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Gravitational radiation in homogeneous cosmological models with space-sections of constant negative curvature

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Abstract. The propagation of type N gravitational radiation in homogeneous cosmological models with space-sections of constant negative curvature is considered. It is found that the propagation of radiation distorts the isotropy of the fluid pressure in the models permanently.

1. *Introduction.* It is well known that the combination of general relativity, the cosmological principle and Weyl's postulate leads to the Robertson–Walker line element

$$ds^2 = dt^2 - e^{g(t)}(dx^2+dy^2+dz^2)/\{1+k(x^2+y^2+z^2)\}^2, \tag{1}$$

$k\,(=\pm 1, 0)$ being the curvature of the space-sections $t=$ constant. Infeld and Schild ((2)) have shown that all models represented by (1) can be brought to the conformally flat form. If we consider the line-element

$$ds^2 = e^{2H}\{dt^2-dx^2-dy^2-dz^2\}, \tag{2}$$

it represents for (I) $H=(1-S^2)^{-1}G(t/1-S^2)$, $S=\sqrt{(t^2-x^2-y^2-z^2)}$, (II) $H=H(S)$ and (III) $H=H(t)$, the three classes of Infeld and Schild models which are equivalent metrically, though not topologically to the Robertson–Walker metrics with $k=+1$, -1, 0, respectively.

In a recent paper ((3)) the propagation of type N gravitational radiation in Einstein–de Sitter and steady-state cosmological models (for which $k=0$) is studied. It was shown that the propagation of full retarded radiation is self-consistent in steady state cosmology but not in Einstein–de Sitter cosmology while the situation is reversed with full advanced radiation.

Though, qualitatively all evolutionary cosmological models based on the Robertson–Walker line-element (1) are alike, it would be interesting to investigate the propagation of gravitational radiation in the models with space-sections of non-vanishing curvature. In the present note we consider the propagation of type N gravitational radiation in class II cosmological models of Infeld and Schild (for which the curvature of the space-sections is negative).

2. *Field equations.* In an earlier note (Krishna Rao ((4))) a particular solution of the Einstein–Rosen ((1)) space-time corresponding to Lichnerowicz's ((5)) 'total radiation' is given. The metric form can be expressed in cylindrical polar coordinates r, ϕ, z and time t as

$$ds^2 = e^{2f}(dt^2-dr^2)-r^2\,d\phi^2-dz^2, \tag{3}$$

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where f is a function of $(r \pm t)$. The Weyl tensor of the space-time (3) is of type N according to Petrov's ((8)) classification.

Now, if we consider the metric forms

$$ds^2 = e^{2H}\{e^{2f}(dt^2 - dr^2) - r^2 d\phi^2 - dz^2\}, \tag{4}$$

where $H = H[\sqrt{(t^2 - r^2 - z^2)}]$, the Weyl tensors of the space-times represented by (4) still belong to type N and therefore, by making use of the statement of Pirani ((9)) can be interpreted as class II cosmological models of Infeld and Schild pervaded by gravitational radiation, the radiation being retarded or advanced according as f is a function of the argument $(r-t)$ or $(r+t)$, respectively.

For the space-time, given by (4), the non-vanishing components of the Einstein tensor G_{ij} connected with the Ricci tensor of the space-time R_{ij} by the relation

$$G_{ij} = R_{ij} - \tfrac{1}{2}g_{ij}R,$$

where $R = g^{ij}R_{ij}$ are given by:

$$\left.\begin{aligned}
G_{11} &= (1/S^2)[A\bar{S}^2 + 2(2+e^{2f})H'S + 2Br^2] + C,\\
G_{22} &= (r^2/e^{2f}S^2)[A\bar{S}^2 + 2(2+e^{2f})H'S],\\
G_{33} &= (1/e^{2f}S^2)[A\bar{S}^2 + 6H'S + 2e^{2f}Bz^2],\\
G_{44} &= -(1/S^2)[A\bar{S}^2 + 2(2+e^{2f})H'S - 2Bt^2] + C,\\
G_{13}/r &= -G_{34}/t = 2Bz/S^2,\\
G_{14} &= -(2Brt/S^2) - C,
\end{aligned}\right\} \tag{5}$$

where

$$A = (2H'' + H'^2 - 2H'/S),$$

$$B = (H'' - H'^2 - H'/S),$$

$$C = f'\left[\frac{2H'}{S}(r+t) \mp \frac{1}{r}\right]$$

and

$$\bar{S} = \sqrt{(t^2 - r^2 - e^{2f}z^2)}.$$

Here and in what follows a prime for H represents a differentiation with respect to S, $f' \equiv \partial f/\partial(r \pm t)$ as the case may be and whenever $\mp$ or $\pm$ occurs, always the upper and lower signs correspond to retarded and advanced solutions, respectively.

If we try to fit in (5) with a distribution consisting of a mixture of incoherent matter (without internal stresses) and radiation or a perfect fluid and radiation, we find that either $f = 0$ or f and H are connected by a relation which is impossible. However, we can fit in (5) with a mixture of nonperfect fluid and radiation, the pressure in the r-direction being different from that in the ϕ, z-directions. With the help of Lichnerowicz's ((6)) energy momentum tensor for an anisotropic fluid distribution the required form of T_{ij} can be written as

$$\left.\begin{aligned}
T_{ij} = (\rho + p)u_i u_j - g_{ij}p + (q-p)v_i v_j + \sigma w_i w_j,\\
u_i u^i = 1, \quad v_i v^i = -1, \quad w_i w^i = 0,
\end{aligned}\right\} \tag{6}$$

where q = pressure in the r-direction, p = pressure in the ϕ, z-directions, ρ and σ being the densities of matter and radiation respectively.

From (5) and (6) with the help of the field equations of general relativity

$$G_{ij} = -8\pi T_{ij},$$

we get

$$8\pi q = -(1/e^{2H+2f}S^2)[3H'^2\bar{S}^2+(2PSz/Q)+6H'S],$$

$$8\pi p = -(1/e^{2H+2f}S^2)[A\bar{S}^2+2(2+e^{2f})H'S],$$

$$8\pi\rho = (1/e^{2H+2f}S^2)[A\bar{S}^2-(2PSz/Q)+2(2+e^{2f})H'S],$$

$$8\pi\sigma = (1/S^2)[\{(1-e^{-2f})BH'(t^2+r^2)/Qz\} + \{(PSz/Q)-B\bar{S}^2+(1-e^{2f})H'S\}(3t^2-r^2)/(t^2-r^2)]+C,$$

$$u^i = (1/e^{H+f}\sqrt{P}S)[Br,0,e^{2f}QS,Bt],$$

$$v^i = (1/e^{H+f}\sqrt{(t^2-r^2)})[(2t^2-r^2)^{\frac{1}{2}},0,0,t],$$

$$w^i = (1/e^{2H+2f})[\pm 1,0,0,1],$$

where

$$P = \left[\left(\frac{B^2\bar{S}^2}{S^2}\right)+(e^{2f}-1)\left\{\frac{2BH'}{S}-(1-e^{-2f})\frac{H'^2}{z^2}\right\}\right],$$

and

$$Q = [(Bz/S)-(1-e^{-2f})H'/z].$$

3. *Conclusion.* From the expressions, for the physical variables given in the last section, it is evident that there is no clear-cut picture here to determine which one of the retarded and advanced solutions is consistent with Infeld and Schild's cosmological models of class II. But one thing which can be noted is that while in the case of Einstein–de Sitter and steady state models the source of radiation can be made to exhaust after a definite time (which can be seen from the results of (3)), in the present solution the source continues indefinitely. This can be proved as follows: if the physical picture of the model given in the last section is viewed as a situation wherein some catastrophe develops on the z-axis in the Infeld and Schild class II cosmological model and the emission of gravitational radiation continues for a constant time T, then we must have a cylindrical shell of constant thickness containing a mixture of anisotropic fluid and gravitational radiation sandwiched in between two material domains. Naturally on the outer side of the shell we shall have a class II cosmological model of Infeld and Schild and on the inner side the distribution left out after the propagation of gravitational radiation. At both these interfaces the necessary jump conditions of O'Brien and Synge ((7)) for the coordinates used here must be satisfied. But in the case of our solution the jump conditions can be made to satisfy only at the outer interface and not at the inner interface. This shows that the catastrophe developed on the z-axis is not temporary but a permanent one. Therefore, as the radiation moves outwards the isotropy of the fluid pressure in the models is distorted and there is no return to the original form.

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