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Finite quasisimple groups of 2×2 matrices over a division ring

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1. *Introduction*

In 1955 [**1**], Amitsur determined all the finite groups G that can be embedded in the multiplicative group $T^* = GL(1, T)$ of some division ring T of characteristic zero. If G can be so embedded, then the rational span of G in T is a division ring of finite dimension over $\mathbb{Q}$, and G acts on it by right multiplication in such a way that every non-trivial element operates fixed point freely. The finite groups admitting such a representation had earlier been determined by Zassenhaus [**24**; **4**, XII. 8], and Amitsur begins by quoting Zassenhaus' results, which show in particular that the only perfect group that can be embedded in the multiplicative group of a division ring of characteristic zero is $SL(2, 5)$. The more difficult part of Amitsur's paper is the determination of the possible soluble groups. Here the main tool is Hasse's theory of cyclic algebras over number fields.

In this paper we begin the investigation of the finite groups G that can be embedded in $GL(2, T)$, for some division ring T of characteristic zero. We consider only the case when G is *quasisimple*, in the sense that G is perfect and $G/Z(G)$ is (non-abelian) simple, where $Z(G)$ is the centre of G. Most of the arguments here are group-theoretic, and the role previously played by Zassenhaus' theorem on fixed-point-free representations is now taken over by the Gorenstein–Harada theorem [**7**] on finite simple groups of sectional 2-rank at most 4. For, by Lemma 2.1 below and a theorem of MacWilliams [**17**], all the groups we meet will have this property. Our proof consists of going through the Gorenstein–Harada list, ruling out the various groups as the simple part of G, until only the two possibilities below remain; these are easily shown to be realizable.

THEOREM. *Let G be a finite quasisimple group. Then there exists a division ring T of characteristic zero such that G can be embedded in $GL(2, T)$ if and only if $G \cong SL(2, 5)$ or $SL(2, 9)$.*

Note in particular that G cannot be non-abelian simple. When we pass to 3×3 matrices we obtain at least one simple group, the alternating group Alt(5) of degree 5. More surprisingly, perhaps, we find also the covering group of the Hall-Janko group [**23**]. It might be interesting to determine exactly which quasisimple groups occur here.

As in Amitsur's case, it is relatively straightforward to determine the possible finite quasisimple groups that can be embedded in some $GL(2, T)$, given the present state of knowledge about finite simple groups. Banieqbal [**2**] has gone on to give an essentially complete description of *all* the finite groups that can be embedded in some $GL(2, T)$. This is much more difficult. Many cases arise, and careful arguments with crossed products over number fields are frequently involved. He has also given a proof of our theorem, avoiding the use of heavy group-theoretic machinery, in the case when T is finite-dimensional over a field of p-adic numbers. Such a proof for general T appears elusive.

2. *Proof of theorem*

We begin by introducing the Gorenstein–Harada theorem into the picture as follows.

LEMMA 2.1. *Let T be a division ring of characteristic zero and G be a finite subgroup of $GL(2,T)$. Then*

(i) *Every abelian subgroup of G has rank at most* 2,

(ii) *G has sectional* 2*-rank at most* 4,

(iii) *Every non-abelian composition factor of G occurs on the following list:*

(*a*) *Groups of odd characteristic:*

$$PSL(n,q)\ (n \leqslant 5);\quad PSp(4,q);\quad PSU(n,q^2)\ (n \leqslant 5);\quad G_2(q);\quad {}^2G_2(3^{2m+1});\quad {}^3D_4(q^3),$$

where q is an odd prime power in all cases.

(*b*) *Groups of even characteristic:*

$$PSL(2,8),\ PSL(2,16),\ PSL(3,4),\ PSU(3,4^2),\ {}^2B_2(2^3).$$

(*c*) *Alternating groups $Alt(n)$* $(7 \leqslant n \leqslant 11)$.

(*d*) *Sporadic groups M_{11}, M_{12}, M_{22}, M_{23} (Mathieu groups), J_1, J_2, J_3 (Janko groups), McLaughlin group, Lyons group.*

Proof. (i) follows from [**10**], lemma 3·1, and (ii) then follows from a theorem of MacWilliams [**17**]. We obtain (iii) from (ii) and the Gorenstein–Harada theorem [**7**].

Notes. (i) The Gorenstein–Harada list is actually a proper subset of the above list since in some cases there are further congruence conditions on q for the groups of odd characteristic. These need not concern us.

(ii) The notation we use for groups of Lie type and classical groups follows Carter [**5**]. The groups J_1, J_2, J_3 are the Janko groups of orders $2^3.3.5.7.11.19$, $2^7.3^3.5^2.7$ and $2^7.3^5.5.17.19$ respectively.

(iii) Each group occurs on the list in only one guise. For example, Alt(5) appears as $PSL(2,5)$, as does $PSL(2,4)$.

We require also the following facts. We denote the set of irreducible complex characters of G by Irr(G), and if $\chi \in \mathrm{Irr}(G)$, $m_{\mathbb{Q}}(\chi)$ denotes the Schur index of G over $\mathbb{Q}$.

LEMMA 2·2. *Let G be as in Lemma* 2·1. *Then*

(i) *If $m_{\mathbb{Q}}(\chi) = 1$ for every $\chi \in Irr(G)$, then G can be embedded in $GL(2,\mathbb{C})$,*

(ii) *If p is an odd prime, then every p-subgroup of G is abelian,*

(iii) *If every Sylow subgroup of G is elementary abelian, then G can be embedded in $GL(2,\mathbb{C})$.*

Proof. (i) See [**10**], lemma 2·3.

(ii) Let P be a subgroup of G of odd prime power order. By a well-known result of Roquette [**19**] or ([**12**], p. 168), $m_{\mathbb{Q}}(\chi) = 1$ for all $\chi \in \mathrm{Irr}(P)$. Hence, by (i), P can be embedded in $GL(2,\mathbb{C})$. The irreducible constituents of the corresponding two-dimensional representation of P must be linear, since their degrees divide $|P|$, hence P is abelian.

(iii) In this case also, $m_{\mathbb{Q}}(\chi) = 1$ for all $\chi \in \mathrm{Irr}(G)$ ([**12**], p. 165).

Finally, for easy reference we list the Schur multipliers of some of the Gorenstein–Harada groups.

Group	Schur multiplier	Reference
$PSL(2,5)$, $PSL(2,7)$	$\mathbb{Z}_2$	[**11**], V. 25. 7
$PSL(2,9)$	$\mathbb{Z}_6$	[**11**], V. 25. 7
$PSL(2,8)$	1	[**11**], V. 25. 7
$PSU(3,4^2)$	1	[**8**]
$Alt(7)$	$\mathbb{Z}_6$	[**3**], [**20**]
${}^2B_2(2^3) = Sz(8)$	$\mathbb{Z}_2 \times \mathbb{Z}_2$	[**4**], XI. 3. 12
M_{11}	1	[**4**], XII. 1. 16
J_1	1	[**13**]

Proof of Theorem. First let T be a division ring of characteristic zero, G be a finite quasisimple subgroup of $GL(2,T)$, and $H = G/Z(G)$. Then H is one of the groups listed in Lemma 2·1 (iii), and we go through the various possibilities.

(*a*) *H has odd characteristic.* Except for $PSL(2,q)$, all the listed groups have a non-abelian Sylow p-subgroup, where q is a power of p, and hence so does G. For PSp and PSU see Huppert ([**11**], II. 10. 12) and for the remaining groups see Carter ([**5**] 5·2, 13·6). Hence $H \cong PSL(2,q)$ for some q, and by ([**10**], Cor. 2·5), $q \leqslant 9$. Thus $q = 5$, 7 or 9, as $PSL(2,3)$ is not simple.

Suppose $q = 5$. From the above list of Schur multipliers, the Schur covering group of $PSL(2,5)$ is $SL(2,5)$, so G is isomorphic to $SL(2,5)$ or $PSL(2,5)$, and we just have to exclude the latter possibility. Now $PSL(2,5) \cong \mathrm{Alt}(5) \geqslant \mathrm{Alt}(4)$, so if $G \cong PSL(2,5)$, then Lemma 2·2 (iii) tells us that Alt(4) can be embedded in $GL(2,\mathbb{C})$. As the unique faithful irreducible character of Alt(4) has degree 3, this is not so.

Now suppose $q = 7$. The Schur covering group of $PSL(2,7)$ is $SL(2,7)$, from the above list. Now $SL(2,7)$ contains a non-abelian group B of order 21 coming from the upper triangular matrices, and so does $PSL(2,7)$. If $H \cong PSL(2,7)$, then Lemma 2·2(iii) tells us that B can be embedded in $GL(2,\mathbb{C})$, which consideration of the degrees of the characters shows is not so.

Finally, consider the case $q = 9$. The Schur multiplier of $PSL(2,9)$ is $\mathbb{Z}_6$, so in this case $Z(G)$ has order dividing 6. Now if 3 divides $|Z(G)|$, then G will have a non-abelian Sylow 3-subgroup. For if K is any finite group and P is a Sylow p-subgroup of K, then $P \cap K' \cap Z(K) = P' \cap Z(K)$, as can be established by considering the transfer of K into P/P' (see [**11**], IV. 2. 2). Hence, by Lemma 2·2 (ii), $|Z(G)| = 1$ or 2, and we just have to exclude the first possibility. This can be done by noting that $m_{\mathbb{Q}}(\chi) = 1$ for all $\chi \in \mathrm{Irr}(PSL(2,q))$ [**15**] or [**21**], using Lemma 2·2 (i), and referring to the character table of $PSL(2,9)$ ([**16**], § 38) to see that the smallest degree of a faithful irreducible character of $PSL(2,9)$ is 5. Alternatively, we note that $PSL(2,9)$ contains a subgroup, coming from the upper triangular matrices, which is an extension of a self-centralizing elementary abelian normal subgroup of order 9 by a cyclic group of order 4, and it is not hard to see, by analysing the rational group ring of this group, that it cannot be embedded in $GL(2,T)$.

This concludes the discussion of the case of odd characteristic.

(*b*) *H has even characteristic.* Suppose first $H \cong PSL(2,8)$. Then the Schur multiplier of H is trivial, so $G \cong PSL(2,8) = SL(2,8)$, which contains an elementary abelian

2-subgroup of rank 3. This is impossible, by Lemma 2·1 (i). Similarly, H cannot be isomorphic to $PSL(2, 16)$. Now $PSL(3, 4)$ contains $PSL(3, 2)$, which is isomorphic to $PSL(2, 7)$. If $H \cong PSL(3, 4)$, then G will contain a non-trivial image of the Schur covering group of $PSL(2, 7)$, and this has previously been ruled out. Next suppose $H \cong PSU(3, 4^2)$. The Schur multiplier of this group is trivial, so $G \cong PSU(3, 4^2)$. Let S be a Sylow 2-subgroup of G. From Huppert ([**11**], II. 10. 12), $Z(S)$ is elementary abelian of order 4 and $N_G(S)$ contains an element of order 3 that operates non-trivially on $Z(S)$. Thus, G contains a copy of Alt(4), which we saw was impossible in discussing $PSL(2, 5)$.

Finally suppose H is the Suzuki group ${}^2B_2(2^3)$. The Schur multiplier of this group is $\mathbb{Z}_2 \times \mathbb{Z}_2$. The Sylow 2-subgroup of H contains an elementary abelian subgroup U of order 8 operated on by an element t of order 7 that permutes the involutions of U transitively ([**4**], XI. 3. 1). Let V be the inverse image of U in the Schur covering group $\hat{H}$ of H. Then $\hat{H}$ contains an element of order 7, also denoted by t, and V contains a normal t-invariant subgroup V_0 such that V/V_0 is elementary abelian of order 8, V_0 is elementary abelian of order 4, and t operates irreducibly on V/V_0 and trivially on V_0. Let W be a subgroup of order 2 of V_0. Then as t operates irreducibly on V/V_0, the centre of V/W is either V/W itself, or V_0/W. In the latter case, V_0/W would be extraspecial, which is impossible as $|V/V_0|$ is not a square. So V/W is abelian, and since this holds for all such W, so is V. But then every non-trivial image of $\hat{H}$ contains an abelian 2-subgroup of rank 3, and so cannot be embedded in $GL(2, T)$.

(*c*) *H is alternating*. Consider first the case $H \cong$ Alt(7). The Schur multiplier of H has order 6. If 3 divides $|Z(G)|$, then the Sylow 3-subgroup of G is non-abelian, as in the discussion of $PSL(2, 9)$, and so this possibility is excluded. So $|Z(G)|$ is 1 or 2. Now Alt(7) contains a non-abelian group B of order 21, and G will also contain this group. As in the discussion of $PSL(2, 7)$, this is impossible. If H is alternating of degree greater than 7, then G will contain a non-trivial image of the Schur covering group of Alt(7), and so this too is impossible.

(*d*) *H is sporadic*. The Schur multiplier of the Mathieu group M_{11} is trivial, and this group contains a copy of $PSL(2, 9)$, ([**4**], XII. 1) which we have seen cannot be embedded in a $GL(2, T)$. Hence H cannot be M_{11}. Since M_{12} contains a subgroup isomorphic to M_{11}, the possibility $H \cong M_{12}$ is also ruled out. The groups M_{22} and M_{23} both contain $PSL(3, 4)$ and hence $PSL(3, 2)$, and are therefore both ruled out as H because of previous parts of the proof. The group J_1 has trivial Schur multiplier, and in it, the centralizer of an involution t is isomorphic to $\langle t \rangle \times$ Alt(5). Thus, if $H \cong J_1$, then $G \geqslant$ Alt(5), and this has been shown to be impossible. The remaining groups have a non-abelian Sylow 3-subgroup, and are ruled out as H for that reason. In fact $J_2 \geqslant PSU(3, 3^2)$ [**9**], and for J_3, see [**14**], in particular Lemma 5·4 and top of page 56. The McLaughlin group contains $PSU(4, 3^2)$ [**18**], and in the Lyons group the centralizer of an involution is isomorphic to the covering group of Alt(11) [**16**].

To complete the proof of the theorem we must show that $SL(2, 5)$ and $SL(2, 9)$ can both be faithfully represented by 2×2 matrices over some division ring of characteristic zero. In fact the real quaternions $\mathbb{H}$ will do. It is well known that $SL(2, 5)$ is isomorphic to a subgroup of $H^* = GL(1, H)$. This can be obtained from the exact sequence

$$1 \to \mathbb{Z}_2 \to SU(1, \mathbb{H}) \to SO(3, \mathbb{R}) \to 1,$$

where $SU_1(\mathbb{H})$ is the group of unit quaternions, by taking the inverse image of the octahedral group Alt(5). We also have the exact sequence

$$1 \to \mathbb{Z}_2 \to SU(2, \mathbb{H}) \to SO(5, \mathbb{R}) \to 1$$

(see, for example, [**22**], p. 105, for these sequences).

The group Alt(6) has a faithful representation of degree 5 over $\mathbb{Q}$ obtained from the elements of coefficient sum zero in the permutation module arising from the natural permutation representation. Hence Alt(6) $\cong PSL(2, 9)$ can be embedded in $SO(5, \mathbb{R})$. Its inverse image in $SU(2, \mathbb{H})$ must be $SL(2, 9)$ or $PSL(2, 9) \times \mathbb{Z}_2$; the latter has been ruled out above, so we obtain a copy of $SL(2, 9)$ in $SU_2(\mathbb{H})$.

An alternative argument using Schur indices can be given as follows. From the character table of $SL(2, q)$ ([**6**], § 38) we see that $SL(2, 9)$ has two irreducible characters, η_1 and η_2, of degree 4. These characters are rational-valued, and $m_{\mathbb{Q}}(\eta_i) = 2$ $(i = 1, 2)$ [**21**] or [**15**]. By the properties of the Schur index there exists an irreducible $\mathbb{Q}S$-module $(S = SL(2, 9))$ V such that $W = V \otimes_{\mathbb{Q}} \mathbb{C}$ affords the character $2\eta_1$. Thus $\dim_{\mathbb{Q}} V = 8$. Also if D is the division ring $\mathrm{End}_{\mathbb{Q}S} V$ then $D \otimes_{\mathbb{Q}} \mathbb{C} \cong \mathrm{End}_{\mathbb{C}S} (V \otimes \mathbb{C})$, which is isomorphic to $M_2(\mathbb{C})$ as $V \otimes \mathbb{C}$ has two isomorphic irreducible constituents. Thus $\dim_{\mathbb{Q}} D = 4$ and $\dim_D V = 2$. Hence, we can obtain an embedding of S in $GL(2, D)$. Much more information can be obtained from Janusz's paper [**15**].

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