcal{Y}_{\mathcal{I}}$ is precisely the partial simultaneous resolution for which Theorem 2.6 below holds.

2.5 Surface deformations via simultaneous partial resolution

Fix a subset $\mathcal{I} \subseteq \Delta$ and consider the composition

$$\mathsf{s}_{\mathcal{I}} = \mathsf{h}_{\mathcal{I}} \circ \mathsf{g}_{\mathcal{I}} \colon \mathcal{Y}_{\mathcal{I}} \to \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}}$$

from (2.E). This is a versal deformation of the surface $Y_{\mathcal{I}}$.

DEFINITION 2.5. *The standard discriminant locus*

$$\mathcal{D}_{\mathcal{I}} \subseteq \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}}$$

is the set of points $p \in \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}}$ such that the fibre $\mathsf{s}_{\mathcal{I}}^{-1}(p)$ contains a complete curve.

There is a similar definition of a discriminant locus associated to any simultaneous partial resolution: the word standard in Definition 2.5 emphasises the choice made in (2.E). The following discussion draws heavily on [KM92, Theorem 1] as used in [BKL01, Proposition 2.2], while also incorporating Lemma 2.3 above to relate the resulting combinatorics to the enhanced movable cone.

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To set notation, recall that $\mathcal{D}_\alpha \subseteq \mathfrak{h}$ is the hyperplane perpendicular to α, and let $\mathcal{D}_{\alpha,\mathbb{C}} \subseteq \mathfrak{h}_\mathbb{C}$ denote its complexification. Recall from Notation 2.2 that for $\mathcal{I} \subseteq \Delta$ there is a quotient map $\pi_\mathcal{I} \colon \mathfrak{h} \to \mathfrak{h}_\mathcal{I}$, where the vector space $\mathfrak{h}_\mathcal{I}$ has basis $\{\pi_\mathcal{I}(\alpha_i) \mid i \in \mathcal{I}^c\}$, and that there is a natural identification

$$\begin{aligned} \mathfrak{h}_\mathcal{I} &\cong \mathrm{A}_1(Y_\mathcal{I})_\mathbb{R} \\ \pi_\mathcal{I}(\alpha_i) &\mapsto \mathrm{C}_i. \end{aligned} \tag{2.F}$$

Every restricted positive root $\pi_\mathcal{I}(\alpha) \in \mathfrak{h}_\mathcal{I}$ has non-negative integer coefficients, and so may be interpreted as a curve class $\pi_\mathcal{I}(\alpha) = \beta \in \mathrm{A}_1(Y_\mathcal{I})$.

Theorem 2.6 (Katz–Morrison). *For any subset $\mathcal{I} \subseteq \Delta$, the following statements hold.*

1. *The standard discriminant locus $\mathcal{D}_\mathcal{I} \subseteq \mathfrak{h}_\mathbb{C}/W_\mathcal{I}$ from Definition 2.5 decomposes as*
$$\mathcal{D}_\mathcal{I} = \bigcup_{\pi_\mathcal{I}(\alpha) \neq 0} \mathcal{D}_{\alpha,\mathbb{C}}/W_\mathcal{I}$$
where the union is over all positive roots α such that $\pi_\mathcal{I}(\alpha) \neq 0$. The irreducible components $\mathcal{D}_{\alpha,\mathbb{C}}/W_\mathcal{I} \subseteq \mathcal{D}_\mathcal{I}$ are indexed by the restricted positive roots $\pi_\mathcal{I}(\alpha)$.
2. *For $p \in \mathcal{D}_{\alpha,\mathbb{C}}/W_\mathcal{I}$ the fibre $\mathsf{s}_\mathcal{I}^{-1}(p)$ is a deformation of $Y_\mathcal{I}$ containing a complete curve of class $\beta := \pi_\mathcal{I}(\alpha)$. If in addition p does not belong to any other component of $\mathcal{D}_\mathcal{I}$, then this is the only complete curve in $\mathsf{s}_\mathcal{I}^{-1}(p)$.*
3. *If $\beta \in \mathrm{A}_1(Y_\mathcal{I})$ is not a restricted positive root, then there are no deformations of $Y_\mathcal{I}$ containing a complete curve of class β.*

Proof. (1) If $\mathcal{I} = \emptyset$ then by [KM92, Theorem 1(3)] (see also [BKL01, Proposition 2.2]) there is a decomposition of the standard discriminant locus

$$\mathcal{D}_\emptyset = \bigcup_\alpha \mathcal{D}_{\alpha,\mathbb{C}} \subseteq \mathfrak{h}_\mathbb{C}$$

where the union is over all positive roots α. The analogous decomposition for general $\mathcal{I}$ follows by considering the standard simultaneous resolution $\mathcal{Z}$ of the standard simultaneous partial resolution $\mathcal{Y}_\mathcal{I}$ from 2.4

$$\begin{array}{ccc} \mathcal{Z} & \longrightarrow & \mathcal{Y}_\mathcal{I} \\ \downarrow{\scriptstyle \mathsf{t}_\mathcal{I}} & & \downarrow{\scriptstyle \mathsf{s}_\mathcal{I}} \\ \mathfrak{h}_\mathbb{C} & \xrightarrow{\phi_\mathcal{I}} & \mathfrak{h}_\mathbb{C}/W_\mathcal{I}. \end{array}$$

The map $\mathcal{Z} \to \mathcal{Y}_\mathcal{I}$ is given by blowing down $\mathcal{Z}$ at the curves in $\mathcal{I}$ and then taking the quotient by $W_\mathcal{I}$.

Fixing a point $p \in \mathfrak{h}_\mathbb{C}$, it follows that the fibre $\mathsf{s}_\mathcal{I}^{-1}(\phi_\mathcal{I}(p))$ contains a complete curve if and only if the fibre $\mathsf{t}_\mathcal{I}^{-1}(p)$ contains a complete curve which is not blown down. Again by [KM92, Theorem 1(3)], the fibre $\mathsf{t}_\mathcal{I}^{-1}(p)$ contains a complete curve if and only if $p \in \mathcal{D}_{\alpha,\mathbb{C}}$ for some positive root α, and this curve is not blown down if and only if $\pi_\mathcal{I}(\alpha) \neq 0$. This produces the desired decomposition of $\mathcal{D}_\mathcal{I}$. It then follows from Lemma 2.3 that the components $\mathcal{D}_{\alpha,\mathbb{C}}/\mathcal{W}_\mathcal{I}$ are indexed by the restricted positive roots $\pi_\mathcal{I}(\alpha)$, i.e. $\mathcal{D}_{\alpha,\mathbb{C}}/W_\mathcal{I} = \mathcal{D}_{\alpha',\mathbb{C}}/W_\mathcal{I}$ if and only if $\pi_\mathcal{I}(\alpha) = \pi_\mathcal{I}(\alpha')$.

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(2)(3) First recall what it means for a curve in $\mathsf{s}_\mathcal{J}^{-1}(p)$ to have class $\beta \in \mathrm{A}_1(Y_\mathcal{J})$. The inclusion of the central fibre $\mathsf{i}_0 \colon \mathsf{s}_\mathcal{J}^{-1}(0) = Y_\mathcal{J} \hookrightarrow \mathcal{Y}_\mathcal{J}$ induces an isomorphism

$$\mathsf{i}_{0\star} \colon \mathrm{A}_1(Y_\mathcal{J}) \xrightarrow{\sim} \mathrm{A}_1(\mathcal{Y}_\mathcal{J}).$$

Now consider an arbitrary fibre $\mathsf{s}_\mathcal{J}^{-1}(p)$ with inclusion $\mathsf{i}_p \colon \mathsf{s}_\mathcal{J}^{-1}(p) \hookrightarrow \mathcal{Y}_\mathcal{J}$. If $\mathrm{C} \subseteq \mathsf{s}_\mathcal{J}^{-1}(p)$ is a complete curve, then C has class $\beta \in \mathrm{A}_1(Y_\mathcal{J})$ if $\mathsf{i}_{p\star}\mathrm{C} = \mathsf{i}_{0\star}\beta$. The same definition applies to the full simultaneous resolution $\mathcal{Z}$. Both (2) and (3) are known for $\mathcal{Z}$ by *loc. cit.*, and the general case follows by tracking curve classes from $\mathcal{Z}$ to $\mathcal{Y}_\mathcal{J}$. □

Remark 2.7. Consider classes $\beta, \beta' \in \mathrm{A}_1(Y_\mathcal{J})$ which are proportional but distinct, i.e. $k\beta = k'\beta'$ for some distinct integers $k, k' \geq 1$. If both β, β' are restricted positive roots, then the lifts α, α' will not be proportional, by the root system axioms. In particular, the corresponding components of the standard discriminant locus $\mathcal{D}_\mathcal{J}$ will be distinct. Every component of the discriminant locus therefore corresponds to a unique curve class β. This is in contrast to the components of the hyperplane arrangement $\mathcal{H}_\mathcal{J} \subseteq \Theta_\mathcal{J}$ from Section 2.3.

Remark 2.8. The description of the standard discriminant locus $\mathcal{D}_\mathcal{J}$ in Theorem 2.6 is a union over positive roots $\alpha \in \mathfrak{h}$ such that $\pi_\mathcal{J}(\alpha) \neq 0$. The complementary union of hyperplane quotients

$$\mathcal{E}_\mathcal{J} = \bigcup_{\pi_\mathcal{J}(\alpha)=0} \mathcal{D}_{\alpha,\mathbb{C}}/W_\mathcal{J}$$

parametrise points $p \in \mathfrak{h}_\mathbb{C}/W_\mathcal{J}$ such that the fibre $\mathsf{s}_\mathcal{J}^{-1}(p)$ is singular. Clearly

$$\mathcal{D}_\emptyset/W_\mathcal{J} = \mathcal{D}_\mathcal{J} \cup \mathcal{E}_\mathcal{J}.$$

For a generic point $p \in \mathcal{E}_\mathcal{J}$, the fibre $\mathsf{s}_\mathcal{J}^{-1}(p)$ contains a single A_1 singularity. The locus $\mathcal{E}_\mathcal{J}$ will play a less central role than $\mathcal{D}_\mathcal{J}$. The translation between our notation and that of [BKL01, Section 2] is as follows: $\mathcal{D}_\mathcal{J} = D^{\mathrm{curv}}$, $\mathcal{E}_\mathcal{J} = D^{\mathrm{sing}}$ and $\mathcal{D}_\mathcal{J} \cup \mathcal{E}_\mathcal{J} = D$.

2.6 3-fold perturbations via surface deformations

Given the flopping contraction $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, a choice of local equation for the hypersurface $\operatorname{Spec} \mathcal{R}/g \subseteq \operatorname{Spec} \mathcal{R}$ produces a flat family $\operatorname{Spec} \mathcal{R} \to \mathcal{D}\mathrm{isc}$ over a formal disc, with central fibre an ADE surface singularity. By composition this produces a flat family $\mathcal{X} \to \mathcal{D}\mathrm{isc}$ with central fibre the partial resolution $Y \cong Y_\mathcal{J}$ of the ADE singularity [Rei83]. This exhibits $\mathcal{X}$, respectively $\operatorname{Spec} \mathcal{R}$, as the total space of a one-parameter deformation of the surface $Y_\mathcal{J}$, respectively $\operatorname{Spec} \mathcal{R}/g$. These deformations are induced by an associated classifying map

$$\mu \colon \mathcal{D}\mathrm{isc} \to \mathfrak{h}_\mathbb{C}/W_\mathcal{J}$$

where $\mathfrak{h}_\mathbb{C}/W_\mathcal{J}$ is as described in the previous subsection, and the contraction $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is obtained from the simultaneous partial resolution of Section 2.4 by base change

$$\begin{array}{ccc}
\mathcal{X} & \longrightarrow & \mathcal{Y}_\mathcal{J} \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathsf{g}_\mathcal{J}} \\
\operatorname{Spec} \mathcal{R} & \longrightarrow & \operatorname{Spec} \mathcal{V}_\mathcal{J} \\
\downarrow & & \downarrow{\scriptstyle \mathsf{h}_\mathcal{J}} \\
\mathcal{D}\mathrm{isc} & \xrightarrow{\mu} & \mathfrak{h}_\mathbb{C}/W_\mathcal{J}.
\end{array}$$

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The central fibre of $\operatorname{Spec}\mathcal{R}\to\mathcal{D}\mathrm{isc}$ is the ADE singularity corresponding to Δ, so $\mu(0)=0$. On the other hand, since $\mathcal{X}$ contains no complete curves outside of Y the map μ does not intersect the discriminant locus away from the origin, so $\mu^{-1}(\mathcal{D}_{\mathcal{J}})=\mu^{-1}(0)=0$.

As in [BKL01, below Lemma 2.7] there exists a one-parameter perturbation of μ

$$(\mu_t)_{t\in[0,\varepsilon]}\colon \mathcal{D}\mathrm{isc}\times[0,\varepsilon]\to\mathfrak{h}_{\mathbb{C}}/W_{\mathcal{J}}$$

such that $\mu_0=\mu$ and for $t\neq 0$ the following transversality condition is satisfied

$$\mu_t \text{ intersects } \mathcal{D}_{\mathcal{J}}\cup\ \mathcal{E}_{\mathcal{J}} \text{ transversely and away from codimension two strata} \tag{2.G}$$

where $\mathcal{E}_{\mathcal{J}}$ is defined in Remark 2.8. Furthermore, making ε smaller if necessary, we can assume that $\mu_t^{-1}(\mathcal{D}_{\mathcal{J}})$ is bounded away from the boundary of $\mathcal{D}\mathrm{isc}$.

The spaces $\mathcal{X}_{t\neq 0}$ give generic perturbations of the target $\mathcal{X}_0=\mathcal{X}$. The following is well-known, and will be used to reduce the enumerative geometry of $\mathcal{X}_t$, locally, to that of the Atiyah flop.

Lemma 2.9. *For any $t\neq 0$, the total space $\mathcal{X}_t$ of the family of surfaces associated to μ_t is a smooth 3-fold. Further, every complete curve in $\mathcal{X}_t$ is isolated, smooth, and rational, with normal bundle isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-1)\oplus\mathcal{O}_{\mathbb{P}^1}(-1)$.*

Proof. This is essentially [BKL01, Proposition 2.2], which itself is extracted from the proof of [KM92, Theorem 1]. □

Write $\mathfrak{X}$ for the 4-dimensional total space of the entire family $(\mu_t)_t$. Then as explained in [Wil92, Section 3], pulling back along inclusions of fibres induces isomorphisms

$$\mathrm{H}^2(Y_{\mathcal{J}};\mathbb{Z})\xleftarrow{\cong}\mathrm{H}^2(\mathcal{X};\mathbb{Z})\xleftarrow{\cong}\mathrm{H}^2(\mathfrak{X};\mathbb{Z})\xrightarrow{\cong}\mathrm{H}^2(\mathcal{X}_t;\mathbb{Z}) \tag{2.H}$$

for any t. Any class L in $\mathrm{H}^2(\mathcal{X},\mathbb{Z})\cong\mathrm{Pic}(\mathcal{X})$ thus induces an invertible sheaf $\mathcal{L}$ on $\mathfrak{X}$ with $\mathcal{L}|_{\mathcal{X}_0}=L$. Similarly, pushing forward curve classes along the inclusion of fibres induces isomorphisms

$$\mathrm{A}_1(Y_{\mathcal{J}})\xrightarrow{\cong}\mathrm{A}_1(\mathcal{X})\xrightarrow{\cong}\mathrm{A}_1(\mathfrak{X})\xleftarrow{\cong}\mathrm{A}_1(\mathcal{X}_t) \tag{2.I}$$

for any t. Given $\beta\in\mathrm{A}_1(\mathcal{X})$ we abuse notation and let $\beta\in\mathrm{A}_1(\mathcal{X}_t)$ denote the image of β under the composition of the natural isomorphisms above. Further, combining (2.I) and (2.F) it makes sense to ask when curve classes are restricted roots.

Corollary 2.10. *Fix $t\neq 0$ and $\beta\in\mathrm{A}_1(\mathcal{X})$ non-zero. Then the following statements hold:*

1. *If β is not a restricted positive root, i.e. there does not exist a positive root α with $\pi_{\mathcal{J}}(\alpha)=\beta$, then there is no complete curve in $\mathcal{X}_t$ of class β.*
2. *If β is a restricted positive root, with $\pi_{\mathcal{J}}(\alpha)=\beta$, then the number of complete curves in $\mathcal{X}_t$ of class β is equal to $|\mu_t^{-1}(\mathcal{D}_{\alpha,\mathbb{C}}/W_{\mathcal{J}})|$ and is always ≥ 1.*

Proof. This follows from Theorem 2.6, the only new claim being that $|\mu_t^{-1}(\mathcal{D}_{\alpha,\mathbb{C}}/W_{\mathcal{J}})|\geq 1$. We observed above that $\mu(0)=0$. Since $\mu=\mu_0$ and $0\in\mathcal{D}_{\alpha,\mathbb{C}}/W_{\mathcal{J}}$ for every positive root α, it follows that $|\mu_0^{-1}(\mathcal{D}_{\alpha,\mathbb{C}}/W_{\mathcal{J}})|\geq 1$ for every positive root α. This quantity does not change under a small perturbation of μ_0 to μ_t. □

3. Curve counting and hyperplane arrangements

The previous section established strong control over generic perturbations of the target geometry $\mathcal{X}$. This section exploits this, and describes the qualitative structure of the systems of enumerative

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invariants attached to $\mathcal{X}$. We show that these are intimately related to the combinatorial hyperplane arrangements from Section 2.3, with the infinite affine arrangement in particular giving the pole locus of the so-called quantum potential.

3.1 Gopakumar–Vafa

Curve counting invariants of $\mathcal{X}$ will be defined using the perturbed target $\mathcal{X}_t$ (for some fixed $t \neq 0$) constructed in Section 2.6. Given a curve class $\beta \in \mathrm{A}_1(\mathcal{X})$ the associated genus-zero Gopakumar–Vafa (GV) invariant

$$n_\beta = n_{\beta,\mathcal{X}} \in \mathbb{Z}_{\geq 0}$$

is defined as the number of complete curves in $\mathcal{X}_t$ of class β. By Corollary 2.10 this is zero if β is not a restricted positive root, and otherwise is equal to the number of intersection points of μ_t with the appropriate component of the discriminant locus, i.e.

$$n_\beta = \left|\mu_t^{-1}(\mathcal{D}_{\alpha,\mathbb{C}}/W_\jmath)\right| \tag{3.A}$$

where α is any positive root with $\pi_\jmath(\alpha) = \beta$. This number is independent of the choice of small perturbation μ_t.

In what follows, for a curve class β consider the dual hyperplane $\mathsf{H}_\beta \subseteq \Theta_\jmath$.

COROLLARY 3.1. *If $\beta \in \mathrm{A}_1(\mathcal{X})$ then n_β is non-zero if and only if β is a restricted positive root, equivalently if and only if H_β belongs to the enhanced finite arrangement $\mathcal{H}_\jmath$.*

Proof. This follows immediately from Corollary 2.10, together with the definition of enhanced finite arrangement in Section 2.3. □

Note that H_β and $\mathsf{H}_{2\beta}$ should be considered as different hyperplanes in the enhanced finite arrangement. See also the discussion in Section 2.3, and Remark 2.7.

Remark 3.2. It follows from Corollary 3.1 that there are only finitely many non-zero GV invariants. There is already a known range outside of which the GV invariants are guaranteed to vanish. Indeed, every simple root $i \in \Delta$ has an associated length δ_i, given by the coefficient of α_i in the maximal root, and writing $\beta = \Sigma_i m_i \mathrm{C}_i$ it is known that $n_\beta = 0$ unless $m_i \leq \delta_i$ for all i. However, this bound is far from sharp, while Corollary 3.1 provides a precise characterisation.

3.2 Gromov–Witten

We refer to [CK99, Section 7] for an introduction to Gromov–Witten theory. For every non-zero curve class $\beta \in \mathrm{A}_1(\mathcal{X})$ there is an associated genus-zero Gromov–Witten (GW) invariant

$$N_\beta = N_{\beta,\mathcal{X}} \in \mathbb{Q}$$

defined as the virtual degree of the corresponding moduli space of stable maps to $\mathcal{X}$. By deformation invariance this coincides with the virtual degree of the moduli space of stable maps to $\mathcal{X}_t$ for $t \neq 0$, as constructed in Section 2.6.

The latter space decomposes as a disjoint union of spaces of stable maps to $\mathbb{P}^1$, and applying the Aspinwall–Morrison multiple cover formula [AM93, Voi96] for the local invariants of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ gives the following relationship between the GW invariants and the GV invariants from Section 3.1, namely

$$N_\beta = \sum_{d|\beta} \frac{n_{\beta/d}}{d^3}. \tag{3.B}$$

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More generally, given $k \geq 0$ and homogeneous classes $\gamma_1, \dots, \gamma_k \in \mathrm{H}^\star(\mathcal{X}; \mathbb{C})$, the associated GW invariant with cohomological insertions at marked points is defined to be

$$\langle \gamma_1, \dots, \gamma_k \rangle^{\mathcal{X}}_{0,k,\beta} := \int_{[\overline{\mathcal{M}}_{0,k}(\mathcal{X},\beta)]^{\mathrm{virt}}} \prod_{i=1}^{k} \mathrm{ev}_i^\star \gamma_i$$

and provides a virtual count of rational curves in $\mathcal{X}$ of class β passing through the cycles $\gamma_1, \dots, \gamma_k$. Note that in particular $N_\beta = \langle \rangle^{\mathcal{X}}_{0,0,\beta}$. Since $\mathcal{X}$ is a Calabi–Yau 3-fold, the invariant vanishes unless the input data satisfies the dimension constraint

$$\sum_{i=1}^{k} \deg \gamma_i = 2k. \tag{3.C}$$

The cohomology of $\mathcal{X}$ is well-understood, see e.g. [Cai05, 5.2]. In particular

$$\mathrm{H}^0(\mathcal{X}; \mathbb{C}) = \mathbb{C} \cdot \mathbb{1}, \quad \mathrm{H}^1(\mathcal{X}; \mathbb{C}) = 0, \quad \mathrm{H}^2(\mathcal{X}; \mathbb{C}) = \mathrm{Pic}\, \mathcal{X} \otimes \mathbb{C}.$$

Moreover, as we work in the complete local setting, by e.g. [VdB04, 3.4.4] $\mathrm{Pic}\, \mathcal{X}$ is dual to the group $\mathrm{A}_1(\mathcal{X})$ of curve classes, since there is a basis of divisor classes $\mathrm{Pic}\, \mathcal{X} = \langle D_i \mid i \in \mathcal{I}^{\mathrm{c}} \rangle_{\mathbb{Z}}$ which satisfies $D_i \cdot \mathrm{C}_j = \delta_{ij}$.

Given a GW invariant $\langle \gamma_1, \dots, \gamma_k \rangle^{\mathcal{X}}_{0,k,\beta}$, if any $\gamma_i = \mathbb{1}$ then the invariant vanishes by the string equation. It follows from (3.C) that the invariant vanishes unless each $\gamma_i \in \mathrm{H}^2(\mathcal{X}; \mathbb{C})$. But then the γ_i are divisors, and the k-pointed invariants with divisorial insertions are related to the 0-pointed invariants by the divisor equation

$$\langle D_{j_1}, \dots, D_{j_k} \rangle^{\mathcal{X}}_{0,k,\beta} = \left(\prod_{i=1}^{k} D_{j_i} \cdot \beta \right) N_\beta.$$

In this way, the non-zero GW invariants are controlled entirely by the N_β, which by (3.B) are controlled entirely by the GV invariants n_β. The latter constitutes a finite list of numbers.

3.3 Quantum cohomology

As is well known, the GW invariants form the structure constants for quantum cohomology. The information defining quantum cohomology is equivalent to the *quantum potential*, defined in our setting as

$$\Phi^{\mathcal{X}}_{\mathsf{t}}(\gamma_1, \gamma_2, \gamma_3) := \sum_{\substack{\beta \in \mathrm{A}_1(\mathcal{X}) \\ \beta \neq 0}} \sum_{k \geq 0} \frac{1}{k!} \langle \gamma_1, \gamma_2, \gamma_3, \mathsf{t}, \dots, \mathsf{t} \rangle^{\mathcal{X}}_{0,k+3,\beta}. \tag{3.D}$$

Here we exclude the case $\beta = 0$ from the sum, since for non-compact $\mathcal{X}$ such invariants are not defined (see Remark 3.7). We view (3.D) as a family of multilinear maps

$$\Phi^{\mathcal{X}}_{\mathsf{t}} \colon \mathrm{H}^\star(\mathcal{X}; \mathbb{C})^{\otimes 3} \to \mathbb{C}$$

parametrised by the formal variable $\mathsf{t} \in \mathrm{H}^\star(\mathcal{X}; \mathbb{C})$. By the earlier dimension arguments, we see that the quantum potential only depends on the component of t in the $\mathrm{H}^2(\mathcal{X}; \mathbb{C})$ direction. Thus we may assume $\mathsf{t} \in \mathrm{H}^2(\mathcal{X}; \mathbb{C})$ and write

$$\mathsf{t} = (\mathsf{t}_i)_{i \in \mathcal{I}^{\mathrm{c}}} = \sum_{i \in \mathcal{I}^{\mathrm{c}}} \mathsf{t}_i D_i.$$

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The parameter space for the quantum potential is thus co-ordinatised by the t_i. An alternative co-ordinate system is given by the Novikov parameters, defined by

$$\mathsf{q}_i := \exp(\mathsf{t}_i).$$

The following result resembles expressions appearing in earlier work [Mor96, Wil92], but is more explicit, being given in terms of canonical bases for $\mathrm{H}^2(\mathcal{X};\mathbb{C})$ and $\mathrm{A}_1(\mathcal{X})$. This refined information will allow us in Corollary 3.4 to pinpoint the non-vanishing terms using Dynkin combinatorics and, in Corollary 5.13, to track the change in quantum potential under iterated flops.

THEOREM 3.3. *The quantum potential has a natural analytic continuation over the parameter space, given as a finite sum of terms indexed by the non-vanishing GV invariants*

$$\Phi^{\mathcal{X}}_{\mathsf{t}}(\gamma_1,\gamma_2,\gamma_3) = \sum_{\beta=(m_i)} n_\beta(\gamma_1\cdot\beta)(\gamma_2\cdot\beta)(\gamma_3\cdot\beta)\frac{\prod_{i\in\mathcal{J}^c}\mathsf{q}_i^{m_i}}{1-\prod_{i\in\mathcal{J}^c}\mathsf{q}_i^{m_i}}. \tag{3.E}$$

The sum is over non-zero curve classes

$$\beta = (m_i)_{i\in\mathcal{J}^c} = \sum m_i \mathrm{C}_i \in \mathrm{A}_1(\mathcal{X}).$$

Each term is a cubic polynomial in the input variables, multiplied by a specific rational function in the Novikov parameters and weighted by the GV invariant n_β. We will also use the term 'quantum potential' to refer to this analytic continuation.

Proof. Write the formal parameter t and the curve class β as sums

$$\mathsf{t} = \sum_{i\in\mathcal{J}^c} \mathsf{t}_i D_i, \qquad \beta = \sum_{i\in\mathcal{J}^c} m_i \mathrm{C}_i.$$

Applying the divisor equation together with the multiple cover formula (3.B) then gives

$$\begin{aligned}
\Phi^{\mathcal{X}}_{\mathsf{t}}(\gamma_1,\gamma_2,\gamma_3) &= \sum_\beta \sum_{k\geq 0} \frac{1}{k!}\langle\gamma_1,\gamma_2,\gamma_3,\mathsf{t},\dots,\mathsf{t}\rangle^{\mathcal{X}}_{0,k+3,\beta} \\
&= \sum_\beta \langle\gamma_1,\gamma_2,\gamma_3\rangle^{\mathcal{X}}_{0,3,\beta}\left(\sum_{k\geq 0}\tfrac{(\mathsf{t}\cdot\beta)^k}{k!}\right) \\
&= \sum_\beta N_\beta(\gamma_1\cdot\beta)(\gamma_2\cdot\beta)(\gamma_3\cdot\beta)\exp\left(\sum_{i\in\mathcal{J}^c} m_i\mathsf{t}_i\right) \\
&= \sum_\beta n_\beta \sum_{d\geq 1}\frac{1}{d^3}(\gamma_1\cdot d\beta)(\gamma_2\cdot d\beta)(\gamma_3\cdot d\beta)\exp\left(\sum_{i\in\mathcal{J}^c} dm_i\mathsf{t}_i\right) \\
&= \sum_\beta n_\beta(\gamma_1\cdot\beta)(\gamma_2\cdot\beta)(\gamma_3\cdot\beta)\sum_{d\geq 1}\exp\left(\sum_{i\in\mathcal{J}^c} m_i\mathsf{t}_i\right)^d \\
&= \sum_\beta n_\beta(\gamma_1\cdot\beta)(\gamma_2\cdot\beta)(\gamma_3\cdot\beta)\sum_{d\geq 1}(\Pi_{i\in\mathcal{J}^c}\mathsf{q}_i^{m_i})^d.
\end{aligned}$$

Note that this sum is finite, since $n_\beta = 0$ for all but finitely many β (Corollary 3.1). For fixed inputs $(\gamma_1,\gamma_2,\gamma_3)$, the above is a formal power series in the Novikov parameters. It is the Taylor series for the following rational function, expanded about the point $(\mathsf{q}_i)_i = (0,\dots,0)$, equivalently $(\mathsf{t}_i)_i = (-\infty,\dots,-\infty)$,

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$$\Phi_{\mathsf{t}}^{\mathcal{X}}(\gamma_1,\gamma_2,\gamma_3)=\sum_{\beta} n_\beta(\gamma_1\cdot\beta)(\gamma_2\cdot\beta)(\gamma_3\cdot\beta)\frac{\prod_{i\in\mathcal{J}^c}\mathsf{q}_i^{m_i}}{1-\prod_{i\in\mathcal{J}^c}\mathsf{q}_i^{m_i}}.$$

This expression provides a natural analytic continuation of the quantum potential beyond the radius of convergence $\{|\mathsf{q}_i|<1\mid i\in\mathcal{J}^c\}$. □

Recall that after combining (2.I) with (2.F), we can ask which curve classes are restricted roots.

Corollary 3.4. *Under the uniformly rescaled co-ordinates* $\mathsf{p}_i:=\mathsf{t}_i/2\pi\sqrt{-1}$ *on* $\mathrm{H}^2(\mathcal{X};\mathbb{C})$, *the pole locus of the quantum potential is given by*

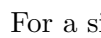
$$\bigcup_{\beta=(m_i)}\left\{\textstyle\sum_{i\in\mathcal{J}^c} m_i\mathsf{p}_i\in\mathbb{Z}\right\}$$

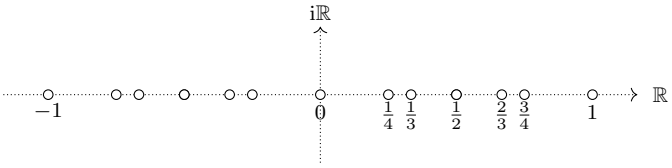
where the union is over all restricted positive roots β. *This is precisely the complexification of* $\mathcal{H}_{\mathcal{J}}^{\mathsf{aff}}$ *under the natural identification* $\mathrm{H}^2(\mathcal{X};\mathbb{R})\cong\Theta_{\mathcal{J}}$ *dual to (2.F).*

Proof. The pole locus of (3.E) is the union of loci in the parameter space given by

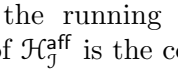
$$\prod_{i\in\mathcal{J}^c}\mathsf{q}_i^{m_i}=1\ \Leftrightarrow\ \sum_{i\in\mathcal{J}^c} m_i\mathsf{t}_i\in 2\pi\sqrt{-1}\cdot\mathbb{Z}\ \Leftrightarrow\ \sum_{i\in\mathcal{J}^c} m_i\mathsf{p}_i\in\mathbb{Z},$$

where $\beta=(m_i)_{i\in\mathcal{J}^c}=\sum m_i\mathrm{C}_i$ is such that n_β is non-zero. The first statement then follows from Corollary 3.1, since n_β is non-zero if and only if β is a restricted positive root. The second is an immediate consequence of the definition of $\mathcal{H}_{\mathcal{J}}^{\mathsf{aff}}$ in (2.D). □

Example 3.5. For a single-curve flop with  , the complexification of $\mathcal{H}_{\mathcal{J}}^{\mathsf{aff}}$ is



extended to infinity in both directions. The non-zero GV invariants are $n_{k\mathrm{C}}$ for $1\leq k\leq 4$.

Example 3.6. In the running Example 1.1, namely, a two-curve flop with  , the complexification of $\mathcal{H}_{\mathcal{J}}^{\mathsf{aff}}$ is the complexification of the real arrangement in (1.C).

Remark 3.7. The definition of the quantum cohomology algebra requires a perfect pairing on cohomology in order to raise indices, but since $\mathcal{X}$ is non-compact here, such a pairing does not exist. This technical issue is often circumvented by localising to a torus-fixed locus which is compact; see e.g. [BG08, CIJ18]. Since our geometries do not always carry a suitable torus action, instead we simply equate 'quantum cohomology' with the data of the quantum potential (3.D), as this is consistent with other approaches [LR01]. In cases where a natural quantum cohomology algebra can be defined, our results apply equally well to that algebra. The only modification required is to reinstate the $\beta=0$ terms in the quantum potential that encode the given perfect pairing.

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4. Flops via simultaneous partial resolutions

This section constructs flops and describes how their dual graph changes, via simultaneous partial resolutions, completing the work of Pinkham [Pin83]. As a consequence we obtain an explicit change-of-basis matrix, which in Section 5.1 is used to track the change of GV invariants under iterated flop.

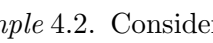
The construction requires some more Dynkin notation, so for a subset $\mathcal{I} \subseteq \Delta$, $j \in \mathcal{I}$ and $i \in \mathcal{I}^c$, write

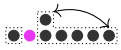
$$\mathcal{I} + i = \mathcal{I} \cup \{i\} \quad \text{and} \quad \mathcal{I} - j = \mathcal{I} \setminus \{j\}.$$

Further, to every Dynkin diagram Γ is an associated *Dynkin involution*, which we will denote ι_Γ. For Type A_n and E_6, this is the obvious reflection, for E_7 and E_8 it is trivial, and for D_n the behaviour depends on the parity of n; see e.g. [IW, (1.2.B)]. If Γ is a disjoint union of Dynkin diagrams, then ι_Γ by definition acts separately on each component. Further, if Δ is ADE and if Γ is a subset of Δ, then automatically the subgraph Γ is a disjoint union of ADE diagrams, so there is an associated ι_Γ.

Notation 4.1 ([IW, 1.16)]. For $i \in \mathcal{I}^c$, the wall crossing $\omega_i(\mathcal{I})$ is defined by the rule

$$\omega_i(\mathcal{I}) := \mathcal{I} + i - \iota_{\mathcal{I}+i}(i) \subseteq \Delta.$$

Example 4.2. Consider the running Example 1.1, namely,  , where by convention, $\mathcal{I}$ equals the six black dots. There are two choices for $i \in \mathcal{I}^c$, namely, the two pink nodes. Let i be the rightmost. Then $\mathcal{I} + i$ equals the black dots in the following:

$\mathcal{I} + i =$ 

The black dots form $A_1 \times A_6$, so applying the Dynkin involution $\iota_{\mathcal{I}+i}$ illustrated, we see that $\iota_{\mathcal{I}+i}(i)$ is the top node. Thus, for this choice of i,

$$\omega_i(\mathcal{I}) = \mathcal{I} + i - \iota_{\mathcal{I}+i}(i) = \text{}.$$

Consider now the fixed flopping contraction $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, which slices under (2.A) to give $Y_{\mathcal{I}} \to \mathbb{C}^2/G$. Pick a flopping curve C_i in $\mathcal{X}$. This corresponds to a choice of $i \in \mathcal{I}^c$, so we can form $\omega_i(\mathcal{I})$. In what follows, consider $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}$, where $\ell_{\mathcal{I}}$ and $\ell_{\mathcal{I}+i}$ are the longest elements in the parabolic subgroups $W_{\mathcal{I}}$ and $W_{\mathcal{I}+i}$, respectively.

LEMMA 4.3. *The left action by $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}$ induces an isomorphism $\mathfrak{h}_{\mathbb{C}}/W_{\omega_i(\mathcal{I})} \to \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}}$.*

Proof. Writing $w = \ell_{\mathcal{I}}\ell_{\mathcal{I}+i}$, the point is that $wW_{\omega_i(\mathcal{I})} = W_{\mathcal{I}}w$ (see e.g. [IW, 1.20(1)(a)]). The action by w is an isomorphism $\mathfrak{h}_{\mathbb{C}} \to \mathfrak{h}_{\mathbb{C}}$, and under this isomorphism, any orbit $(W_{\omega_i(\mathcal{I})})p$ gets sent to $w(W_{\omega_i(\mathcal{I})})p = (W_{\mathcal{I}})wp$, which is an orbit under $W_{\mathcal{I}}$. □

In what follows, we will always consider left actions. When labelling arrows in commutative diagrams, we will often write e.g. $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}\cdot$ to denote the morphism given by left multiplication by $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}$.

Note that $\ell_{\omega_i(\mathcal{I})}\ell_{\mathcal{I}+i}\ell_{\mathcal{I}}\ell_{\mathcal{I}+i} = 1$ [IW, 1.2(3)], and thus $\ell_{\omega_i(\mathcal{I})}\ell_{\mathcal{I}+i} \colon \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{I}} \to \mathfrak{h}_{\mathbb{C}}/W_{\omega_i(\mathcal{I})}$ is the inverse map. Recall the notation in (2.E). Using the universal property of the pullback gives

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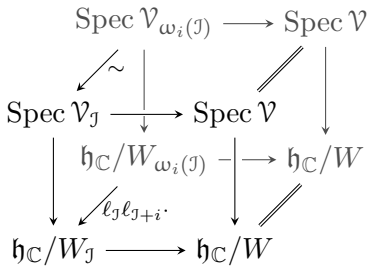
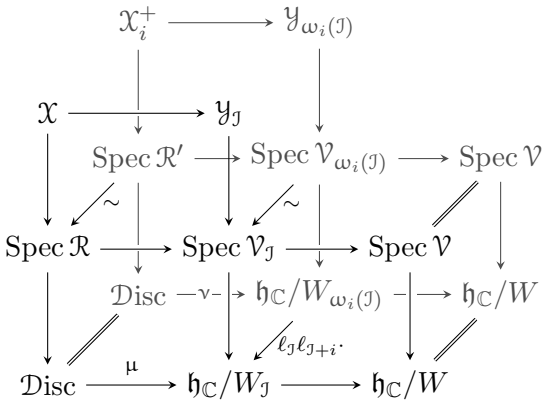
a non-obvious isomorphism between $\mathcal{V}_{\mathcal{J}}$ and $\mathcal{V}_{\omega_i(\mathcal{J})}$, which sits in the following commutative diagram:



Now let $\mathcal{Y}_{\mathcal{J}}$, respectively $\mathcal{Y}_{\omega_i(\mathcal{J})}$, be the standard simultaneous resolution associated to $\mathcal{J}$, respectively $\omega_i(\mathcal{J})$. As explained in Section 2.6, $\mathcal{X}$ can be obtained from $\mu\colon \mathcal{D}\mathrm{isc} \to \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{J}}$ by pulling back $\mathcal{Y}_{\mathcal{J}}$. Hence, setting $\nu = (\ell_{\omega_i(\mathcal{J})}\ell_{\mathcal{J}+i}) \circ \mu = (\ell_{\mathcal{J}}\ell_{\mathcal{J}+i})^{-1} \circ \mu$ and pulling back to $\mathcal{Y}_{\omega_i(\mathcal{J})}$ constructs a variety $\mathcal{X}_i^+$ sitting within the following commutative diagram:



(4.A)

Composing the map $\mathcal{X}_i^+ \to \operatorname{Spec}\mathcal{R}'$ with the isomorphism $\operatorname{Spec}\mathcal{R}' \to \operatorname{Spec}\mathcal{R}$ yielding a morphism $\mathcal{X}_i^+ \to \operatorname{Spec}\mathcal{R}$.

Theorem 4.4. *With notation as above, $\mathcal{X}_i^+ \to \operatorname{Spec}\mathcal{R}$ is the flop of $\mathcal{X}$ at the curve C_i. In particular, the following statements hold:*

1. *$\omega_i(\mathcal{J}) \subseteq \Delta$ is the Dynkin data associated to the flopping contraction $\mathcal{X}_i^+ \to \operatorname{Spec}\mathcal{R}$.*
2. *All other crepant resolutions of $\operatorname{Spec}\mathcal{R}$ can be obtained from the fixed μ by post-composing with $x^{-1}\colon \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{J}} \to \mathfrak{h}_{\mathbb{C}}/W_{\mathcal{K}}$ and pulling back along $\mathcal{Y}_{\mathcal{K}}$, as the pair $(x, \mathcal{K})$ ranges over the (finite) indexing set $\mathsf{Cham}(\Delta, \mathcal{J})$ of Notation 2.2.*

Proof. Since the exceptional locus of $\mathsf{g}_{\mathcal{J}}\colon \mathcal{Y}_{\mathcal{J}} \to \operatorname{Spec}\mathcal{V}_{\mathcal{J}}$ has codimension two, $\mathrm{Cl}(\mathcal{V}_{\mathcal{J}}) \cong \mathrm{Cl}(\mathcal{Y}_{\mathcal{J}})$. But $\mathcal{Y}_{\mathcal{J}}$ is smooth, so the latter is isomorphic to $\mathrm{Pic}(\mathcal{Y}_{\mathcal{J}})$, which in turn is isomorphic to $\mathbb{Z}^{|\mathcal{J}^c|}$ based on divisors dual to the $|\mathcal{J}^c|$ curves above the origin. Choosing this basis, we may write $\mathrm{Cl}(\mathcal{V}_{\mathcal{J}}) \cong \bigoplus_{j\in\mathcal{J}^c} \mathbb{Z}e_j^\star$, and, further, by definition $\mathcal{Y}_{\mathcal{J}}$ is obtained by blowing up any element in the characteristic cone

$$C_{\mathcal{J}} := \{(z_j) \mid z_j > 0 \text{ for all } j\} \subseteq \bigoplus_{j\in\mathcal{J}^c} \mathbb{Z}e_j^\star.$$

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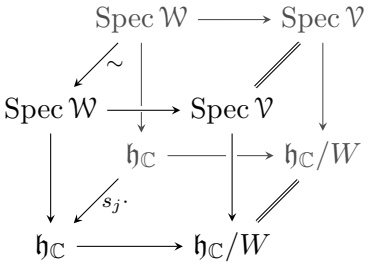
Applying a similar analysis to the standard simultaneous resolution $\mathcal{Z} \to \operatorname{Spec} \mathcal{W}$, we have $\operatorname{Cl}(\mathcal{W}) \cong \bigoplus_{j\in\Delta} \mathbb{Z}e_j^\star$, and since by construction $Y_\mathcal{I}$ is obtained from $\mathcal{Z}$ by simultaneously blowing down curves, it is clear that

$$\begin{array}{ccc} \operatorname{Cl}(\mathcal{V}_\mathcal{I}) & \hookrightarrow & \operatorname{Cl}(\mathcal{W}) \\ \sim\downarrow & & \sim\downarrow \\ \bigoplus_{j\in\mathcal{I}^c} \mathbb{Z}e_j^\star & \hookrightarrow & \bigoplus_{j\in\Delta} \mathbb{Z}e_j^\star \end{array} \tag{4.B}$$

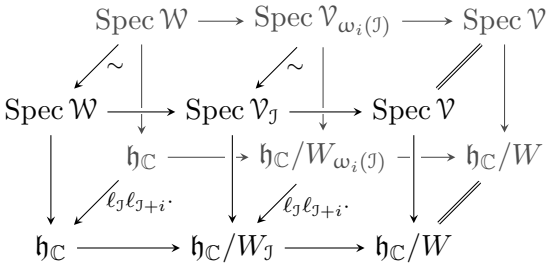
where the bottom morphism is the obvious inclusion induced by the inclusion $\mathcal{I}^c \subseteq \Delta$. There is a natural identification of $\operatorname{Cl}(\mathcal{V}_\mathcal{I})$, respectively $\operatorname{Cl}(\mathcal{W}_\mathcal{I})$, with the lattice inside $\Theta_\mathcal{I}$, respectively Θ.

By the universal property of the pullback, the action of any $s_j \in W$ on $\mathfrak{h}_\mathbb{C}$ induces an automorphism of $\mathcal{W}$.

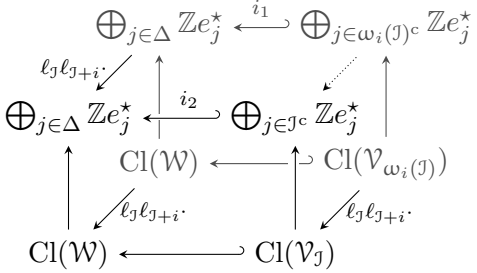


The effect on homology of $\mathcal{Z}$ is via the Weyl reflection s_j on roots [Rei83], and thus the action on $\operatorname{Cl}(\mathcal{W})$ is the dual, namely, the action of the Weyl reflection s_j on coroots.

Composing these s_j, we can consider the action of $\ell_\mathcal{I}\ell_{\mathcal{I}+i}$. We already know that the bottom-right square in (4.A) commutes, and hence the bottom two squares in the following are well defined and commute.



The universal property of pullbacks give the induced isomorphisms in the top squares. The top-left square induces the following bottom commutative square on class groups, and the top square is obtained by applying (4.B) to both $\mathcal{I}$ and to $\omega_i(\mathcal{I})$.



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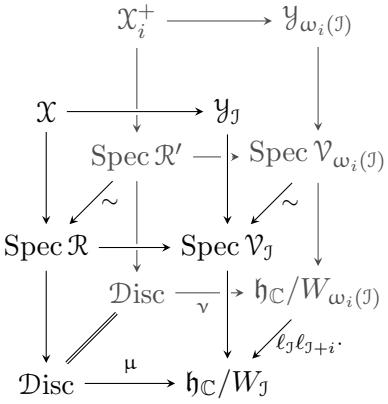
In the above diagram, every non-hooked arrow is an isomorphism. By construction, the dotted arrow takes the characteristic cone

$$C_{\omega_i(\mathcal{J})} := \{(z_j) \mid z_j > 0 \text{ for all } j\} \subseteq \bigoplus_{j \in \omega_i(\mathcal{J})^c} \mathbb{Z}e_j^\star$$

of $Y_{\omega_i(\mathcal{J})}$ to the region $\ell_{\mathcal{J}}\ell_{\mathcal{J}+i}C_{\omega_i(\mathcal{J})} \subseteq \bigoplus_{j\in\mathcal{J}^c} \mathbb{Z}e_j^\star$. Since $\ell_{\mathcal{J}}\ell_{\mathcal{J}+i}$ acts via the action on coroots, this matches the conventions in [IW, Section 3]. It thus immediately follows from [IW 1.20(1)(d)] that $\ell_{\mathcal{J}}\ell_{\mathcal{J}+i}C_{\omega_i(\mathcal{J})}$ and $C_{\mathcal{J}}$ are neighbouring regions, adjacent via the wall $z_i = 0$.

We next restrict this information to 3-folds. As in (4.A), consider the following commutative diagram.



(4.C)

As explained by Pinkham [Pin83], $\mathrm{Cl}(\mathcal{R}) \cong \mathrm{Cl}(\mathcal{V}_{\mathcal{J}})$, and further $\mathrm{Cl}(\mathcal{V}_{\mathcal{J}}) \cong \bigoplus_{j\in\mathcal{J}^c} \mathbb{Z}e_j^\star$ as explained above. Hence $\mathrm{Cl}(\mathcal{R}) \cong \bigoplus_{j\in\mathcal{J}^c} \mathbb{Z}e_j^\star$, and under *this* choice of basis $\mathcal{X}$ is obtained as the blowup of the characteristic cone $C_{\mathcal{J}}$. The same analysis holds for $\mathcal{X}_i^+$, which is the blowup of the characteristic cone $C_{\omega_i(\mathcal{J})}$ under the choice of basis $\mathrm{Cl}(\mathcal{R}') \cong \bigoplus_{j\in\omega_i(\mathcal{J})^c} \mathbb{Z}e_j^\star$ induced from $\mathrm{Cl}(\mathcal{R}') \cong \mathrm{Cl}(\mathcal{V}_{\omega_i(\mathcal{J})})$.

Pulling across the middle horizontal plane in (4.C) it thus follows that the map $\mathcal{X}_i^+ \to \operatorname{Spec}\mathcal{R}$ is obtained by blowing up the region $\ell_{\mathcal{J}}\ell_{\mathcal{J}+i}C_{\omega_i(\mathcal{J})}$ in $\mathrm{Cl}(\mathcal{R}) \cong \bigoplus_{j\in\mathcal{J}^c} \mathbb{Z}e_j^\star$. Since these are neighbouring regions, separated by the codimension one wall $e_i^\star = 0$, it is implicit in [Pin83] (see also [Wem18]) that $\mathcal{X}_i \to \operatorname{Spec}\mathcal{R}$ is the flop at the curve C_i. Since $\mathcal{X}_i^+$ is obtained from $\mathcal{Y}_{\omega_i(\mathcal{J})}$ via pullback, the statement on Dynkin data follows.

The final statement about all other crepant resolutions follows by iterating over all possible simple flops. Indeed, the finite indexing set $\mathsf{Cham}(\Delta, \mathcal{I})$ is precisely the combinatorial object which indexes all the chambers [IW, Section 1], and each chamber $(x, \mathcal{K})$ can be obtained from $(1, \mathcal{I})$ by iteratively applying the wall crossing rule [IW, 1.20(2)]. □

5. Applications

As before, consider a smooth 3-fold flopping contraction $\mathcal{X} \to \operatorname{Spec}\mathcal{R}$. The main applications of the previous sections are to GV and GW invariants, the Crepant Transformation Conjecture, and to the associated contraction algebras.

5.1 Tracking GV invariants under flop

The benefit of Theorem 4.4 is that both $\mathcal{X}$ and a flop $\mathcal{X}_i^+$ can be perturbed using essentially the same classifying map, and thus their curve invariants can be easily compared. This requires three combinatorial results.

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LEMMA 5.1. *$\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}\colon \mathfrak{h}\to\mathfrak{h}$ induces an isomorphism $\mathsf{M}_i\colon \mathfrak{h}_{\omega_i(\mathcal{I})}\to\mathfrak{h}_{\mathcal{I}}$.*

Proof. Set $w=\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}$ then it suffices to prove that $w\cdot$ restricts to an isomorphism between the subspace spanned by $\{\alpha_j \mid j\in\omega_i(\mathcal{I})\}$ and the subspace spanned by $\{\alpha_j \mid j\in\mathcal{I}\}$. To see this, recall that $\ell_\Gamma\alpha_i=-\alpha_{\iota_\Gamma(i)}$ for all Dynkin Γ and all $i\in\Gamma$, where ι_Γ is the Dynkin involution on Γ.

For all $j\in\omega_i(\mathcal{I})\subseteq\mathcal{I}+i$, it follows that $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}\alpha_j=-\ell_{\mathcal{I}}\alpha_{\iota_{\mathcal{I}+i}(j)}$. Now $\iota_{\mathcal{I}+i}\colon \omega_i(\mathcal{I})\to\mathcal{I}$ is a bijection. Indeed, $\iota_{\mathcal{I}+i}\colon \mathcal{I}+i\to\mathcal{I}+i$ is a bijection, sending $\iota_{\mathcal{I}+i}(i)$ to i, and so removing these elements gives the claimed bijection. Hence for all $j\in\omega_i(\mathcal{I})$, it follows that $\iota_{\mathcal{I}+i}(j)\in\mathcal{I}$ and so $\ell_{\mathcal{I}}\alpha_{\iota_{\mathcal{I}+i}(j)}=-\alpha_{\iota_{\mathcal{I}}\iota_{\mathcal{I}+i}(j)}$. Combining gives $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}\alpha_j=\alpha_{\iota_{\mathcal{I}}\iota_{\mathcal{I}+i}(j)}$ for all $j\in\omega_i(\mathcal{I})$. Since $\iota_{\mathcal{I}}\iota_{\mathcal{I}+i}(j)\in\mathcal{I}$, this proves the claim. □

Write M_i for the induced isomorphism in Lemma 5.1, so that the following diagrams commute.

$$\begin{array}{ccc} \mathfrak{h} & \xrightarrow{\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}\cdot} & \mathfrak{h} \\ {\scriptstyle \pi_{\omega_i(\mathcal{I})}}\downarrow & & \downarrow{\scriptstyle \pi_{\mathcal{I}}} \\ \mathfrak{h}_{\omega_i(\mathcal{I})} & \xrightarrow{\mathsf{M}_i} & \mathfrak{h}_{\mathcal{I}} \end{array} \qquad \begin{array}{ccc} \mathfrak{h} & \xrightarrow{\ell_{\omega_i(\mathcal{I})}\ell_{\mathcal{I}+i}\cdot} & \mathfrak{h} \\ {\scriptstyle \pi_{\mathcal{I}}}\downarrow & & \downarrow{\scriptstyle \pi_{\omega_i(\mathcal{I})}} \\ \mathfrak{h}_{\mathcal{I}} & \xrightarrow{\mathsf{M}_i^{-1}} & \mathfrak{h}_{\omega_i(\mathcal{I})} \end{array} \tag{5.A}$$

In essence, the bases of $\mathfrak{h}_{\omega_i(\mathcal{I})}$ and $\mathfrak{h}_{\mathcal{I}}$ really differ at only one element. Indeed, setting $e_j=\pi_{\mathcal{I}}(\alpha_j)$, then $\{e_j \mid j\notin\mathcal{I}\}$ is a basis for $\mathfrak{h}_{\mathcal{I}}$. On the other hand, for $\mathfrak{h}_{\omega_i(\mathcal{I})}$ we abuse notation, setting $e_j=\pi_{\omega_i(\mathcal{I})}(\alpha_j)$ whenever $j\notin\mathcal{I}+i$, and $e_i=\pi_{\omega_i(\mathcal{I})}(\alpha_{\iota_{\mathcal{I}+i}(i)})$. Then $\{e_j, e_i \mid j\notin\mathcal{I}+i\}$ is a basis for $\mathfrak{h}_{\omega_i(\mathcal{I})}$.

LEMMA 5.2. *For $i\in\mathcal{I}^{\mathrm{c}}$ the action of M_i is given in terms of the above bases as*

$$e_k\mapsto\begin{cases} e_k+\lambda_k e_i & \text{if } k\notin\mathcal{I}+i \\ -e_i & \text{if } k=i \end{cases}$$

for some $\lambda_k\in\mathbb{Z}_{\geq 0}$.

Proof. In (5.A), given any $\sum a_i\alpha_i$, since $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}$ consists only of reflections s_i with $i\in\mathcal{I}+i$, the map $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}$ cannot change the coefficient of any a_j with $j\notin\mathcal{I}+i$. The claim that the induced map M_i sends $e_k\mapsto e_k+\lambda_k e_i$ if $k\notin\mathcal{I}+i$ follows. We next claim that λ_k is positive. In the decomposition of $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}\alpha_k$ into simple roots, there is at least some positive coefficient (namely, the coefficient of α_k, which is 1). Hence, all coefficients must be positive, in particular, the coefficient of $\alpha_{\iota_{\mathcal{I}+i}(i)}$. But under the induced map, this coefficient is what gives λ_k in the claim.

Now, as in Lemma 5.1, for all Dynkin Γ and all $i\in\Gamma$, $\ell_\Gamma\alpha_i=-\alpha_{\iota_\Gamma(i)}$. Thus since $\iota_{\mathcal{I}+i}(i)\in\mathcal{I}+i$, it follows that

$$\mathsf{M}_i e_i \overset{(5.A)}{=} \pi_{\mathcal{I}}(\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}\alpha_{\iota_{\mathcal{I}+i}(i)})=\pi_{\mathcal{I}}(-\ell_{\mathcal{I}}\alpha_i)=\pi_{\mathcal{I}}(-\alpha_i)=-e_i,$$

where we have used the facts that $\ell_{\mathcal{I}}$ changes coefficients only in $\mathcal{I}$ and that $\pi_{\mathcal{I}}$ forgets these. □

COROLLARY 5.3. *If $\beta\in\mathfrak{h}_{\omega_i(\mathcal{I})}$ is a restricted root, the following hold.*

1. *If $\beta\in\mathbb{Z}e_i$, say $\beta=ze_i$, then $\mathsf{M}_i\cdot(ze_i)=-ze_i$.*
2. *If $\beta\notin\mathbb{Z}e_i$, then all entries of $\mathsf{M}_i\cdot\beta$ are positive.*

Proof. The first part is an immediate consequence of the $k=i$ case in Lemma 5.2. For the second part, by Lemma 5.2 the only coefficient of $\beta=\sum\mu_i e_i$ that can change under M_i is the coefficient on e_i. Hence, provided there is some other positive coefficient μ_k, this survives under

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M_i, so $\mathsf{M}_i \cdot \beta$ has at least one positive entry. Now, by assumption, β is a restricted root, say, $\pi_{\omega_i(\mathcal{J})}(\alpha) = \beta,$. Under (5.A), $w \cdot \alpha$ is a root restricting to $\mathsf{M}_i \cdot \beta$, and this root $w \cdot \alpha$ must contain at least one positive coefficient, since $\mathsf{M}_i \cdot \beta$ does. Hence, all must be positive. In particular, all entries of $\mathsf{M}_i \cdot \beta$ must also be positive. □

The following is one of our main results. In order to obtain a unified statement, set $|\mathsf{M}_i \cdot \beta|$ to be the curve class obtained from $\mathsf{M}_i \cdot \beta$ by making every coefficient positive.

Theorem 5.4. *With the notation as above, for any curve class* $\beta \in \mathrm{A}_1(\mathcal{X}_i^+) \cong \mathfrak{h}_{\omega_i(\mathcal{J})}$,

$$\begin{aligned} n_{\beta,\mathcal{X}_i^+} &= \begin{cases} n_{\beta,\mathcal{X}} & \text{if } \beta \in \mathbb{Z}e_i = \mathbb{Z}\mathrm{C}_i^+ \\ n_{\mathsf{M}_i\cdot\beta,\mathcal{X}} & \text{else} \end{cases} \\ &= n_{|\mathsf{M}_i\cdot\beta|,\mathcal{X}}. \end{aligned}$$

Proof. With respect to the notation in (4.A), perturbing μ to μ_t gives, by composition, a perturbation of ν to ν_t.

Set $w = \ell_{\mathcal{J}}\ell_{\mathcal{J}+i}$. Then for any positive root α for which $\pi_{\omega_i(\mathcal{J})}(\alpha) = \beta$,

$$\begin{aligned} n_{\beta,\mathcal{X}_i^+} &= \left|\nu_t^{-1}(\mathcal{D}_{\alpha,\mathbb{C}}/W_{\omega_i(\mathcal{J})})\right| && \text{(by (3.A) applied to } \mathcal{X}_i^+) \\ &= \left|\mu_t^{-1}(\mathcal{D}_{w\cdot\alpha,\mathbb{C}}/W_{\mathcal{J}})\right|. && \text{(since } \nu = (w\cdot)^{-1}\circ\mu) \end{aligned}$$

Now by (5.A) we have $\pi_{\mathcal{J}}(w \cdot \alpha) = \mathsf{M}_i \circ \pi_{\omega_i(\mathcal{J})}(\alpha) = \mathsf{M}_i \cdot \beta$, so $w \cdot \alpha$ is a lift of $\mathsf{M}_i \cdot \beta$, albeit not necessarily a positive one.

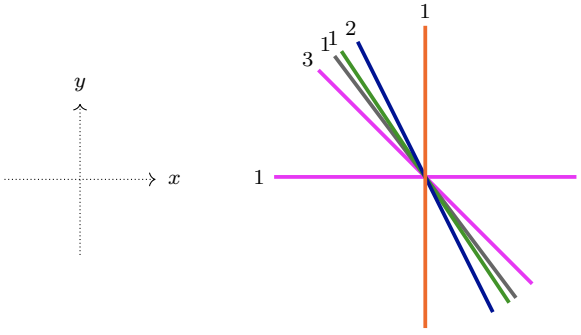
Case 1. If $\beta \notin \mathbb{Z}\mathrm{C}_i^+$, then by Corollary 5.3 all entries of $\mathsf{M}_i \cdot \beta$ are positive, and further as argued in the proof, $w \cdot \alpha$ is a positive root restricting to $\mathsf{M}_i \cdot \beta$. (3.A) then implies that $|\mu_t^{-1}(\mathcal{D}_{w\cdot\alpha,\mathbb{C}}/W_{\mathcal{J}})| = n_{\mathsf{M}_i\cdot\beta,\mathcal{X}}$.

Case 2. If $\beta \in \mathbb{Z}\mathrm{C}_i^+$, then by Corollary 5.3, $\mathsf{M}_i \cdot \beta = -\beta$. Arguing as above, it follows that $w \cdot \alpha$ is negative root restricting to $\mathsf{M}_i \cdot \beta = -\beta$, and thus $-w \cdot \alpha$ is positive root restricting to β. But negating a root does not affect the hyperplane, and combining this fact with (3.A), it follows that

$$|\mu_t^{-1}(\mathcal{D}_{w\cdot\alpha,\mathbb{C}}/W_{\mathcal{J}})| = |\mu_t^{-1}(\mathcal{D}_{-w\cdot\alpha,\mathbb{C}}/W_{\mathcal{J}})| = n_{\beta,\mathcal{X}}.$$

This covers both cases. For the final equality, note in case 1 that $|\mathsf{M}_i \cdot \beta| = \mathsf{M}_i \cdot \beta$ since all coefficients are already positive, and in case 2 note that $|\mathsf{M}_i \cdot \beta| = |-\beta| = \beta$. □

Example 5.5. Consider the running Example 1.1. Then after flop of the right pink curve, by Theorem 4.4 and Example 4.2 we obtain  . Hence, the restricted roots and thus the curve classes giving non-zero GV invariants, on the flopped space $\mathcal{X}_i^+$, are as follows, where the hyperplanes are drawn in $\Theta_{\omega_i(\mathcal{J})}$:



Restricted Root	
01	
11, 22, 33	
43	
32	
21, 42	
10	

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Write 1 for the leftmost pink node, 2 for the orange node, and $2'$ for the rightmost pink node (in $\mathcal{I}$). Under this wall crossing, $\ell_{\mathcal{I}}\ell_{\mathcal{I}+i}$ is very large; however the morphism

$$\mathsf{M}_i\colon \mathfrak{h}_{\omega_i(\mathcal{I})} \to \mathfrak{h}_{\mathcal{I}}$$

is easily described: in the notation of Lemma 5.2, $\lambda_1 = 1$ and thus M_i sends $\mu_1 e_1 + \mu_2 e_2 \mapsto \mu_1 e_1 + (\mu_1 - \mu_2)e_{2'}$. Under the dual transformation between the hyperplane arrangements in Example 1.1 and here, the pictures are drawn so that hyperplanes are sent to hyperplanes in such a way that the colours are preserved.

Indeed, M_i sends $01 \mapsto 0\,{-1}$, with all other restricted roots being permuted; e.g. $31 \mapsto 32$. In particular, by Theorem 5.4, the GV invariants on $\mathcal{X}_i^+$ can be obtained from the GV invariants on $\mathcal{X}$ as follows:

GV on $\mathcal{X}_i^+$	GV on $\mathcal{X}$	
01	01	–
10	11	–
$21, 42$	$21, 42$	–
32	31	–
43	41	–
$11, 22, 33$	$10, 20, 30$	–

Remark 5.6. As explained in the introduction, the finite arrangement $\mathcal{H}_{\mathcal{I}}$ equals the movable cone. The multiplicities of the restricted roots are assigned to each wall, and this enhancement is required in order to describe the curve-counting invariants. It is possible, albeit not a priori obvious, to enhance the movable cone without Dynkin combinatorics. Given a chamber corresponding to some crepant resolution $\mathcal{X}^\dagger \to \operatorname{Spec} \mathcal{R}$, say, then the multiplicities on the walls of that chamber turn out to correspond to the lengths of all the individual single-curve contractions obtained from $\mathcal{X}^\dagger$. The issue with this method is that, whilst it explains walls, it does not explain hyperplanes: it is not so clear that every chamber touching the hyperplane containing the said wall should be enriched with the same scheme-theoretic length. This geometric fact falls out from our approach.

5.2 Tracking fundamental regions

The previous subsection tracked GV invariants from $\mathcal{X}$ to $\mathcal{X}_i^+$. As with the movable cone, it is possible to fix $\mathcal{X}$ and track all other crepant resolutions back to $\mathcal{X}$.

As notation, recall that the fixed $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ has an associated $\Theta_{\mathcal{I}}$ in Notation 2.2, and recall from (5.A) that there is a map $\mathsf{M}_i^{-1}\colon \mathfrak{h}_{\mathcal{I}} \to \mathfrak{h}_{\omega_i(\mathcal{I})}$. Write

$$\mathsf{N}_i\colon \Theta_{\omega_i(\mathcal{I})} \to \Theta_{\mathcal{I}}$$

for the dual. Below, $\Theta_{\mathcal{I}}$ will be temporarily be written $\Theta_{\mathcal{X}}$, to allow for the flexibility of considering another crepant resolution $\mathcal{Y} \to \operatorname{Spec} \mathcal{R}$, which has associated $\Theta_{\mathcal{Y}}$.

DEFINITION 5.7. *Let $\mathcal{Y} \to \operatorname{Spec} \mathcal{R}$ be a crepant resolution. Consider a chain of flops, each flopping a single irreducible curve, that links $\mathcal{Y}$ to $\mathcal{X}$, and the resulting maps*

$$\Theta_{\mathcal{Y}} \xrightarrow{\mathsf{N}_{i_1}} \dots \xrightarrow{\mathsf{N}_{i_t}} \Theta_{\mathcal{X}}.$$

The composition will be called the comparison map and will be written $\mathsf{N}\colon \Theta_{\mathcal{Y}} \to \Theta_{\mathcal{X}}$.

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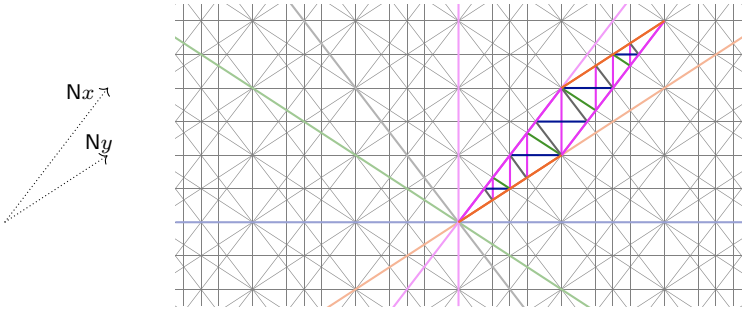
By [HW23, 4.8], the comparison map N is independent of the choice of chain of flops.

Definition 5.8. *Given a crepant resolution $\mathcal{Y} \to \operatorname{Spec} \mathcal{R}$, the fundamental region $\mathsf{Fund}_{\mathcal{Y}}$ of $\Theta_{\mathcal{Y}}$ is defined as the intersection of the infinite hyperplane arrangement inside $\Theta_{\mathcal{Y}}$, with the unit box $\{(\vartheta_i) \mid 0 \leq \vartheta_i \leq 1 \text{ for all } i\}$.*

Proposition 5.9. *For any crepant resolution $\mathcal{Y} \to \operatorname{Spec} \mathcal{R}$, $\mathsf{N}(\mathsf{Fund}_{\mathcal{Y}})$ generates $\Theta_{\mathcal{J}}$ via translation. Furthermore, two $\mathsf{N}(\mathsf{Fund}_{\mathcal{X}_1})$ and $\mathsf{N}(\mathsf{Fund}_{\mathcal{X}_2})$ share a codimensional wall if and only if $\mathcal{X}_1$ and $\mathcal{X}_2$ are connected by a flop at a single curve.*

Proof. Since the axes belong to the finite hyperplane arrangement in $\Theta_{\mathcal{Y}}$ and the definition of the infinite arrangement involves translating this finite collection of hyperplanes over $\mathbb{Z}$ or at worst $\frac{1}{k}\mathbb{Z}$ (see Subsections 1.2 and 2.3), it is clear that the fundamental region $\mathsf{Fund}_{\mathcal{Y}}$ generates the arrangement in $\Theta_{\mathcal{Y}}$. The first statement then follows, since N is known to preserve the infinite arrangements [IW, Section 9]. Since the only codimensional wall that the fundamental regions can share belong to the finite arrangement, the last statement is really a statement on the movable cone, which is e.g. [Wem18, Sections 5–6]. □

Example 5.10. Write $\mathcal{Y} \to \operatorname{Spec} \mathcal{R}$ for the crepant resolution obtained after flop in Example 5.5. Then the region $\mathsf{N}(\mathsf{Fund}_{\mathcal{Y}})$ is illustrated below, where for clarity we have illustrated the images of the x and y co-ordinates in Example 5.5 under the map N.



It is visually clear that both $\mathsf{Fund}_{\mathcal{X}}$ in (1.D) and $\mathsf{N}(\mathsf{Fund}_{\mathcal{Y}})$ above individually generate $\mathcal{H}_{\mathcal{J}}^{\mathsf{aff}}$, via translation, and that $\mathsf{Fund}_{\mathcal{X}}$ and $\mathsf{N}(\mathsf{Fund}_{\mathcal{Y}})$ are different.

Remark 5.11. The above example gives a visual proof of the Crepant Transformation Conjecture of Section 5.3. The regions $\mathsf{Fund}_{\mathcal{X}}$ and $\mathsf{N}(\mathsf{Fund}_{\mathcal{Y}})$ are different. But they generate the same object, namely, $\mathcal{H}_{\mathcal{J}}^{\mathsf{aff}}$, which by Corollary 3.4 is the pole locus of the GW quantum potential. Thus, although the curve invariants of $\mathcal{X}$ and $\mathcal{Y}$, captured in the fundamental regions, are technically different, after a change in variables (namely, N), they can be compared, where they generate the same object.

Remark 5.12. The matrix N_i appears via moduli tracking in the HomMMP [Wem18, 5.4] and via the K-theory of contraction algebras [AW22, 2.4]. In contrast, M_i from (5.A) is the dual, and it arises from the change in dimension vector [Wem18, 5.4], or in the K-theory of projective modules [HW23, 3.2]. For more details, see [AW22, 2.4] and references therein.

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5.3 Crepant transformation conjecture

We make no attempt at a comprehensive summary of the Crepant Transformation Conjecture (CTC) and instead refer the reader to [CR13, CIJ18, BG09, Lee11]. For a pair of smooth varieties related via a sequence of flops, the conjecture asserts that their quantum potentials should coincide, under a suitable identification of (co)homologies and analytic continuation in the Novikov parameters. There has been extensive work on this conjecture within both algebraic and symplectic geometry [LR01, McL20, LLW10, LLW16a, LLW16b, LLQW16].

Here we prove the CTC for germs of isolated 3-fold flops, as a direct application of the expression for the quantum potential in Theorem 3.3, together with the construction of flops via simultaneous partial resolutions in Theorem 4.4. This gives the first algebraic-geometric proof of the CTC for flops of arbitrary type (for recent symplectic developments; see [McL20]).

As before, consider a curve $\mathrm{C}_i \subseteq \mathcal{X}$, and let $\mathcal{X}_i^+$ be the flop of $\mathcal{X}$ at C_i. Recall that the following vector spaces are based on the sets of exceptional curves:

$$\mathfrak{h}_{\mathcal{I},\mathbb{C}} = \mathrm{A}_1(\mathcal{X})_{\mathbb{C}} = \langle \mathrm{C}_j \mid j \in \mathcal{I}^{\mathrm{c}} \rangle_{\mathbb{C}}, \qquad \mathfrak{h}_{\omega_i(\mathcal{I}),\mathbb{C}} = \mathrm{A}_1(\mathcal{X}_i^+)_{\mathbb{C}} = \left\langle \mathrm{C}_j, \mathrm{C}_i^+ \mid j \notin \mathcal{I} + i \right\rangle_{\mathbb{C}}.$$

where as in Lemma 5.2, we abuse notation by denoting the flopped curve C_i^+ instead of $\mathrm{C}^+_{\iota_{\mathcal{I}+i}(i)}$. As explained in Section 5.1, there is an explicit transformation matrix

$$\mathsf{M}_i \colon \mathrm{A}_1\left(\mathcal{X}_i^+\right)_{\mathbb{C}} \to \mathrm{A}_1(\mathcal{X})_{\mathbb{C}}.$$

This is the complexification of the matrix M_i from earlier, but we use the same symbol. Let N_i be the matrix dual to M_i^{-1}, which can be viewed as the linear map

$$\mathsf{N}_i \colon \mathrm{H}^2\left(\mathcal{X}_i^+; \mathbb{C}\right) \to \mathrm{H}^2(\mathcal{X}; \mathbb{C})$$

with the property that $\mathsf{N}_i\gamma \cdot \beta = \gamma \cdot \mathsf{M}_i^{-1}\beta$ for $\gamma \in \mathrm{H}^2(\mathcal{X}_i^+; \mathbb{C})$ and $\beta \in \mathrm{A}_1(\mathcal{X})_{\mathbb{C}}$. We notate the Novikov co-ordinates on the parameter spaces for the quantum potentials by

$$\{\mathsf{q}_j \mid j \in \mathcal{I}^{\mathrm{c}}\} \text{ on } \mathrm{A}_1(\mathcal{X})_{\mathbb{C}},$$
$$\{\mathsf{r}_j \mid j \in \mathcal{I}^{\mathrm{c}}\} \text{ on } \mathrm{A}_1(\mathcal{X}_i^+)_{\mathbb{C}}.$$

The matrix M_i^{-1} defines a monomial co-ordinate transformation relating the two sets of Novikov parameters. Writing monomials as

$$\mathsf{q}^{\beta} := \prod_{j \in \mathcal{I}^{\mathrm{c}}} \mathsf{q}_j^{m_j}, \qquad \mathsf{r}^{\beta} := \prod_{j \in \mathcal{I}^{\mathrm{c}}} \mathsf{r}_j^{m_j}$$

this is given by the equation

$$\mathsf{q}^{\beta} = \mathsf{r}^{\mathsf{M}_i^{-1}\beta}. \tag{5.B}$$

By Lemma 5.2, $\mathsf{q}_i = \mathsf{r}_i^{-1}$ whereas every other q_j is a monomial in the r_j with non-negative coefficients.

Lastly, recall the quantum potentials of $\mathcal{X}$ and $\mathcal{X}_i^+$ from Section 3.3, which to ease notation will be written Φ and Φ^+, namely,

$$\Phi_{\mathsf{q}}(\gamma_1, \gamma_2, \gamma_3) := \Phi_{\mathsf{q}}^{\mathcal{X}}(\gamma_1, \gamma_2, \gamma_3),$$
$$\Phi_{\mathsf{r}}^{+}(\gamma_1, \gamma_2, \gamma_3) := \Phi_{\mathsf{r}}^{\mathcal{X}_i^+}(\gamma_1, \gamma_2, \gamma_3).$$

The equation (5.B) will be used to express the quantum potential of $\mathcal{X}$ in the variables r, and this will be denoted $\Phi_{\mathsf{r}}(\gamma_1, \gamma_2, \gamma_3)$.

COROLLARY 5.13 (Crepant Transformation Conjecture). *On the* r *parameter space, the quantum potentials of* $\mathcal{X}$ *and* $\mathcal{X}_i^+$ *coincide, up to the following explicit term, which does not depend*

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on the Novikov parameters

$$\Phi^+_{\mathsf{r}}(\gamma_1,\gamma_2,\gamma_3)-\Phi_{\mathsf{r}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3)=-(\gamma_1\cdot \mathrm{C}_i^+)(\gamma_2\cdot \mathrm{C}_i^+)(\gamma_3\cdot \mathrm{C}_i^+)\sum_{k\geq 1}k^3 n_{k\mathrm{C}_i,\mathcal{X}}. \tag{5.C}$$

More precisely, the pole loci are transformed into each other via (5.B), and away from these, the analytic continuations constructed in Theorem 3.3 coincide.

Remark 5.14. The quantum potentials of $\mathcal{X}$ and $\mathcal{X}_i^+$ have no constant terms in their respective Novikov parameters, due to the absence of a perfect pairing on cohomology (see Remark 3.7). However, the change of parameters (5.B) introduces constant terms into $\Phi_{\mathsf{r}}(\gamma_1,\gamma_2,\gamma_3)$, which form the right-hand side of (5.C). It follows that the quantum potentials agree once these extraneous constant terms are removed from $\Phi_{\mathsf{r}}(\gamma_1,\gamma_2,\gamma_3)$. In particular, Φ^+ can be effectively reconstructed from Φ. In situations where an ordinary cup product can be defined, the additional terms on the right-hand side account for the defect between the cup products on $\mathcal{X}$ and $\mathcal{X}_i^+$; see e.g. [Mor96, Subsection 4.3 and Equation (4.4)].

Remark 5.15. The expansion points for the quantum potentials differ, as

$$(\mathsf{r}_j)_{j\in\mathcal{I}^c}=(0,\dots,0)\ \Leftrightarrow\ (\mathsf{q}_j)_{j\in\mathcal{I}^c}=(0,\dots,0,\infty,0,\dots,0),$$

with ∞ in the ith position. Thus, the term $\Phi_{\mathsf{r}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3)$ is analytically continued from $\mathsf{q}_i=0$ to $\mathsf{q}_i=\infty$, the analytic continuation occurring precisely in the Novikov parameter corresponding to the flopped curv

Proof of 5.13. We explicitly match both sides, using our knowledge of the structure of the quantum potentials (Theorem 3.3) and the behaviour of the GV invariants under the flop (Theorem 5.4).

Separating curve classes according to whether or not they are a multiple of C_i, the quantum potential for $\mathcal{X}$ may be written as the sum of contributions

$$\Phi_{\mathsf{q}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3)=\mathsf{G}_{\mathsf{q}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3)+\mathsf{H}_{\mathsf{q}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3),$$

where

$$\mathsf{G}_{\mathsf{q}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3)=\sum_{k\geq 1}n_{k\mathrm{C}_i,\mathcal{X}}\,(\mathsf{N}_i\gamma_1\cdot k\mathrm{C}_i)(\mathsf{N}_i\gamma_2\cdot k\mathrm{C}_i)(\mathsf{N}_i\gamma_3\cdot k\mathrm{C}_i)\frac{\mathsf{q}_i^k}{1-\mathsf{q}_i^k}$$

$$\mathsf{H}_{\mathsf{q}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3)=\sum_{\substack{\beta\in \mathrm{A}_1(\mathcal{X})\\ \beta\neq k\mathrm{C}_i}}n_{\beta,\mathcal{X}}\,(\mathsf{N}_i\gamma_1\cdot\beta)(\mathsf{N}_i\gamma_2\cdot\beta)(\mathsf{N}_i\gamma_3\cdot\beta)\frac{\mathsf{q}^\beta}{1-\mathsf{q}^\beta}.$$

Similarly, write the quantum potential of $\mathcal{X}_i^+$ as

$$\Phi^+_{\mathsf{r}}(\gamma_1,\gamma_2,\gamma_3)=\mathsf{G}^+_{\mathsf{r}}(\gamma_1,\gamma_2,\gamma_3)+\mathsf{H}^+_{\mathsf{r}}(\gamma_1,\gamma_2,\gamma_3),$$

where

$$\mathsf{G}^+_{\mathsf{r}}(\gamma_1,\gamma_2,\gamma_3)=\sum_{k\geq 1}n_{k\mathrm{C}_i^+,\mathcal{X}_i^+}\,(\gamma_1\cdot k\mathrm{C}_i^+)(\gamma_2\cdot k\mathrm{C}_i^+)(\gamma_3\cdot k\mathrm{C}_i^+)\frac{\mathsf{r}_i^k}{1-\mathsf{r}_i^k}$$

$$\mathsf{H}^+_{\mathsf{r}}(\gamma_1,\gamma_2,\gamma_3)=\sum_{\substack{\beta\in \mathrm{A}_1(\mathcal{X})\\ \beta\neq k\mathrm{C}_i}}n_{\beta,\mathcal{X}_i^+}\,(\gamma_1\cdot\beta)(\gamma_2\cdot\beta)(\gamma_3\cdot\beta)\frac{\mathsf{r}^\beta}{1-\mathsf{r}^\beta}.$$

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We begin with the G terms. Using (5.B) to change variables from q to r gives

$$\begin{aligned}\mathsf{G}_{\mathsf{r}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3) &= \sum_{k\geq 1} n_{k\mathrm{C}_i,\mathcal{X}}\,(\mathsf{N}_i\gamma_1\cdot k\mathrm{C}_i)(\mathsf{N}_i\gamma_2\cdot k\mathrm{C}_i)(\mathsf{N}_i\gamma_3\cdot k\mathrm{C}_i)\frac{\mathsf{r}_i^{-k}}{1-\mathsf{r}_i^{-k}}\\ &= (\gamma_1\cdot \mathrm{C}_i^+)(\gamma_2\cdot \mathrm{C}_i^+)(\gamma_3\cdot \mathrm{C}_i^+)\sum_{k\geq 1} k^3 n_{k\mathrm{C}_i,\mathcal{X}}\frac{1}{1-r_i^k}.\end{aligned}$$

where the second equality follows from $\mathsf{N}_i\gamma\cdot \mathrm{C}_i = \gamma\cdot \mathsf{M}_i^{-1}\mathrm{C}_i = -\gamma\cdot \mathrm{C}_i^+$ and the equality

$$\frac{r_i^{-k}}{1-r_i^{-k}} = \frac{1}{r_i^k-1}.$$

Using $n_{k\mathrm{C}_i,\mathcal{X}} = n_{k\mathrm{C}_i^+,\mathcal{X}_i^+}$ by Theorem 5.4, the difference $\mathsf{G}_{\mathsf{r}}^+(\gamma_1,\gamma_2,\gamma_3) - \mathsf{G}_{\mathsf{r}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3)$ is equal to

$$\begin{aligned}&(\gamma_1\cdot \mathrm{C}_i^+)(\gamma_2\cdot \mathrm{C}_i^+)(\gamma_3\cdot \mathrm{C}_i^+)\sum_{k\geq 1} k^3 n_{k\mathrm{C}_i,\mathcal{X}}\left(\frac{r_i^k}{1-r_i^k}-\frac{1}{1-r_i^k}\right)\\ =& -(\gamma_1\cdot \mathrm{C}_i^+)(\gamma_2\cdot \mathrm{C}_i^+)(\gamma_3\cdot \mathrm{C}_i^+)\sum_{k\geq 1} k^3 n_{k\mathrm{C}_i,\mathcal{X}}.\end{aligned}$$

We next examine the H terms. Note that for $\beta\in \mathrm{A}_1(\mathcal{X})$, we have $\beta\in\mathbb{Z}\mathrm{C}_i$ if and only if $\mathsf{M}_i^{-1}\beta\in\mathbb{Z}\mathrm{C}_i^+$. Consequently,

$$\begin{aligned}\mathsf{H}_{\mathsf{r}}(\mathsf{N}_i\gamma_1,\mathsf{N}_i\gamma_2,\mathsf{N}_i\gamma_3) &= \sum_{\substack{\beta\in \mathrm{A}_1(\mathcal{X})\\ \beta\neq k\mathrm{C}_i}} n_{\beta,\mathcal{X}}\,(\mathsf{N}_i\gamma_1\cdot\beta)(\mathsf{N}_i\gamma_2\cdot\beta)(\mathsf{N}_i\gamma_3\cdot\beta)\frac{\mathsf{r}^{\mathsf{M}_i^{-1}\beta}}{1-\mathsf{r}^{\mathsf{M}_i^{-1}\beta}}\\ &= \sum_{\substack{\beta\in \mathrm{A}_1(\mathcal{X}_i^+)\\ \beta\neq k\mathrm{C}_i^+}} n_{\mathsf{M}_i\beta,\mathcal{X}}\,(\mathsf{N}_i\gamma_1\cdot\mathsf{M}_i\beta)(\mathsf{N}_i\gamma_2\cdot\mathsf{M}_i\beta)(\mathsf{N}_i\gamma_3\cdot\mathsf{M}_i\beta)\frac{r^\beta}{1-r^\beta}\\ &= \sum_{\substack{\beta\in \mathrm{A}_1(\mathcal{X}_i^+)\\ \beta\neq k\mathrm{C}_i^+}} n_{\beta,\mathcal{X}_i^+}\,(\gamma_1\cdot\beta)(\gamma_2\cdot\beta)(\gamma_3\cdot\beta)\frac{r^\beta}{1-r^\beta}\\ &= \mathsf{H}_{\mathsf{r}}^+(\gamma_1,\gamma_2,\gamma_3).\end{aligned}$$

where the penultimate equality holds since $\mathsf{N}_i\gamma\cdot\mathsf{M}_i\beta = \gamma\cdot\mathsf{M}_i^{-1}\mathsf{M}_i\beta = \gamma\cdot\beta$ and $n_{\mathsf{M}_i\beta,\mathcal{X}} = n_{\beta,\mathcal{X}_i^+}$ again by Theorem 5.4. Combining the comparison of the G terms with the comparison of the H terms gives (5.C), as required. □

5.4 The contraction algebra under flop

The flopping contraction $\mathcal{X}\to\operatorname{Spec}\mathcal{R}$ has an associated contraction algebra $\mathrm{A}_{\mathrm{con}}$, defined using noncommutative deformation theory [DW16, DW19]. After flopping a single curve C_i to obtain $\mathcal{X}_i^+\to\operatorname{Spec}\mathcal{R}$, noncommutative deformation theory associates to this another contraction algebra, written $\nu_i\mathrm{A}_{\mathrm{con}}$. The algebra $\nu_i\mathrm{A}_{\mathrm{con}}$ can be intrinsically obtained from $\mathrm{A}_{\mathrm{con}}$ via a certain mutation procedure, and in fact $\mathrm{A}_{\mathrm{con}}$ and $\nu_i\mathrm{A}_{\mathrm{con}}$ are derived equivalent [Aug20b]. Both $\mathrm{A}_{\mathrm{con}}$ and $\nu_i\mathrm{A}_{\mathrm{con}}$ are finite dimensional algebras [DW16, 2.13].

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Fix the GV invariants n_β associated to $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$, and then Toda's dimension formula (see A.4) asserts that

$$\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}} = \sum_{\beta \in \mathrm{A}_1(\mathcal{X})} n_\beta (\beta \cdot \mathbb{1})^2$$

In many, but not all, cases, the dimension of $\mathrm{A}_{\mathrm{con}}$ is in fact enough to recover the n_β. The next result asserts that the numbers n_β associated to $\mathrm{A}_{\mathrm{con}}$, together with the matrix M_i^{-1}, completely determine the dimension of $\nu_i \mathrm{A}_{\mathrm{con}}$.

COROLLARY 5.16. *Under mutation at vertex i, equivalently flop at C_i,*

$$\dim_{\mathbb{C}} \nu_i \mathrm{A}_{\mathrm{con}} = \sum_{\beta \in \mathrm{A}_1(\mathcal{X})} n_\beta ((\mathsf{M}_i^{-1} \beta) \cdot \mathbb{1})^2$$

where M_i is the explicit matrix in (5.A).

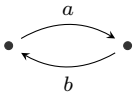
Proof. Combining previous results, it follows that

$$\begin{aligned}
\dim_{\mathbb{C}} \nu_i \mathrm{A}_{\mathrm{con}} &= \sum_{\gamma \in \mathrm{A}_1(\mathcal{X}_i^+)} n_\gamma (\gamma \cdot \mathbb{1})^2 && \text{(by A.4)} \\
&= \sum_{\beta \in \mathrm{A}_1(\mathcal{X})} n_\beta \left(|\mathsf{M}_i^{-1} \beta| \cdot \mathbb{1} \right)^2 && (\gamma = |\mathsf{M}_i^{-1} \beta| \text{ in Theorem 5.4}) \\
&= \sum_{\beta \in \mathrm{A}_1(\mathcal{X})} n_\beta \left((\mathsf{M}_i^{-1} \beta) \cdot \mathbb{1} \right)^2 && \text{(by Corollary 5.3)}
\end{aligned}$$

where the point is that by Corollary 5.3, the sign issue doesn't matter once we square. □

In particular, it is possible to compute the dimension of $\nu_i \mathrm{A}_{\mathrm{con}}$ without first having to present it.

Example 5.17. As in [SW23], consider the cA_2 example $\mathcal{R}_k = \frac{\mathbb{C}[\![u,v,x,y]\!]}{uv - xy(x^k + y)}$ for $k \geq 1$ and the specific crepant resolution $\mathcal{X} \to \operatorname{Spec} \mathcal{R}_k$ described in [SW23, 3.1], obtained first by blowing up (u, y) and then (u, x). In this case, as explained in [SW23, Subsection 6.2] $\mathrm{A}_{\mathrm{con}}$ can be presented as



$$(ab)^k a = 0 = b(ab)^k.$$

We can immediately read off the GV invariants, namely, $n_{1,0} = 1$, $n_{0,1} = 1$, and $n_{1,1} = k$. Thus $\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}} = n_{1,0} \cdot (1+0)^2 + n_{0,1} \cdot (0+1)^2 + n_{1,1} \cdot (1+1)^2$, which equals $1 + 1 + 4k = 4k + 2$.

We now flop the right-hand curve. In this Type A example, M_i^{-1} sends $(1,0) \mapsto (1,1)$, $(1,1) \mapsto (1,0)$ and $(0,1) \mapsto (0,-1)$. Thus, by Corollary 5.16,

$$\dim_{\mathbb{C}} \nu_i \mathrm{A}_{\mathrm{con}} = n_{1,0} \cdot (1+1)^2 + n_{0,1} \cdot (0+1)^2 + n_{1,1} \cdot (0-1)^2,$$

which equals $4 + 1 + k = k + 5$. In particular, $\nu_i \mathrm{A}_{\mathrm{con}} \ncong \mathrm{A}_{\mathrm{con}}$ provided that $k \geq 2$.

Appendix A. Toda's dimension formula

This appendix contains a proof of the general form of Toda's dimension formula, which relates the dimension of the contraction algebra to a weighted sum of GV invariants. This formula first

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appeared in [Tod15] for single-curve flops, then in [Tod18] in general. Alas, the GV invariants in [Tod18] are defined with respect to moduli spaces of the contraction algebra, and not via perturbation as done here (Subsection 3.1), and furthermore [Tod18, Subsection 4.4] contains no proof. As such, we briefly sketch the argument to convince the reader (and ourselves!) that the formula contains nothing specific to single-curve flops.

As notation, let $f\colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$ be a smooth 3-fold flopping contraction, with contraction algebra $\mathrm{A}_{\mathrm{con}}$. Write $\mathcal{V}$ for the Van den Bergh tilting bundle on $\mathcal{X}$ [VdB04] which generates zero perverse sheaves, and set $\mathrm{A} = \operatorname{End}_{\mathcal{X}}(\mathcal{V})$.

LEMMA A.1. $\mathbf{R}\mathrm{Hom}_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}, \mathrm{A}) \cong \mathrm{M}[-3]$ *for some* $\mathrm{A}_{\mathrm{con}}$*-bimodule* M *for which* $\dim_{\mathbb{C}} \mathrm{M} = \dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}}$.

Proof. First, by CY duality

$$\operatorname{Ext}^i_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}, \mathrm{A}) \cong \operatorname{Ext}^{3-i}_{\mathrm{A}}(\mathrm{A}, \mathrm{A}_{\mathrm{con}})^\star,$$

which is zero unless $i = 3$, when it equals $\operatorname{Hom}_{\mathrm{A}}(\mathrm{A}, \mathrm{A}_{\mathrm{con}})^\star \cong (\mathrm{A}_{\mathrm{con}})^\star$. In particular, it follows that the cohomology of $\mathbf{R}\mathrm{Hom}_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}, \mathrm{A})$ is concentrated in a single degree (namely, three), where as a vector space it has dimension $\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}}$. Truncating in the category of bimodules then yields the result. □

Remark A.2. $M \cong \mathrm{A}_{\mathrm{con}}$ as $\mathrm{A}_{\mathrm{con}}$-bimodules [Aug20b, 2.6], but we will not need this fact.

In what follows, set $\mathrm{B} := \mathrm{A} \otimes \mathrm{A}^{\mathrm{op}}$, so that B-modules are the same as A-bimodules. The following is then [Tod15, 2.3] adapted to our setting.

COROLLARY A.3. $\mathbf{R}\mathrm{Hom}_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}, \mathrm{A}) \otimes^{\mathbf{L}}_{\mathrm{B}} (\mathcal{V}^\star \boxtimes \mathcal{V}) \cong \mathcal{G}[-2]$ *for some* $\mathcal{G} \in \operatorname{coh} \mathcal{X} \times \mathcal{X}$ *which admits a filtration*

$$0 = \mathcal{G}_0 \subset \ldots \subset \mathcal{G}_{\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}}} = \mathcal{G}$$

such that each $\mathcal{G}_t/\mathcal{G}_{t-1}$ *is isomorphic to* $\mathcal{O}_{\mathrm{C}_i}(-1) \boxtimes \mathcal{O}_{\mathrm{C}_j}(-1)$ *for some* i, j *with* $1 \leq i, j \leq n$.

Proof. As notation, let $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_n$ be the simple left A-modules, and $\mathcal{S}_0, \mathcal{S}_1, \ldots, \mathcal{S}_n$ be the simple right A-modules. All have dimension one, as a vector space, and by convention $\mathcal{T}_0$ (respectively $\mathcal{S}_0$) is the only simple which is not an $\mathrm{A}_{\mathrm{con}}$-module.

Now M from Lemma A.1 is a finite dimensional B-module, so it is filtered by finite dimensional simples. But these all have the form $\mathcal{T}_i \otimes_{\mathbb{C}} \mathcal{S}_j$ for some $0 \leq i, j \leq n$ (see e.g. [[EGH+11], 3.10.2]). Since M is an $\mathrm{A}_{\mathrm{con}}$-bimodule, $i = 0$ or $j = 0$ is not possible. Hence M admits a filtration with quotients all of the form $\mathcal{T}_i \otimes_{\mathbb{C}} \mathcal{S}_j$ where $i, j \neq 0$. The length of the filtration must be $\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}}$, since each $\mathcal{T}_i \otimes_{\mathbb{C}} \mathcal{S}_j$ is one-dimensional.

Now, as observed by Toda [Tod15]

$$\begin{aligned}
&\mathbf{R}\mathrm{Hom}_{\mathcal{X}\times\mathcal{X}}(\mathcal{V}^\star \boxtimes \mathcal{V}, \mathcal{O}_{\mathrm{C}_i}(-1) \boxtimes \mathcal{O}_{\mathrm{C}_j}(-1)[-2]) \\
\cong &\mathbf{R}\mathrm{Hom}_{\mathcal{X}}(\mathcal{V}^\star, \mathcal{O}_{\mathrm{C}_i}(-1)[1]) \otimes_{\mathbb{C}} \mathbf{R}\mathrm{Hom}_{\mathcal{X}}(\mathcal{V}, \mathcal{O}_{\mathrm{C}_j}(-1))[-3] \\
\cong &\mathcal{T}_i \otimes_{\mathbb{C}} \mathcal{S}_j[-3] \qquad \text{(by [VdB04, 3.5.6, 3.5.7])}
\end{aligned}$$

Since $\mathbf{R}\mathrm{Hom}_{\mathrm{A}}(\mathrm{A}_{\mathrm{con}}, \mathrm{A}) \cong \mathrm{M}[-3]$, applying the inverse functor $- \otimes^{\mathbf{L}}_{\mathrm{B}} \mathcal{V}^\star \boxtimes \mathcal{V}$ and inducting along the filtration of $\mathrm{M}[-3]$ gives the result. □

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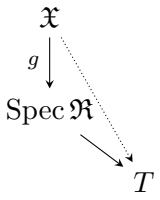
Now as explained in Section 2.6, there exists a flat deformation



for some Zariski open neighbourhood T of $0 \in \mathbb{A}^1$, such that

- the central fibre $g_0 \colon \mathfrak{X}_0 \to \operatorname{Spec} \mathfrak{R}_0$ is isomorphic to $f \colon \mathcal{X} \to \operatorname{Spec} \mathcal{R}$.
- all other fibres $g_t \colon \mathfrak{X}_t \to \operatorname{Spec} \mathfrak{R}_t$ are flopping contractions with exceptional locus a disjoint union of $(-1,-1)$-curves.

Regarding the flopping curves $\mathrm{C}_1, \dots, \mathrm{C}_n$ of f as curves in the central fibre of $\mathfrak{X} \to T$, and thus as a curve in $\mathfrak{X}$, then the GV invariant n_β is defined in Section 3.1 to be the number of g_t-exceptional $(-1,-1)$-curves C such that for every line bundle $\mathcal{L}$ on $\mathfrak{X}$,

$$\deg(\mathcal{L}|_{\mathrm{C}}) = \beta \cdot (\deg(\mathcal{L}|_{\mathrm{C}_1}), \dots, \deg(\mathcal{L}|_{\mathrm{C}_n})) := \sum_{i=1}^{n} \beta_i \deg(\mathcal{L}|_{\mathrm{C}_i}).$$

Here we are using the identification (2.H) of line bundles on each fibre of $\mathfrak{X} \to T$ with line bundles on the total space $\mathfrak{X}$.

Theorem A.4 (Toda). *If $\mathcal{X} \to \operatorname{Spec} \mathcal{R}$ is a smooth 3-fold flopping contraction, then*

$$\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}} = \sum_{\beta \in \mathrm{A}_1(\mathcal{X})} n_\beta (\beta \cdot \mathbb{1})^2$$

where the n_β are the GV invariants defined as in Section 3.1.

Proof. The proof very closely follows the single-curve strategy in [Tod15], and so we only outline the parts where some care is required.

The 4-fold $g \colon \mathfrak{X} \to \operatorname{Spec} \mathfrak{R}$ is a flopping contraction, and so by [Che02, Section 6] g admits a flop $\mathfrak{X}^+ \to \operatorname{Spec} \mathfrak{R}$ and a derived equivalence $\mathrm{D}^b(\mathfrak{X}) \to \mathrm{D}^b(\mathfrak{X}^+)$. Flopping back gives another equivalence $\mathrm{D}^b(\mathfrak{X}^+) \to \mathrm{D}^b(\mathfrak{X})$, and thus the composition gives rise to an autoequivalence

$$\psi \colon \mathrm{D}^b(\operatorname{coh} \mathfrak{X}) \to \mathrm{D}^b(\operatorname{coh} \mathfrak{X})$$

with FM kernel $\mathcal{P} \in \mathrm{D}^b(\mathfrak{X} \times_T \mathfrak{X})$, say. Define $\mathcal{P}_t = \mathbf{L} j_t^\star \mathcal{P}$, where j_t is the inclusion $\mathfrak{X}_t \times \mathfrak{X}_t \to \mathfrak{X} \times_T \mathfrak{X}$.

On one hand, for $t = 0$, as in [Tod15, (18)] the restriction of ψ to the zero fibre results in the NC twist functor of [DW16], so by uniqueness of FM kernels

$$\mathcal{P}_0 \cong \operatorname{Cone}(\mathcal{F}_0[-2] \to \mathcal{O}_{\Delta_{\mathfrak{X}_0}})$$

where $\mathcal{F}_0 = \mathcal{G}$ in Corollary A.3.

On the other hand, for $t \neq 0$, the birational map from $\mathfrak{X}_t$ to $\mathfrak{X}_t$ is the composition of the flops of all the curves in that fibre. Since all the curves have normal bundle $(-1,-1)$, the restriction of ψ to the neighbouring fibre $\mathfrak{X}_t$ results in composition of the corresponding (classical) spherical twists. Grouping the curves in $\mathfrak{X}_t$ together via their curve class, namely $\{\mathrm{C}_{\beta,i} \mid 1 \leq i \leq n_\beta\}_\beta$,

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then again as in [Tod15, (19)], uniqueness of FM kernels yields

$$\mathcal{P}_t \cong \mathrm{Cone}(\mathcal{F}_t[-2] \to \mathcal{O}_{\Delta_{\mathfrak{X}_t}})$$

where $\mathcal{F}_t \in \mathrm{coh}\,\mathfrak{X}_t \times \mathfrak{X}_t$ is the sheaf now defined by

$$\mathcal{F}_t = \bigoplus_{\beta} \bigoplus_{i=1}^{n_\beta} \mathcal{O}_{\beta,i}(-1) \boxtimes \mathcal{O}_{\beta,i}(-1) \tag{A.A}$$

where $\mathcal{O}_{\beta,i}$ is the structure sheaf of $\mathrm{C}_{\beta,i}$. Using [Tod15, 3.1, 3.2], which are general, it follows that $\mathcal{F}_t$ with $t \neq 0$ is a flat deformation of $\mathcal{F}_0$. Since both $\mathcal{F}_0$ and $\mathcal{F}_t$ have compact supports, their Hilbert polynomials must therefore be equal. In particular, let $\mathcal{L}$ be the g-ample line bundle on $\mathfrak{X}$ such that $\deg(\mathcal{L}|_{\mathrm{C}_i}) = 1$ for all $i = 1, \ldots, n$, which exists by (2.H) and the sentence underneath, then

$$\chi(\mathcal{F}_0 \otimes (\mathcal{L} \boxtimes \mathcal{L})) = \chi(\mathcal{F}_t \otimes (\mathcal{L} \boxtimes \mathcal{L})). \tag{A.B}$$

But χ is additive on filtrations, and so by Corollary A.3

$$\chi(\mathcal{F}_0 \otimes (\mathcal{L} \boxtimes \mathcal{L})) = \sum_{i=1}^{\dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}}} \chi(\mathcal{O}_{\mathrm{C}_i})\chi(\mathcal{O}_{\mathrm{C}_j}) = \dim_{\mathbb{C}} \mathrm{A}_{\mathrm{con}}. \tag{A.C}$$

Conversely, given the form of $\mathcal{F}_t$ in (A.A), it is clear that

$$\chi(\mathcal{F}_t \otimes (\mathcal{L} \boxtimes \mathcal{L})) = \sum_{\beta} \sum_{i=1}^{n_\beta} \chi(\mathcal{O}_{\beta,i}(-1) \otimes \mathcal{L})^2 = \sum_{\beta} n_\beta (\beta_1 + \ldots + \beta_n)^2 \tag{A.D}$$

since $\deg(\mathcal{L}|_{\mathrm{C}_{\beta,i}}) = \beta_1 + \ldots + \beta_n$. Combining (A.B), (A.C) and (A.D) gives the result. □

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Conflicts of Interest

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