ON APPROXIMATE SOLUTIONS OF LINEAR DIFFERENTIAL EQUATIONS

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ABSTRACT. Asymptotic approximations of Green's type to solutions of differential equations are studied, with special reference to the uniformity of the approximation given by the first term. In extension to the complex variable this is found to require substantial restrictions on the region considered. An anomaly previously noticed is traced to non-uniformity of approximation. The case where the coefficient χ_0 has a simple zero and χ_1 is not zero is treated by a simple method.

1. The asymptotic solutions of differential equations of the form

$$\frac{d^2y}{dx^2} = (h^2\chi_0 + h\chi_1 + \chi_2)\,y, \tag{1}$$

where χ_0, χ_1, χ_2 are given functions of x and h is large, have received much attention. Langer (7–12), especially, has given sufficient conditions for the existence of solutions in descending powers of h and has shown that they possess an asymptotic property in Poincaré's sense. His discussion includes the case where χ_0 has a zero in the interval of x considered and χ_1 is not zero. On the other hand, his method includes several rather cumbrous transformations, and the final answer needs several references to earlier parts of the work before it can be interpreted. In the case where χ_0 has a zero use is made of Bessel functions of orders $\pm\frac{1}{3}$, thus introducing a singularity at an ordinary point of the original differential equation. This can be avoided by using the Airy integral and its companion function Bi(z). Both these functions have been tabulated for real argument (Miller (13)) and their general behaviour is known over the complex plane (Miller (13); Jeffreys and Jeffreys (5), chap. 17).

Cherry (1) has given an alternative treatment, using the Airy function but not Bi(z); but the latter is much the most convenient second solution of the Airy equation in problems of this type. He does not treat the effect of the term in χ_1 explicitly.

As the names of several modern authors are associated in the literature with solutions of this type, I think that the part played by Green (3) needs emphasis (see also Lamb (6)). Green studied the propagation of long waves in a channel of non-uniform section, under the condition that the period is short enough for the depth and width not to change greatly within a wave-length. He found a solution that is easily seen to be equivalent to the first term of the modern solutions for the case where χ_0 does not vanish in the interval of x considered. Further, his solution brings out the essential principle that the solutions are functions of two variables, x, h, and are required to hold over a fixed interval of x; the asymptotic property holds with respect to the large parameter h (which in Green's problem is the speed of the vibration considered). The resemblance in some respects to the asymptotic solutions of Bessel's equation for fixed order and large argument has led to some confusion of the essentially different natures of the approximations that are attempted.

In practice not much use is made of the later terms of the approximations, partly because the first term usually gives surprising accuracy, partly because the later terms are often rather complicated. The practical procedure, if the first term is not enough, would usually be to take the first term as a first approximation and use numerical integration.

There is a need for a simplified account that will place the emphasis on the first term of the series and give conditions for its validity as an approximation. The method is also being applied to complex variables, whereas it was originally developed only for real variables, and some attention needs to be given to the nature of the region of validity.

2. In what follows I make extensive use of the principle that if

$$\frac{d^2y}{dx^2}+p(x)\frac{dy}{dx}+q(x)\,y = S(x)\,y, \tag{1}$$

and two independent solutions when $S(x) = 0$ are $y_1(x)$, $y_2(x)$, then the complete solution is given by the solutions of the integral equation

$$y(x) = f(x)+\int_a^x K(x,t)\,y(t)\,dt, \tag{2}$$

where a is any conveniently chosen constant, and

$$f(x) = A_1 y_1(x)+A_2 y_2(x), \tag{3}$$

$$K(x,t) = \frac{y_1(x)\,y_2(t)-y_2(x)\,y_1(t)}{y_1'(t)\,y_2(t)-y_2'(t)\,y_1(t)}\,S(t). \tag{4}$$

The Wronskian in the denominator is never zero, and is constant if $p(x) = 0$. A_1, A_2 are arbitrary constants.

If $|f(x)| < M$, $|K(x,t)| < N$ in a bounded region of x, of diameter k, then $f(x)$ can be taken as a first approximation to $y(x)$. Successive substitutions in the integral lead to a series of corrections whose moduli are less than the terms

$$M\{N\,|x-a|\}^n/n! \leqslant N(Mk)^n/n!,$$

and thus always yield a uniformly convergent solution, which is analytic if p, q, S are.

When $p(x)$, $q(x)$ depend not only on x but also on a parameter h, M and N will become $M(h)$, $N(h)$, and so long as these are finite the same argument shows that there is a solution analytic with regard to x for any h. There is nothing in the argument, however, to say that $N(h)$ does not tend to infinity with h, and if it does so the series, though ultimately convergent, does not yield a useful approximation in a few terms. Also the terms are in general unbounded with regard to x in an unbounded region of x.

3. If we transform the independent variable in 1 (1) to ξ and put

$$y = \left(\frac{d\xi}{dx}\right)^{-\frac{1}{2}} z, \tag{1}$$

1 (1) becomes

$$\xi'^2\frac{d^2z}{d\xi^2} = \left(h^2\chi_0+h\chi_1+\chi_2+\frac{\xi'''}{2\xi'}-\frac{3}{4}\frac{\xi''^2}{\xi'^2}\right)z. \tag{2}$$

We take $|\chi_0|$ to have a positive lower bound in the region of x considered (which may be unbounded) and take

$$\xi = \int_0^x \left(\chi_0 + \frac{\chi_1}{h} + \frac{\psi_2}{h^2}\right)^{\frac{1}{2}} dx, \tag{3}$$

where ψ_2 is to some extent at our disposal. It is customary to take either $\psi_2 = 0$ or $\psi_2 = \chi_2$; I have recommended taking

$$\xi = \int_0^x \chi_0^{\frac{1}{2}}\left(1 + \frac{\chi_1}{2h\chi_0}\right) dx. \tag{4}$$

In any case (2) takes the form

$$\frac{d^2z}{d\xi^2} - h^2 z = g(\xi, h)\, z, \tag{5}$$

which we shall take as the fundamental equation. h is taken real and positive. In either case the usual method makes the first terms of the approximate solutions

$$z = \exp(\pm h\xi), \quad y = \chi_0^{-\frac{1}{4}} \exp(\pm h\xi), \tag{6}$$

and the choice of ψ_2 makes only a difference of order $1/h$. But ψ_2 does affect $g(\xi, h)$. Suppose that $\chi_0 = 1$, $\chi_1 = 0$, $\chi_2 = a > 0$. The exact solutions are $\exp\{\pm(h^2+a)^{\frac{1}{2}} x\}$. If we take $\psi_2 = \chi_2 = a$, the first approximations given by (6) are exact solutions. If, however, we take $\psi_2 = 0$ in (3) or use (4), the first approximations are in error by factors approximating to $\exp\{\pm ax/2h\}$, which vary to any extent with x for given h. For some purposes (for instance, in giving precision to the notions of reflected and transmitted waves) it is desirable to have approximations that never become wrong by more than a factor $O(1/h)$ even in an unbounded region of x. An example has been known since 1923 (Jeffreys (4); Jeffreys and Jeffreys (5), p. 523) where it was better to use (4), and this will be re-examined later; the immediate point is that the effect of the choice of ψ_2 on the first approximation is important in some conditions. We shall say that the approximations (6) are *uniformly asymptotic* if the error terms are of the form P/h times the main terms, where P is bounded for x in the region and $h \geqslant h_0 > 0$. The theorem of § 2 might suggest that this condition would hold if $g(\xi, h)$ is uniformly bounded, but the example just given shows that this is false.

With suitable cuts in the x plane (3) will define ξ uniquely in a region of the x plane, and we shall suppose that by excluding singularities of the transformation we can find a region D of ξ such that for points of this region there is a 1–1 correspondence between x and ξ. In this region $g(\xi, h)$ is supposed analytic. We assume further that there is a part of D, say E, such that (i) (5) *holds in* E; (ii) E *contains the whole or part of the real axis of* ξ; (iii) *if* ξ *is in* E, *then all points* η *such that* $\eta = \Re(\xi) + i\theta\Im(\xi)$ $(0 \leqslant \theta \leqslant 1)$ *are also in* E; (iv) *on any paths in* E *parallel to the real or the imaginary axis, for* $h \geqslant h_0$,

$$\int_{\xi_1}^{\xi_2} |\, g(t, h)\, dt\,| < M.$$

These conditions permit E to contain infinite strips enclosing the real or imaginary axes. We show that *in these conditions* (5) *has two solutions* Z_1, Z_2 *analytic in* D *and such that in* E

$$Z_1 = e^{h\xi}(1 + O(1/h)), \quad Z_2 = e^{-h\xi}(1 + O(1/h)). \tag{7}$$

uniformly with respect to ξ.

The analytic property in D follows from the argument of § 2 and will not be discussed further.

If g was zero solutions would be

$$z_1 = e^{h\xi}, \quad z_2 = e^{-h\xi}. \tag{8}$$

Denoting the lower and upper bounds (possibly $\pm\infty$) of $\mathfrak{R}(\xi)$ by A, B, we have that a pair of solutions of (5) are solutions of the integral equations

$$Z_1(\xi) = e^{h\xi} + \int_A^\xi \frac{1}{2h}\{e^{h(\xi-t)} - e^{-h(\xi-t)}\} g(t,h)\, Z_1(t)\, dt, \tag{9}$$

$$Z_2(\xi) = e^{-h\xi} - \int_\xi^B \frac{1}{2h}\{e^{h(\xi-t)} - e^{-h(\xi-t)}\} g(t,h)\, Z_2(t)\, dt, \tag{10}$$

where the path in (9) is from $t = A$ to $\mathfrak{R}(\xi)$ along the real axis and then from $\mathfrak{R}(\xi)$ to ξ parallel to the imaginary axis; in (10) it is from ξ to $\mathfrak{R}(\xi)$ and then from $\mathfrak{R}(\xi)$ to B. In (9) the first approximation to the integral is

$$\frac{1}{2h}\int_A^\xi \{e^{h\xi} - e^{h(2t-\xi)}\} g(t,h)\, dt, \tag{11}$$

and on the path

$$| e^{h\xi} - e^{h(2t-\xi)} | \leqslant 2e^{h\mathfrak{R}(\xi)}. \tag{12}$$

Then the modulus of the first correction is not greater than

$$\frac{1}{h}\, e^{h\mathfrak{R}(\xi)}\left\{\int_A^{\mathfrak{R}(\xi)} + \int_{\mathfrak{R}(\xi)}^{\xi} | g(t,h)\, dt |\right\} \leqslant \frac{2M}{h}\, e^{h\mathfrak{R}(\xi)}. \tag{13}$$

Further substitutions show that

$$e^{-h\xi} Z_1(\xi) = 1 + \Sigma k_r(\xi, h)/h^r, \tag{14}$$

where

$$| k_r | = (2M)^r. \tag{15}$$

Hence for $h > h_0 > 2M$ the error of the first term is uniformly $O(1/h)$. Similar considerations show that in the same conditions the second solution gives

$$e^{h\xi} Z_2(\xi) = 1 + \Sigma l_r(\xi, h)/h^r, \quad | l_r | < (2M)^r, \tag{16}$$

and the error of the first term is again uniformly $O(1/h)$ for $h > h_0$.

The solutions given by (14), (16) are uniformly convergent in the conditions stated, but that does not imply the existence of convergent solutions of the forms

$$e^{\pm h\xi}[1 + \Sigma f_r(\xi)/h^r].$$

Solutions of this form can be obtained in many cases by expanding k_r, l_r in descending powers of h and rearranging; but in fact k_r, l_r in general contain non-zero functions tending to zero exponentially. Such functions and all their derivatives with regard to $1/h$ tend to 0 for $1/h \to 0$, and would therefore vanish if they had expansions in powers of $1/h$. Hence the most that can be expected of expansions rearranged in this way is that they will be asymptotic.

3·1. The condition (iii) is needed because otherwise there might be no path in E that passes from A to ξ without passing through some point β where $\mathfrak{R}(t) > \mathfrak{R}(\xi)$, and then $\exp h\Re(\xi)$ would not be the upper bound of $|\exp h(2t-\xi)|$ on the path. Similarly in (10) the upper bound of $|\exp h(\xi-2t)|$ might not be $\exp(-h\Re(\xi))$. In either condition the successive corrections would ordinarily contain extra factors of the form $\exp[\pm h\Re(\xi-\beta)]$, and at least one of the solutions, though convergent and analytic, would depart from the first approximation by a factor that is not uniformly $O(1/h)$. Thus the main conclusion is not true for parts of D not included in E. This complication does not arise when x is restricted to be real or purely imaginary; it is a new feature introduced by the application to complex values.

It must be recognized that the effect of singularities is an essential feature of the problem. If $g(\xi,h)$ was analytic and bounded in the whole ξ plane, g would be a constant, by Liouville's theorem, and the whole subject becomes trivial. Similarly, condition (iv) applied over the whole plane would give $g=0$. If $g(\xi,h)$ has singularities, on the other hand, not only the asymptotic approximations but the exact solutions will differ according to the paths of integration used; (iii) picks out a type of region such that suitable paths are easily specified, and permits us to establish the required asymptotic property for such regions. If E was identical with D, its boundary would be a wall of singularities.

3·2. It does not appear that condition (iv) can be appreciably lightened for complex arguments. If, however, ξ is restricted to be real, it is enough that $\int_A^\xi g(t,h)\,dt$ shall be bounded, by Abel's lemma for integrals. If ξ is purely imaginary, it is enough that $\int_0^\xi g(t,h)\,dt$ shall be bounded and $g(t,h)$ of uniformly bounded variation; though this condition is not in practice often satisfied when $\int_0^\xi |g(t,h)\,dt|$ is not bounded.*

3·3. These considerations explain an anomaly that has already been noticed (Jeffreys (4); Jeffreys and Jeffreys (5), p. 523). If

$$\theta=(\log x)^2, \tag{17}$$

the equation
$$\frac{d^2y}{dx^2}=(h^2\theta'^2+h\theta'')\,y \tag{18}$$

has an exact solution
$$y_1=\exp(h\theta), \tag{19}$$

and a second solution given by

$$y_2=\left(\frac{1}{\theta'}-\frac{1}{2h}\frac{\theta''}{\theta'^3}+O(h^{-2})\right)e^{-h\theta}. \tag{20}$$

If we use (4)
$$\xi=\int^x\left(\theta'+\frac{\theta''}{2h\theta'}\right)dx=(\log x)^2+\frac{1}{2h}\log\frac{2\log x}{x}, \tag{21}$$

$$\frac{d^2z}{d\xi^2}=\left\{h^2-\frac{\log x-2}{8(\log x)^4}+O\left(\frac{1}{h}\frac{\theta''}{\theta'}\right)\right\}z. \tag{22}$$

* An example, for real variable, is

$$f(x)=\frac{\sin(\log x)}{x\log x};\quad f(1)=1.$$

Then $f(x)$ is of bounded variation in $(1,\infty)$, $\int_1^\infty f(x)\,dx$ converges, but $\int_1^\infty |f(x)|\,dx$ does not converge.

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Then $$\int |g(\xi,h)|\,d\xi \doteqdot \int (\log x)^{-3}\,d(\log x)^2, \tag{23}$$

and converges absolutely in $(1+\epsilon, \infty)$. Hence the first terms of the asymptotic solutions give a uniform approximation, agreeing with (19) and (20). But if we used 1 (3) with $\psi_2 = \chi_2 = 0$, $g(\xi, h)$ would contain a term in $(\log x)^{-2}$ and the integral corresponding to (23) would not converge. This explains why it was found better in this case to use 1 (4) rather than 1 (3) with $\psi_2 = \chi_2$; but it must be emphasized that this was highly exceptional. If possible ψ_2 should be chosen so that (iv) is satisfied; direct inspection of the transformation is probably easier than the application of any general rule.

4. *Solutions when χ_0 has a simple zero.* In this case 3 (3) would give a point where $\xi' = 0$, and $g(\xi, h)$ would in general be unbounded near this point. The solutions for real x, h are of exponential type for $x > 0$, oscillating for $x < 0$. We can reduce the equation approximately to the Airy equation

$$\frac{d^2z}{d\xi^2} = h^2\xi z, \tag{1}$$

whose exact solutions are known; and by comparison with the asymptotic solutions of the original equation we can derive the required connecting formulae. This was apparently first done by Rayleigh (14), who, however, gave only the exponentially decreasing solution explicitly. It is noteworthy that he alluded only incidentally to the possibility of expressing the solutions in terms of Bessel functions of orders $\pm\frac{1}{3}$. Gans (2) gave connecting formulae for both solutions; these were rediscovered by me in 1923 (4). The first study of the convenient second solution Bi (z) was in Miller's introduction (13) to the *British Association tables of* Ai (z) and Bi (z). In my original recommendation for the tabulation I pointed out that it was desirable that for negative z the first term of the asymptotic expansions should have the same amplitude and differ by $\frac{1}{2}\pi$ in phase; the definition of Bi (z) was chosen by Miller so as to satisfy this condition. Bi (z) is real for real z.

We want to transform 1 (1) so that it reduces approximately to the above form. Then ξ must vanish with $\chi_0 + \chi_1/h + \psi_2/h^2$, where ψ_2 is again to some extent at our disposal. If $\chi_0(0) = 0$, $\chi_0'(0) \neq 0$, this occurs at $x = -\alpha \doteqdot -\chi_1(0)/h\chi_0'(0)$; and we take

$$\xi'^2\xi = \chi_0 + \chi_1/h + \psi_2/h^2. \tag{2}$$

Take $\chi_0'(0) > 0$. Then $$\tfrac{2}{3}\xi^{\frac{3}{2}} = \int_{-\alpha}^{x} (\chi_0 + \chi_1/h + \psi_2/h^2)^{\frac{1}{2}}\,dx, \tag{3}$$

which approximates to $\frac{2}{3}\chi_0'^{\frac{1}{2}}(x+\alpha)^{\frac{3}{2}}$ near $x = -\alpha$. Consequently the transformation is 1–1 near $x = -\alpha$. Then we find

$$\frac{d^2z}{d\xi^2} = \{h^2\xi + g(\xi,h)\}\,z \tag{4}$$

$$= \left\{h^2\xi + \frac{1}{2}\frac{\xi'''}{\xi'^3} - \frac{3}{4}\frac{\xi''^2}{\xi'^4}\right\} z. \tag{5}$$

If χ_0, χ_1, ψ_2 are of the form $A + B/x$ for x large, ξ behaves like $(\frac{3}{2}A^{\frac{1}{2}}x)^{\frac{2}{3}}$, $g(\xi, h)$ like $x^{-\frac{4}{3}}$, i.e. like ξ^{-2}. Thus $g(\xi, h)$ will ordinarily be $O(\xi^{-2})$ for $|\xi|$ large.

As for the case of § 3, cuts will be needed to give a 1–1 correspondence between x and ξ for a suitable region D of ξ, and the asymptotic expressions will be limited to a part E of D. We assume E to include a segment $0 \leqslant \xi \leqslant B$ of the real axis, with $B > 0$; if $|\arg \xi| \leqslant \frac{1}{3}\pi$, an arc $t = \rho e^{i\theta}$ with $\rho^{\frac{3}{2}} \cos \frac{3}{2}\theta =$ constant lies in E connecting ξ with ξ_0 on the positive real axis; if $||\arg \xi| - \frac{2}{3}\pi| \leqslant \frac{1}{3}\pi$, a similar arc lies in E and connects ξ with ξ_0 on a line $|\arg \xi| = \frac{2}{3}\pi$, and E contains this arc and the line from ξ_0 to 0. It is also assumed that the values of ξ with these properties include a neighbourhood of $\xi = 0$ and that for integrals on all such arcs and segments $\int_{\xi_1}^{\xi_2} |g(t, h)\, t^{-\frac{1}{2}}\, dt| < M$.

Solutions of (1) are
$$z_1(\xi) = \mathrm{Bi}\,(h^{\frac{2}{3}}\xi), \quad z_2(\xi) = \mathrm{Ai}\,(h^{\frac{2}{3}}\xi), \tag{6}$$
and
$$z_1' z_2 - z_1 z_2' = h^{\frac{2}{3}}/\pi. \tag{7}$$
A pair of solutions of (4) are solutions of
$$Z(\xi) = f(\xi) + \int_a^{\xi} K(\xi, t, h)\, g(t, h)\, Z(t)\, dt, \tag{8}$$
with $f(\xi) = z_1$ or z_2,
$$K(\xi, t, h) = \pi h^{-\frac{2}{3}}\{\mathrm{Bi}\,(h^{\frac{2}{3}}\xi)\,\mathrm{Ai}\,(h^{\frac{2}{3}}t) - \mathrm{Ai}\,(h^{\frac{2}{3}}\xi)\,\mathrm{Bi}\,(h^{\frac{2}{3}}t)\}. \tag{9}$$

It is convenient to record at this point certain identities satisfied by $\mathrm{Ai}\,(z)$ and $\mathrm{Bi}\,(z)$. For integral values of k
$$\mathrm{Ai}\,(ze^{\frac{2}{3}k\pi i}) = e^{\frac{1}{3}k\pi i}\left\{\cos k\pi\,\mathrm{Ai}\,(z) - \frac{i}{\sqrt{3}} \sin \tfrac{1}{3}k\pi\,\mathrm{Bi}\,(z)\right\}, \tag{10}$$
$$\mathrm{Bi}\,(ze^{\frac{2}{3}k\pi i}) = e^{\frac{1}{3}k\pi i}\{-\sqrt{3}\, i \sin \tfrac{1}{3}k\pi\,\mathrm{Ai}\,(z) + \cos \tfrac{1}{3}k\pi\,\mathrm{Bi}\,(z)\}, \tag{11}$$
$$\cos \tfrac{1}{3}k\pi\,\mathrm{Ai}\,(ze^{\frac{2}{3}k\pi i}) + \frac{i}{\sqrt{3}} \sin \tfrac{1}{3}k\pi\,\mathrm{Bi}\,(ze^{\frac{2}{3}k\pi i}) = e^{\frac{1}{3}k\pi i}\,\mathrm{Ai}\,(z), \tag{12}$$
$$\mathrm{Ai}\,(ze^{\frac{2}{3}k\pi i})\,\mathrm{Bi}\,(te^{\frac{2}{3}k\pi i}) - \mathrm{Bi}\,(ze^{\frac{2}{3}k\pi i})\,\mathrm{Ai}\,(te^{\frac{2}{3}k\pi i}) = e^{\frac{2}{3}k\pi i}\{\mathrm{Ai}\,(z)\,\mathrm{Bi}\,(t) - \mathrm{Bi}\,(z)\,\mathrm{Ai}\,(t)\}. \tag{13}$$
It follows that $\mathrm{Bi}\,(z)$ is exponentially large for large $|z|$ for all values of $\arg z$, except for $\arg z = (2k+1)\pi$ or $(2k \pm \frac{1}{3})\pi$; while $|K|$ in (8) is unaltered if $\arg t$ and $\arg z$ are both changed by a multiple of $\frac{2}{3}\pi$. We have
$$\mathrm{Ai}\,(z) \sim \frac{1}{2\sqrt{\pi}} z^{-\frac{1}{4}} \exp\,(-\tfrac{2}{3}z^{\frac{3}{2}}) \quad (|\arg z| < \tfrac{2}{3}\pi - \delta), \tag{14}$$
$$\mathrm{Bi}\,(z) \sim \frac{1}{\sqrt{\pi}} z^{-\frac{1}{4}} \exp\,(\tfrac{2}{3}z^{\frac{3}{2}}) \quad (|\arg z| < \tfrac{1}{3}\pi - \delta). \tag{15}$$

Consider first $|\arg \xi| < \frac{1}{3}\pi - \delta$. In (8) take $a = 0$, $f(\xi) = z_1(\xi)$. Then since
$$|\,\mathrm{Ai}\,(h^{\frac{2}{3}}t)\,\mathrm{Bi}\,(h^{\frac{2}{3}}t)\,|$$
is uniformly less than a fixed multiple of $|h^{\frac{1}{3}} t^{-\frac{1}{2}}|$, if we take a path from 0 to ξ_0 and then on an arc from ξ_0 to ξ, the first term of K will give a correction of order $h^{-1}\,\mathrm{Bi}\,(h^{\frac{2}{3}}\xi)$. The second term gives
$$\int_0^{\xi} \pi h^{-\frac{2}{3}}\{\mathrm{Ai}\,(h^{\frac{2}{3}}\xi)\,\mathrm{Bi}^2\,(h^{\frac{2}{3}}t)\, t^{\frac{1}{2}}\}\,\{t^{-\frac{1}{2}} g(t, h)\}\, dt. \tag{16}$$

The modulus is $$< \pi M \,|\, h^{-\frac{2}{3}} \operatorname{Ai}(h^{\frac{2}{3}}\xi) \operatorname{Bi}^2(h^{\frac{2}{3}}\xi)\, \xi^{\frac{1}{2}} \,|, \tag{17}$$
which is of the same form. By repetition, if h is sufficiently large there is a uniformly convergent solution in this sector satisfying

$$Z_1 = \operatorname{Bi}(h^{\frac{2}{3}}\xi)\{1 + O(1/h)\}. \tag{18}$$

For $|\arg \xi| = \frac{1}{3}\pi$, we can write

$$Z_1 = \operatorname{Bi}(h^{\frac{2}{3}}\xi) + O(1/h). \tag{19}$$

For the second solution we use the equation

$$Z_2 = \operatorname{Ai}(h^{\frac{2}{3}}\xi) - \int_\xi^B \frac{\pi}{h^{\frac{2}{3}}} \{\operatorname{Bi}(h^{\frac{2}{3}}\xi) \operatorname{Ai}(h^{\frac{2}{3}}t) - \operatorname{Ai}(h^{\frac{2}{3}}\xi) \operatorname{Bi}(h^{\frac{2}{3}}t)\}\, g(t,h)\, Z_2(t)\, dt, \tag{20}$$

and take the path to consist of the arc from ξ to ξ_0 on the real axis and the part of the real axis from ξ_0 to B. In the asymptotic expressions for Ai and Bi the factors $\exp(\pm \frac{2}{3} h t^{\frac{3}{2}})$ have constant modulus on the arc, decreasing toward B. Then the first correction is of the form $(\alpha M/h) \exp(-\frac{2}{3} h \xi^{\frac{3}{2}})$, with α bounded; and proceeding to further corrections we find

$$Z_2(\xi) = \operatorname{Ai}(h^{\frac{2}{3}}\xi) + O\{h^{-1} \exp(-\tfrac{2}{3} h \xi^{\frac{3}{2}})\}, \tag{21}$$

which may be written

$$Z_2(\xi) = \operatorname{Ai}(h^{\frac{2}{3}}\xi)\{1 + O(1/h)\} \quad (|\arg z| \leqslant \tfrac{1}{3}\pi - \delta), \tag{22}$$

$$Z_2(\xi) = \operatorname{Ai}(h^{\frac{2}{3}}\xi) + O(1/h) \quad (|\arg z| = \tfrac{1}{3}\pi). \tag{23}$$

As for the case of § 3 these relations will not hold for values of ξ not in E. For these the error may be many times the first term for h large.

Note that a change from B to $B' < B$ introduces a term in $\operatorname{Bi}(h^{\frac{2}{3}}\xi)$, but its coefficient is $O\{h^{-\frac{2}{3}} \operatorname{Ai}(h^{\frac{2}{3}}B')\}$ and the term is $o(1/h)$ of the first term for $|\exp \frac{2}{3} h \xi^{\frac{3}{2}}| < |\exp \frac{2}{3} h B'^{\frac{3}{2}}|$.

In sectors $\frac{1}{3}\pi < |\arg \xi| < \pi$, Ai and Bi are nearly proportional and both large. Then it no longer follows that (17) is $O(\operatorname{Bi}(h^{\frac{2}{3}}\xi))$. But we can proceed as follows. Take $\xi = \eta e^{\frac{2}{3}k\pi i}$, where $|\arg \eta| \leqslant \frac{1}{3}\pi$, $k = \pm 1$; there is an arc $\rho^{\frac{3}{2}} \cos \frac{3}{2}\theta =$ constant passing through ξ and having a point ξ_0 with $\arg \xi_0 = \pm \frac{2}{3}\pi$. E is assumed to contain this arc and the line from ξ_0 to 0. Then with $\eta_0 = \xi_0 e^{-\frac{2}{3}k\pi i}$, $\tau = t e^{-\frac{2}{3}k\pi i}$,

$$\begin{aligned} Z_1(\xi) &= \operatorname{Bi}(h^{\frac{2}{3}}\xi) + \pi h^{-\frac{2}{3}} \int_0^{\xi_0} + \int_{\xi_0}^{\xi} \{\operatorname{Bi}(h^{\frac{2}{3}}\xi) \operatorname{Ai}(h^{\frac{2}{3}}t) - \operatorname{Ai}(h^{\frac{2}{3}}\xi) \operatorname{Bi}(h^{\frac{2}{3}}t)\}\, Z_1(t)\, g(t,h)\, dt \\ &= \operatorname{Bi}(h^{\frac{2}{3}}\xi) + \pi h^{-\frac{2}{3}} e^{\frac{2}{3}k\pi i} \left[\int_0^{\eta_0} + \int_{\eta_0}^{\eta} \{\operatorname{Bi}(h^{\frac{2}{3}}\eta) \operatorname{Ai}(h^{\frac{2}{3}}\tau) - \operatorname{Ai}(h^{\frac{2}{3}}\eta) \operatorname{Bi}(h^{\frac{2}{3}}\tau)\}\, g(t)\, Z_1(e^{\frac{2}{3}k\pi i}\tau)\, d\tau \right] \end{aligned}$$

by (13). Then by (11) we have $\operatorname{Bi}(h^{\frac{2}{3}}t) = O(h^{-\frac{1}{6}} \tau^{-\frac{1}{4}} \exp(\frac{2}{3} h \tau^{\frac{3}{2}}))$ and the previous argument shows that

$$Z_1(\xi) = \operatorname{Bi}(h^{\frac{2}{3}}\xi) + O(h^{-1} \exp(-\tfrac{2}{3} h \xi^{\frac{3}{2}})).$$

Similarly by taking the path from ξ to ξ_0, 0 and B in turn we find

$$Z_2(\xi) = \operatorname{Ai}(h^{\frac{2}{3}}\xi) + O(h^{-1} \exp(-2 h \xi^{\frac{3}{2}})).$$

If $\arg \xi = \pi$ or $\pm \frac{1}{3}\pi$, $\xi_0 = 0$, but the results in this form still hold.

As ξ varies, subject to ξ remaining in E, the paths are deformed continuously, and Z_1, Z_2 always represent the same analytic solutions and are valid in the same region. Small third solutions can be found for $\frac{1}{3}\pi < |\arg \xi| < \pi$ by taking paths in E to $\infty \exp(i \arg \xi)$ if such paths satisfy the remaining conditions, but these are not the analytic continuations of the solution represented by (22).

The Stokes phenomenon arises from the different asymptotic expressions of Ai and Bi in different sectors, and is fully taken into account by the identities (10) to (13). The complication from additional singularities of the transformation has received less attention, but is of comparable importance when the theory is applied to the complex variable.

When $|h^{\frac{2}{3}}\xi|$ is large we can approximate to Ai and Bi by asymptotic expressions, with further errors of order $1/h\xi^{\frac{3}{2}}$.

4·1. It remains to relate the approximate solutions to the original equation. We have, if $\chi = \chi_0 + \chi_1/h + \psi_2/h^2$,

$$y = \xi'^{-\frac{1}{2}} z = \chi^{-\frac{1}{4}} \xi^{\frac{1}{4}} z. \tag{1}$$

For $|\arg \xi| \leqslant \frac{1}{3}\pi$ we put

$$M = \int_{-a}^{x} h\chi^{\frac{1}{2}}\,dx = \tfrac{2}{3} h \xi^{\frac{3}{2}}, \tag{2}$$

and then a pair of solutions are

$$y_2 \sim \tfrac{1}{2}(\chi^{-\frac{1}{4}}\xi^{\frac{1}{4}})\,\xi^{-\frac{1}{4}} \exp(-M) = \tfrac{1}{2}\chi^{-\frac{1}{4}} \exp(-M) \tag{3}$$

$$\sim \sqrt{\pi}\, h^{\frac{1}{6}} \xi'^{-\frac{1}{2}} \operatorname{Ai}(h^{\frac{2}{3}}\xi), \tag{4}$$

$$y_1 \sim \chi^{-\frac{1}{4}} \exp M \sim \sqrt{\pi}\, h^{\frac{1}{6}} \xi'^{-\frac{1}{2}} \operatorname{Bi}(h^{\frac{2}{3}}\xi). \tag{5}$$

For

$$\tfrac{2}{3}\pi + \delta\, |\arg \xi| \leqslant \pi,\ \xi = -\zeta,\ L = \int_0^{\zeta} h(-\chi)^{\frac{1}{2}}\,d\zeta, \tag{6}$$

$$y_1 \sim |\chi|^{-\frac{1}{4}} \cos(L + \tfrac{1}{4}\pi), \tag{7}$$

$$y_2 \sim |\chi|^{-\frac{1}{4}} \sin(L + \tfrac{1}{4}\pi). \tag{8}$$

In these χ can be replaced by χ_0 with an error of order $1/h$. In a neighbourhood of $x = 0$, where Ai and Bi have to be used directly, ξ' can be replaced by its value at $x = 0$, namely, $\{\chi_0'(0)\}^{\frac{1}{3}}$.

The present method has allowed for χ_1 from the start, but it is interesting to examine its effect for large x and compare with the case where χ_0 never vanishes. We have, nearly,

$$M - \int_0^x h\chi_0^{\frac{1}{2}}\,dx = \int_{-a}^{0} h\chi^{\frac{1}{2}}\,dx + \int_0^a (\chi^{\frac{1}{2}} - \chi_0^{\frac{1}{2}})\,dx + \int_a^x h(\chi^{\frac{1}{2}} - \chi_0^{\frac{1}{2}})\,dx. \tag{9}$$

Here a is some suitably chosen constant. Then the first two terms are independent of x; and the last approximates to $\int_a^x (\chi_1/2\chi_0^{\frac{1}{2}})\,dx$. The principal effect of χ_1 for large x is therefore to add the last integral to M. For $x < 0$, if $\chi = -\psi_0 + \chi_1/h$,

$$L - \int_0^x h\psi_0^{\frac{1}{2}}\,dx \sim \int_0^x \frac{\chi_1}{2\psi_0^{\frac{1}{2}}}\,dx. \tag{10}$$

Thus on the negative side, if $\chi_0'(0) > 0$, $\chi_1(0) > 0$, all phases are reduced.

4·2. It has been pointed out by Langer (especially (8)) that care is needed in the use of 4·1 (3), (5), (7), (8) to establish correspondences between solutions on opposite sides of a zero of χ_0. We have, if A and B are constants, a general solution $Ay_1 + By_2$, with asymptotic expressions

$$\tfrac{1}{2}A\chi^{-\frac{1}{4}} \exp M\{1 + O(h^{-1})\} + B\chi^{-\frac{1}{4}} \exp(-M)\{1 + O(h^{-1})\}, \tag{1}$$

$$A\,|\,\chi\,|^{-\frac{1}{4}}\{\cos(L + \tfrac{1}{4}\pi) + O(h^{-1})\} + B\,|\,\chi^{-\frac{1}{4}}\,|\,\{\sin(L + \tfrac{1}{4}\pi) + O(h^{-1})\}. \tag{2}$$

If the solution for $\xi > 0$ is exponentially small, it follows that $A = 0$ and hence the solution must be, to $O(1/h)$, equal to $B\,\chi^{-\frac{1}{4}} \exp(-M)$ and $B\,|\,\chi\,|^{-\frac{1}{4}} \sin(L + \frac{1}{4}\pi)$ on the respective sides. If, however, it is exponentially large, $A \neq 0$, but this information does not determine B, and any multiple of y_2 can be added to the solution for $\xi > 0$ without disturbing the asymptotic solution for $\xi > 0$. Confusion can be avoided if explicit mention is made of the orders of magnitude of the error terms. The difficulty will not arise for an infinite interval, since the relevant solution will be required to be bounded in any case.

5. In actual calculation where the dominant solution contains a large exponential, it is often found worth while to include the small solution; thus for Bi (ξ), where $\Re(\xi)$ is negative and $|\,\Im(\xi)\,|$ is small but not zero, $\xi^{\frac{3}{2}}$ has a small but not zero real part, and it may happen that the 'small' series is larger than the fourth or fifth term in the 'large' series. Stokes's original papers (15, 16) contain a numerical illustration. This fact is essentially related to the non-uniformity of the asymptotic expansion in power series in ξ^{-1} in the neighbourhood of certain values of $\arg\xi$. In asymptotic approximations obtained by the method of steepest descents it can be shown to come from a topological change in the line of steepest descent through a saddle-point when it passes over a second saddle-point. There seems to be no formal theory on the point; this is essentially because Poincaré's definition of the asymptotic property deals with limits as the modulus of the variable tends to infinity, in which case the small series ultimately becomes less than any assigned term in the large series, and no precise alternative definition exists. But it is noteworthy that the approximations to Ai (z) and Bi (z), for z real and negative, arise from paths through two saddle-points and that for $\frac{2}{3}\pi < \arg z < \frac{4}{3}\pi$ they are more accurate than those that include only one path.

6. We have not assumed that $g(\xi, h)$ has a convergent, or even asymptotic, expansion in powers of $1/h$. Apparently a term in $h^{-\frac{1}{2}}$ would vitiate Langer's treatment (though of course an extension of his treatment would cover it); the present argument, however, shows that it does not affect the first term of the solution.

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