A recording gyroscope. By Dr G. F. C. Searle, Peterhouse, University Lecturer in Experimental Physics.

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§ 1. *Introduction.* Gyroscopic effects furnish important instances of dynamical actions, but their general discussion is outside the range of most students reading for Part I of the Natural Sciences Tripos. If, however, we limit ourselves to the case in which the axis of the wheel makes a constant angle with the vertical, the theory becomes elementary and simple devices suffice for recording the movements of the revolving wheel. The recording gyroscope was designed to give students at the Cavendish Laboratory, Cambridge, an opportunity of making practical measurements to verify the theoretical results*.

§ 2. *Relation between couple and precession in simplest case.* The axle of a cycle wheel W (Fig. 1) is rigidly attached to a frame UU which can turn about a horizontal axis or hinge D supported by a block E carried by a vertical shaft S. This shaft

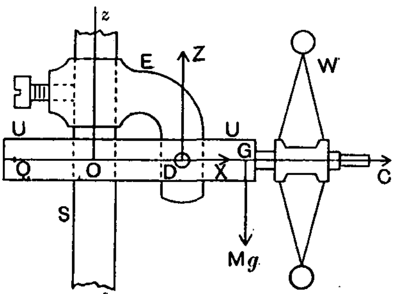

Fig. 1

turns freely in suitable bearings. The axis of the hinge D does not intersect that of the shaft S. The shortest distance between these axes is l. Fig. 1 is diagrammatic; the details of the apparatus are better seen in Figs. 4, 5.

The axis of W intersects that of D at right angles and also intersects the axis Oz of the shaft S in O. The plane containing the axes of D and of W is a plane of symmetry of the frame as is also the plane which is at right angles to the axis of D and contains the axis of W. The centre of gravity of the wheel and frame will, therefore, be at a point G on OC, the axis of W.

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In the present *simple* case, the axis GC of the wheel is horizontal.

The mass of the whole system of wheel and frame is M grms., and the moment of inertia of the wheel alone about its axis is C grm. cm.^{2}

The axle of the wheel turns with angular velocity Ω radians per second about the fixed vertical axis Oz. When Ω is positive, the relation of the motion of the axis OC to the upward direction of Oz is right-handed, i.e. is that of the rotation to the translation of a right-handed screw working in a fixed nut. In the horizontal plane in which OC moves we take a moving line OB always at right angles to OC and such that right-handed rotation about Oz turns OC to OB.

The wheel spins about its axle. In view of the more general theory, the angular velocity of this spin needs careful definition. The azimuthal plane COz turns about Oz with angular velocity Ω. If the wheel turn about its axle p times per second relative to the frame and to the moving plane, its angular velocity about OC relative to the plane is $2\pi p$ radians per second. We count p positive when the angular velocity relative to the frame is related right-handedly to the direction OC. The electro-magnetic gear (§ 4) records p, whether the axle of the wheel be horizontal or not. We shall write

$$2\pi p = \eta. \qquad \text{.............................(1)}$$

The angular velocity Ω of the system about Oz has no component about OC in the present case, since here OC is perpendicular to Oz. Hence, the moving plane COz has no angular velocity about OC. Thus, if ω be the angular velocity of the wheel about OC, we have in this case (but *not* in the general case of § 3)

$$\omega = \eta. \qquad \text{.............................(2)}$$

By symmetry, the azimuthal motion of the frame and the wheel about Oz gives rise to no angular momentum about OC. Hence, the angular momentum about OC is simply $C\eta$. Again, by symmetry, neither the angular velocity Ω nor the angular velocity η gives rise to any angular momentum about OB.

Let the moving axes OC, OB coincide with the fixed axes Ox, Oy (Fig. 2) at time t_0. When, at time t, the axle has reached OC', by turning through ϕ about Oz, and OB has reached OB', the angular momentum about Oy is $C\eta \sin\phi$, since there is no angular momentum about OB'. Hence, if the rate of increase of angular momentum about Oy be R_ϕ when $C'OC = \phi$, we have

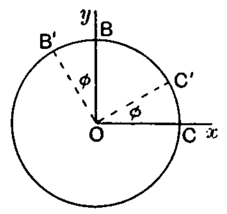

Fig. 2

$$R_\phi = \frac{d}{dt}(C\eta \sin\phi) = C\left(\eta\Omega \cos\phi + \sin\phi \frac{d\eta}{dt}\right), \text{......(3)}$$

since $d\phi/dt = \Omega$. If the rate of increase of angular momentum about Oy, when the axle has the direction Ox and ϕ is zero, be R, we have

$$R = C\eta\Omega. \quad \text{.........................(4)}$$

It thus appears that when, as in this case, there is no angular momentum about OB' for any value of ϕ, the value of R depends only upon η and Ω and not upon their rates of change.

The forces acting on the system of frame and wheel are its weight Mg dynes acting at G and the horizontal and vertical forces X, Z exerted on the system by the hinge, as shown in Fig. 1. These pass through the axis of the (frictionless) hinge. In practice, the slight vibrations due to lack of perfect balance of the wheel cause the hinge friction to be very slight.

The centre of gravity has no vertical acceleration and hence $Z = Mg$. The two vertical forces thus constitute a couple Mgh, where h is the distance of G from the axis of the hinge. The moment of this couple about Oy is Mgh when $\phi = 0$, and, with the arrangement of Fig. 1, is positive. The couple generates angular momentum about the line Oy at the rate Mgh when $\phi = 0$. Hence,

$$C\eta\Omega = R = Mgh. \quad \text{.......................(5)}$$

The value of $\eta\Omega$ in terms of measured quantities is given by (16) below.

The preponderance Mgh is found by hanging such a mass m from a point Q of the frame on the straight line CGO as will produce equilibrium, the wheel and frame being at rest. If $QD = q$, we have $Mgh = mgq$, and thus

$$C\eta\Omega = mgq. \quad \text{.........................(6)}$$

We are not here concerned with the force X, but

$$X = -M(h + l)\Omega^2.$$

If the wheel be spun about its axis, which is horizontal, and if the frame and wheel be then given such a precession about Oz that $\Omega = Mgh/C\eta$, the state of motion so established will be steady and will continue indefinitely in the absence of frictional forces. The method of starting the precession is described in § 4.

The effect of the earth's rotation is considered in § 9 and is shown to be inappreciable.

In practice, η diminishes while a record is being taken. But the observer applies such a couple to the shaft S that the axle of the wheel remains horizontal. This couple has, of course, no component about any horizontal axis. Axle friction does not concern us, for it is internal to the system of frame and wheel. The resistance of the air has, by symmetry, no moment about OB. Hence (5) remains true at each instant.

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§ 3. *General case of motion with constant slope of axle.* In general, the axle is not horizontal but makes angle θ with the horizontal. We now take as axes for frame and wheel (1) GC, (2) GA in the plane COz, as shown, and (3) GB perpendicular to the plane CGA and pointing away from the reader. These right-handed axes are principal axes of the system of frame and wheel. The moments of inertia of the wheel and of the frame about GC are C, C' and about GA are A, A'.

The angular velocity Ω about Oz, or about any parallel axis, may be resolved into $\Omega \sin \theta$ about GC and $-\Omega \cos \theta$ about GA.

The wheel makes p revs. per sec. relative to the frame and to the moving plane COz, and this plane has angular velocity $\Omega \sin \theta$

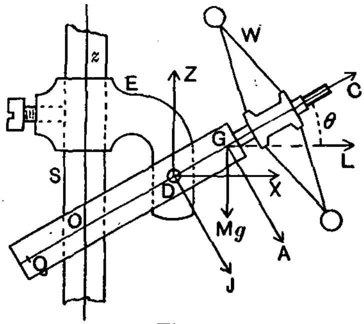

Fig. 3

about GC. As in § 2, we denote $2\pi p$ by η. Hence, if the resultant angular velocity of the wheel about GC be ω, we have

$$\omega = \eta + \Omega \sin \theta. \quad \text{.......................(7)}$$

The angular velocity of the wheel about GA is $-\Omega \cos \theta$.

By symmetry, neither the angular velocity $\Omega \sin \theta$ of the frame nor the angular velocity $\eta + \Omega \sin \theta$ of the wheel about GC gives rise to any angular momentum about either GA or GB. The corresponding angular momenta about GC are $C' \Omega \sin \theta$ and $C(\eta + \Omega \sin \theta)$. By symmetry, the angular velocity $-\Omega \cos \theta$ of frame and wheel about GA gives rise to no angular momentum about either GB or GC. The corresponding angular momentum about GA is $-(A + A')\Omega \cos \theta$. Hence, if the angular momentum of the wheel and frame about the horizontal line GL in the plane COz be F, we have

$$F = C\eta \cos \theta + (C + C' - A - A')\Omega \cos \theta \sin \theta. \quad \text{......(8)}$$

Provided that θ be constant, neither frame nor wheel has angular momentum about GB and this is true whether η and Ω be constant or variable.

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By the method of § 2, if the rate of increase of angular momentum about an axis with which GB is instantaneously coincident be R, we have

$$R = F\Omega. \quad \text{.........................(9)}$$

The forces applied to the system are its weight Mg acting at G and the horizontal and vertical forces X, Z exerted by the hinge, each being in the plane COz. The friction between the wheel and its axle is internal to the system. Then

$$X = -M(l + h\cos\theta)\,\Omega^2, \qquad Z = Mg.$$

The moment of these forces about GB is R, and thus

$$F\Omega = R = Mgh\cos\theta + M(l + h\cos\theta)\,\Omega^2 h\sin\theta. \quad \text{...(10)}$$

Hence, by (8), on dividing by $\cos\theta$,

$$Mgh = C\eta\Omega + (C + C' - A - A' - Mh^2)\,\Omega^2\sin\theta - Mlh\Omega^2\tan\theta.$$

If the moments of inertia of the system about the axes DC and DJ (parallel to GA) be K and H, then

$$C + C' = K, \qquad A + A' + Mh^2 = H.$$

Thus

$$Mgh = C\eta\Omega + (K - H)\,\Omega^2\sin\theta - Mlh\Omega^2\tan\theta. \quad \text{...(11)}$$

For given values of η and θ, (11) gives two values for Ω. If there be one *real* value of Ω there will be two real values.

By suitable design, both l and $K - H$ could be made zero. Then (11) would become

$$Mgh = C\eta\Omega, \quad \text{......................(12)}$$

and the product $\eta\Omega$ for a given steady value of θ would be independent of θ.

In the experiment, Ω/η is small and θ is small, and hence the terms in (11) involving Ω^2 have little effect. When θ is small, $\sin\theta = \tan\theta = \theta$, and then

$$Mgh = C\eta\Omega\,\{1 + \theta(K - H - Mhl)/Cn\} = C\eta\Omega\,\{1 + f\theta/n\}, \text{...(13)}$$

where $n = \eta/\Omega$, so that the wheel makes n turns relative to the frame while the frame turns once about the vertical shaft. Thus (12) needs a correction of the first order in θ, but, with the apparatus used in § 8, f/n is so small that moderate accuracy in adjusting the axle to the horizontal suffices.

§ 4. *Apparatus.* The apparatus is shown diagrammatically in Fig. 4. The axle of the cycle wheel W is attached to the frame UU. This frame is supported by a horizontal axle D passing through the block E carried by the vertical steel shaft SS. The axle is clamped by set-screws to the frame and turns in the long plain hole in E with little friction.

The shaft is supported by a conical bearing R, fixed to the floor, and by a plain bearing J, fixed to a table projecting from the wall of the laboratory. When the wheel is at rest, its axle can be made horizontal by suspending a suitable load, by means of a knife-edged hanger, from the groove Q. The moment of the weight of the wheel and frame about D is then equal to that of the mass hanging from Q. If this mass be m, the latter moment is $mg\,.\,QD$ or mgq (§ 2).

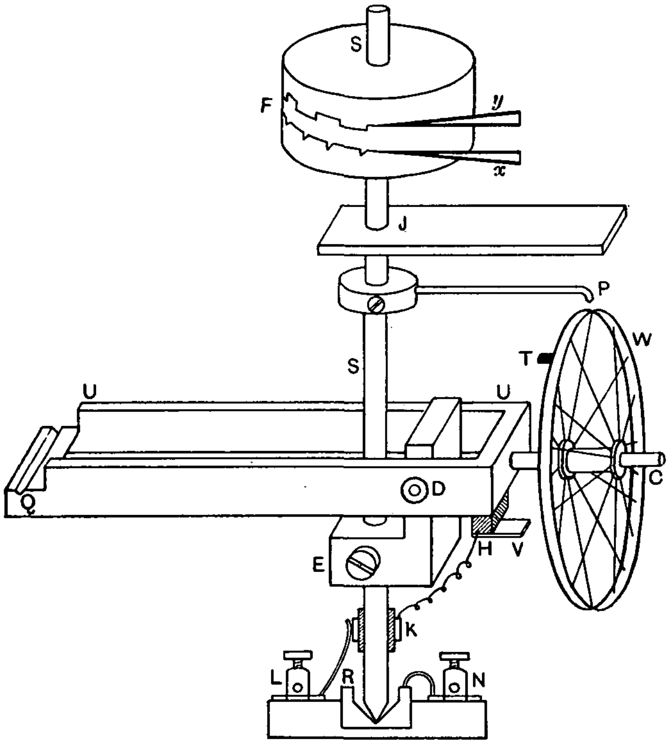

Fig. 4

The upper end of the shaft carries a drum F, on which the records are taken automatically.

To operate the apparatus, the end C of the axle is held in one hand, the axle being horizontal, and the wheel W is given a vigorous spin with the other. If the operator now move the axle in a horizontal plane, he will be able, after a little practice, to give the system such an angular velocity about the shaft S that, when he releases the grip of his hand on the axle, the wheel will precess about S with the axle practically horizontal. Success depends upon so adjusting this angular velocity about S that the force exerted by the axle on the hand is felt to vanish. When this sensation is reached, the grip is released and the hand is removed. Any small oscillations soon disappear under the influence of the slight frictional forces. If the axle be not quite horizontal, an adjustment is made by applying—steadily, not in jerks—a small couple to the shaft by the fingers. If the couple assist the precession, the end C of the axle will rise; a reverse couple will cause C to fall.

Automatic devices are used in recording the two angular velocities η and Ω. The frame carries an insulating block H from which projects a suitable spring, and the wheel carries a projecting tooth; the spring and the tooth are represented diagram-

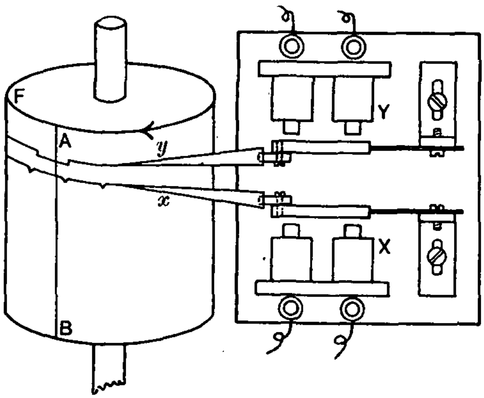

Fig. 5

matically by V and T in Fig. 4. Actually the tooth projects from the hub. The tooth makes contact with the spring once in each revolution of the wheel relative to the frame. This contact puts the spring into electrical connexion with the axle of the wheel and so with the shaft S. A thin, flexible, wire from V passes to a collar K (Fig. 4) fixed to the shaft by an insulating bush. A brush pressing on K connects it with the terminal L. The terminal N is in electrical connexion with the shaft. If a battery be connected to L, N, a current will pass whenever the tooth touches the spring.

The battery circuit includes the windings of an electromagnet X (Fig. 5). When the current passes, the armature is attracted and moves a style x. The electromagnet Y, with its style y, is operated by a pendulum which makes and breaks a second electrical circuit.

The two electromagnets are mounted on a sliding piece which has, when released, a slow vertical motion controlled by a piston working in a cylinder containing very viscous oil.

By a simple geometrical slide, the styles can be brought into contact with the drum F. The drum carries smoked paper, and thus the movements of the styles are recorded. By aid of a line AB ruled on the drum parallel to the shaft, the number of revolutions made by the wheel relative to the frame and also the number of beats made by the pendulum during one revolution of the drum are determined. Fractional parts are, in each case, found by interpolation.

Let the periodic time of the pendulum be T seconds, and in one particular revolution of the drum let there be N periods. Then, if the time of that one revolution of the drum be D seconds, $D = NT$. Hence, if the mean angular velocity of the drum during the interval D be Ω_{av}, we have

$$\Omega_{av} = 2\pi/D = 2\pi/(NT). \qquad (14)$$

Let the wheel make n revolutions relative to the frame while the drum makes the one revolution. Then, if the mean relative angular velocity of the wheel during the interval D be η_{av}, we have

$$\eta_{av} = 2\pi n/D = 2\pi n/(NT). \qquad (15)$$

It is shown in § 7 that, if η, Ω be the values at any instant during the interval D, we may write

$$\eta\Omega = \eta_{av}\Omega_{av} = 4\pi^2 n/D^2 = 4\pi^2 n/(N^2T^2), \qquad (16)$$

with only a second-order error when the friction is small.

Hence, by (5), we may write

$$Mgh = C\eta\Omega = 4\pi^2 Cn/(N^2T^2). \qquad (17)$$

§ 5. *Experimental details.* The shaft S must be vertical. If, when the wheel is not spinning, the system turn about the axis of S to take up a definite position, the shaft is not vertical. The error is corrected by adjusting one of the bearings. The adjustment may be tested by a level resting on an adjustable table fixed to the shaft.

The shaft carries a pointer, which is set so that its tip, P, is vertically above one edge of the rim of the wheel when the axle is horizontal. If the highest and lowest points of that edge be equidistant from the shaft when P is vertically above the selected edge, the adjustment is correct.

A slip of paper, with a hard smooth surface, is wrapped round the drum. The overlapping ready-gummed end is stuck to the slip itself. The drum is held over a smoky paraffin flame to smoke the paper and is then fixed to the shaft.

The pendulum is started and the battery switch is closed. One observer then starts the wheel and gives it approximately the precessional motion which will cause the axle to remain horizontal as the wheel spins. A second observer applies a small couple to the vertical shaft by his fingers so as to keep the axle horizontal. The couple is applied continuously while the record is taken and not spasmodically; small errors of level as indicated by the pointer P are of less consequence than sudden changes of precessional motion. When the wheel is spinning correctly, the first observer releases the slider carrying the electromagnets and brings the styles against the drum. If the speed of the slider has been suitably adjusted, the drum will make 5 or 6 revolutions while the styles move from the top to the bottom of the drum. The drum is removed from the shaft and is mounted on a short shaft which can rest, with its axis horizontal, in V's in a suitable cradle standing on a horizontal plate. By a scribing block sliding on the plate, a line parallel to the axis of the shaft is drawn upon the smoked paper. A second line, to furnish an independent set of readings, may be drawn approximately diametrically opposite the first line. The record is "fixed" by a weak solution of shellac

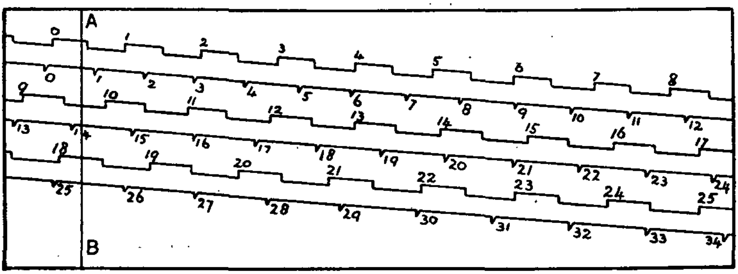

Fig. 6

in methylated spirit. When the paper is dry, it is cut along a generating line of the cylinder and is then ready for measurement.

The general appearance of the record, when removed from the drum, is shown in Fig. 6. The pendulum completes its circuit for *about* half a complete vibration during each complete vibration. The contacts are made at the times indicated by the points marked 0, 1, 2,.... The contact made by the wheel is of short duration. The contacts are marked 0, 1, 2,....

Interpolation is used to find the position of the vertical line AB, which was marked by the scribing block, relative both to the pendulum signals and to the wheel signals. Thus, for the pendulum signals on the original record, which is reproduced on a smaller scale as Fig. 6, the distance from 0 to 1 is 18·5 mm. and that from 0 to AB is 7·2 mm. Hence AB corresponds to $0 + 7{\cdot}2/18{\cdot}5$ or 0·389. Similarly, from 9 to 10 is 21·0 mm. and from 9 to AB is 14·9 mm. Hence AB now corresponds to $9 + 14{\cdot}9/21{\cdot}0$ or 9·710. Thus the revolution has occupied $(9{\cdot}710 - 0{\cdot}389)\,T$ or $9{\cdot}321\,T$, where T is the period of the pendulum.

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For the wheel record we find that AB corresponds (1) to $0 + 9{\cdot}4/12{\cdot}6$ or $0{\cdot}746$ and (2) to $14 + 2{\cdot}9/15{\cdot}5$ or $14{\cdot}187$. Thus, while the drum made the one turn just considered, the wheel made $14{\cdot}187 - 0{\cdot}746$ or $13{\cdot}441$ turns relative to the frame.

The mean angular velocity of the drum is thus $2\pi/(9{\cdot}321T)$. The mean angular velocity of the wheel relative to the frame is $13{\cdot}441$ times that of the drum and is thus $2\pi \times 13{\cdot}441/(9{\cdot}321T)$.

§ 6. *The moments of inertia.* In order to find the moment of inertia (C) of the wheel about its axis, the wheel is removed from the axle and a suitable fitting, of small and known moment of inertia, is secured in the hub. By this fitting, the wheel is fixed, with its axis vertical, to a *straight* vertical torsion wire. An inertia bar is then substituted for the wheel. The moment of inertia (C) of the wheel is deduced from the periods of torsional vibrations of wheel and bar. The moment of inertia of the frame and axle about the axis of the latter is found in a similar manner, a light fitting, of known moment of inertia, being used to fix the frame to the torsion wire.

The wheel is now replaced on its axle and the frame UU is disconnected from the block E (Fig. 4). A small fitting provided with a knife edge is fixed to the frame. The line of the knife edge intersects the axis of the hinge and is perpendicular to that axis and to the axis of the wheel. The system of wheel and frame is then swung as a pendulum about the knife edge. The fitting is adjusted so that its period about the knife edge is approximately equal to that of the system of fitting, frame and wheel. Then the period T_0, which the frame and wheel would have if the fitting were massless, may be taken as equal to that found for the system of the actual fitting, the frame and the wheel. The moment of inertia H (§ 3) of the system about the knife edge is given by

$$H = T_0^2 Mgh/4\pi^2. \qquad \text{.....................(18)}$$

By § 2, $Mgh = mgq$, where m is the mass which balances the system when hung from the groove Q, and $q = QD$.

§ 7. *Relation of average to actual angular velocities.* Since it is impossible to observe the actual values of η and Ω at any instant, we use a number of *average* values of η and Ω derived from successive *complete* revolutions of the drum.

In the case of the general equation (11), if we maintain θ constant by adjusting Ω to suit η at each instant, a diminution of η will involve a change in Ω. Whether this will be an increase or a decrease will depend upon the numerical values of the quantities appearing in (11). For convenience, we will rewrite (11) in the form

$$G = \eta\Omega + J\Omega^2. \qquad \text{.....................(19)}$$

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When the axis of the wheel is horizontal, $J=0$. For the duration of time, D, occupied by one particular revolution of the drum, it will suffice, when the effects of friction are small, to write $\Omega = \Omega_0(1+\beta t)$, where Ω_0 is the value at the beginning of the revolution when $t=0$. Then, by (19),

$$\eta = \frac{G}{\Omega_0(1+\beta t)} - J\Omega_0(1+\beta t).$$

Since the wheel makes n revolutions relative to the frame in time D,

$$2\pi n = \int_0^D \eta\, dt = \frac{G}{\beta\Omega_0}\log(1+\beta D) - J\Omega_0(D+\tfrac{1}{2}\beta D^2).$$

The drum turns through the one revolution in time D. Hence

$$2\pi = \int_0^D \Omega\, dt = \Omega_0(D+\tfrac{1}{2}\beta D^2),$$

and $$\Omega_0 = 2\pi/(D+\tfrac{1}{2}\beta D^2).$$

Hence $$2\pi n = \frac{GD}{2\pi\beta}(1+\tfrac{1}{2}\beta D)\log(1+\beta D) - 2\pi J.$$

Thus, when βD is small,

$$2\pi n = \frac{GD^2}{2\pi}(1+\tfrac{1}{12}\beta^2 D^2) - 2\pi J. \quad \text{...........(20)}$$

Since (19) holds at each instant,

$$G = \eta_0\Omega_0 + J\Omega_0^2. \quad \text{....................(21)}$$

If we write $$G^* = \frac{2\pi n}{D}\frac{2\pi}{D} + J\frac{4\pi^2}{D^2}, \quad \text{.................(22)}$$

then G^* is the value of G *computed* by using the *average* values $\eta_{av} = 2\pi n/D$ and $\Omega_{av} = 2\pi/D$ in place of η_0 and Ω_0. By (20),

$$G^* = G(1+\tfrac{1}{12}\beta^2 D^2), \quad \text{.................(23)}$$

or $$G = G^*(1-\tfrac{1}{12}\beta^2 D^2). \quad \text{.................(24)}$$

With the apparatus at the Cavendish Laboratory, β is less than ·02, when a second is the unit of time. When the wheel is spun by hand, the greatest value of D is less than 10 seconds, and hence $\beta D <$ ·2. Thus G does not differ from G^* by as much as one part in 300.

When the axis of the wheel is horizontal and, in consequence, $J=0$, (24) becomes, since $D=nT$,

$$\eta_0\Omega_0 = \frac{4\pi^2 n}{N^2T^2}(1-\tfrac{1}{12}\beta^2 D^2). \quad \text{..............(25)}$$

§ 8. *Practical example.*

The wheel had mass 1515 grm. and diameter 64·9 cm. The shortest distance between the centre of its hub and the axis of the hinge was 9·3 cm.

Moments of inertia. The moments of inertia with respect to axis OC (Fig. 3) were found as in § 6.

For wheel, $C = 1{\cdot}11013 \times 10^6$; for frame, $C' = 4{\cdot}488 \times 10^4$ grm. cm.^{2}

Hence . $K = C + C' = 1{\cdot}1550 \times 10^6$ grm. cm.^{2}

The moment of inertia, H, of system of wheel and frame about axis DJ was found as in § 6. Period about knife edge, $T_0 = 1{\cdot}462$ secs.; as below, $Mgh = 1{\cdot}3402 \times 10^7$ dyne cm.

By (18), $H = 7{\cdot}256 \times 10^5$ grm. cm.^{2}

The shortest distance, l, between axes of vertical shaft and hinge was a little uncertain as shaft is not quite straight and is not very stiff. Approximately, $l = 1{\cdot}5$ cm. Thus

$$Mhl = Mgh \, . \, l/g = 1{\cdot}3402 \times 10^7 \times 1{\cdot}5/981 = 2{\cdot}05 \times 10^4 \text{ grm. cm.}^2$$

Hence, see (13),

$$f = (K - H - Mhl)/C = 4{\cdot}089 \times 10^5/(1{\cdot}11013 \times 10^6) = 0{\cdot}368.$$

Thus, by (13),

$$Mgh = C\eta\Omega\,(1 + f\theta/n) = C\eta\Omega\,(1 + {\cdot}368\theta/n) \text{ dyne cm.}$$

The radius of the wheel is 32·5 cm. An error of one cm. horizontally in adjusting rim to pointer P (Fig. 4) makes $\theta = 1/32{\cdot}5$ radians. In measurements recorded below, least value of n was 7·342. With $\theta = 1/32{\cdot}5$, and $n = 7{\cdot}342$, $0{\cdot}368\theta/n = 0{\cdot}0015$. After practice, it is easy to maintain adjustment to within 0·5 cm.; thus error due to obliquity was less than one per thousand.

Precession measurements. These were made with the help of Mr Leslie Bairstow, of King's College. The periodic time of the pendulum was $T = 0{\cdot}71206$ secs.

The second and third columns of Table give respectively swings of pendulum and turns of wheel corresponding to line AB (Fig. 6) ruled on record; fractions were obtained by interpolation (§ 5). Columns "N" and "n" give swings of pendulum and turns of wheel during successive turns of drum.

Turns of drum	Swings of pendulum	Turns of wheel	Per turn of drum		$\frac{n}{N^2}$	$\frac{\beta^2 D^2}{12}$
			Swings, N, of pendulum	Turns, n, of wheel		
0	0·817	0·658	—	—	—	—
1	12·331	21·260	11·514	20·602	·15540	·00176
2	22·569	37·544	10·238	16·284	·15536	·00106
3	31·816	50·848	9·247	13·304	·15559	·00070
4	40·301	62·023	8·485	11·175	·15522	·00049
5	48·163	71·602	7·862	9·579	·15497	·00036
6	55·488	79·946	7·325	8·344	·15551	·00028
7	62·368	87·288	6·880	7·342	·15511	·00022

Mean value of $n/N^2 = 0{\cdot}15531$.

Two other records gave 0·15558 and 0·15551 for n/N^2. The mean of the three values is 0·15547.

Hence, by (16), which omits the correction due to (25),

$$C\eta\Omega = C \times 4\pi^2 (n/N^2)/T^2 = 4\pi^2 \times 1{\cdot}11013 \times 10^6 \times 0{\cdot}15547/({\cdot}71206)^2$$
$$= 1{\cdot}3438 \times 10^7 \text{ dyne cm.}$$

When θ, the angle turned through by drum, is plotted against t, it conforms closely to $\theta = a(t + \frac{1}{2}\gamma t)$. With values in Table, $a = 9{\cdot}82\pi/(60T)$, $\gamma T = 1/79{\cdot}2$, where T secs. is pendulum period. The angular velocity is $\Omega = a(1 + \gamma t)$.

If angular velocity be Ω_0 when $t = rT$ and Ω' when $t = rT + t'$,

$$\Omega' = \Omega_0 \left(1 + \frac{\gamma t'}{1 + \gamma r T}\right) = \Omega_0 (1 + \beta t').$$

Thus $\beta = \gamma/(1 + \gamma r T)$. If time occupied by turn which begins when $t = rT$ be sT, we have $D = sT$; then

$$\beta D = s/\{r + (\gamma T)^{-1}\} = s/(r + 79{\cdot}2).$$

For the first turn, $r = 0$, $s = 11{\cdot}514$, $\beta D = 0{\cdot}145$; for the last turn, $r = 54{\cdot}671$, $s = 6{\cdot}88$, $\beta D = 0{\cdot}051$. The values of $\frac{1}{12}\beta^2 D^2$ are given in Table; their mean is 0·00070. The three records are so nearly alike and the values of n/N^2 are so nearly equal that we apply the mean correction to the mean of n/N^2. Hence, by (25),

$$C\eta\Omega = C \times 4\pi^2 (n/N^2)(1 - 0{\cdot}00070)/T^2 = 1{\cdot}3438 \times 0{\cdot}99930 \times 10^7$$
$$= 1{\cdot}3429 \times 10^7 \text{ dyne cm.}$$

The distance, q, between groove Q (Fig. 4) and axis of hinge was 14·709 cm. On account of hinge friction, the balancing mass was a little uncertain. It was therefore measured ten times. One out of ten different objects of masses unknown to the observer was included in the load, the remainder consisting of known masses. After the balancing values of the latter had been found, the ten objects were weighed. The ten values of the load, m, varied from 927·45 to 929·75 grm. The mean gave $m = 928{\cdot}54$ grm. with a probable error of 0·15 grm. Since Mgh for frame and wheel equals (§ 2) mgq, we have

$$Mgh = 928{\cdot}54 \times 981{\cdot}27 \times 14{\cdot}709 = 1{\cdot}3402 \times 10^7 \text{ dyne cm.}$$

This value differs from $C\eta\Omega$ by 2·0 per thousand.

§ 9. *Effect of the Earth's rotation.*

The earth turns about its axis once in a sidereal day of 86,164 secs. Its angular velocity, Θ, is $2\pi/86164$ or $7{\cdot}292 \times 10^{-5}$ radians per sec. We suppose the precession of the gyroscope about the vertical shaft S (Fig. 4) to be so adjusted by a couple applied to the shaft S that the axis of the wheel is always perpendicular to that shaft.

For the moving origin we take D (Fig. 1), the point in which the axis of the wheel intersects the axis of the hinge. For moving axes x, z, y respectively we take (1) the upward line parallel to the shaft through D, (3) the axis $ODGC$ of the wheel, and (2) a line through D perpendicular to (1) and (3) to complete a right-handed system. At any time the axis (3) makes an angle ψ, measured in the positive direction, with the southward meridian through O. The north latitude of O is λ.

If θ_1, θ_2, θ_3 be the angular velocities of the axes about themselves,

$$\theta_1 = \Theta \sin\lambda + \dot{\psi}, \qquad \theta_2 = -\Theta\cos\lambda \sin\psi, \qquad \theta_3 = -\Theta\cos\lambda\cos\psi.$$

If the components of the velocity of the origin D relative to the centre of the earth be u_1, u_2, u_3, if those of the velocity of G, the centre of gravity, be v_1, v_2, v_3, and if the distance of O from the earth's axis be r, we have

$$u_1 = l\theta_2, \qquad u_2 = -r\Theta\cos\psi - l\theta_1, \qquad u_3 = r\Theta\sin\psi,$$
$$v_1 = (l+h)\,\theta_2, \qquad v_2 = -r\Theta\cos\psi - (l+h)\,\theta_1, \qquad v_3 = r\Theta\sin\psi.$$

The moment of inertia of the wheel about (3) is C and of the frame is C'. The moments of inertia of the wheel and frame system, whose mass is M, about axes through G parallel to (1) and (2) respectively are A_0 and B_0. If h_1, h_2, h_3 be the angular momenta about the axes (1), (2), (3),

$$h_1 = A_0(\Theta\sin\lambda + \dot\psi) - Mhv_2 = (A_0+E)(\Theta\sin\lambda + \dot\psi) + Mhr\Theta\cos\psi,$$
$$h_2 = -B_0\Theta\cos\lambda\sin\psi + Mhv_1 = -(B_0+E)\,\Theta\cos\lambda\sin\psi,$$
$$h_3 = C\omega - C'\Theta\cos\lambda\cos\psi,$$

where
$$E = Mh\,(l+h).$$

Here ω is the angular velocity of the wheel about (3); in the absence of friction, ω remains constant.

The equation of moments about axis (2) is

$$\dot h_2 - h_3\theta_1 + h_1\theta_3 + M(v_1u_3 - v_3u_1) = H_2, \quad \text{.................(26)}$$

where H_2 is the moment about (2) of the forces acting on the system of wheel and frame. These forces arise from the earth's attraction and the reaction of the hinge; the reaction has no moment about (2).

If a particle of one gramme be at rest relative to the earth at O (Fig. 1), the support exerts g dynes along the upward vertical. This force with a force f dynes due to the earth's attraction causes the acceleration $r\Theta^2$ necessary to make the particle move in its circular path of radius r about the earth's axis. The line of action of f intersects the earth's axis in L. If OL make angle ϵ with the vertical through O, we have, by resolving along the vertical,

$$f\cos\epsilon = g + r\Theta^2\cos\lambda.$$

Since the dimensions of the apparatus are very small compared with the earth's radius, we may consider that each gramme of matter in the system of wheel and frame experiences an attraction of f dynes parallel to OL. Hence the resultant force due to attraction passes through G, is parallel to OL and is Mf dynes.

The component $Mf\sin\epsilon$ lies in the plane of (2) and (3) and so has no moment about (2). The moment of $Mf\cos\epsilon$ about (2) is $-Mfh\cos\epsilon$. Thus

$$H_2 = -M(g + r\Theta^2\cos\lambda)\,h.$$

When we substitute in (26) the values of the various quantities, we obtain, after reduction,

$$\dot\psi\,(C\omega + U\Theta\cos\lambda\cos\psi) = Mgh - C\omega\Theta\sin\lambda - V\Theta^2\cos\lambda\sin\lambda\cos\psi, \text{...(27)}$$

where
$$U = A_0 + B_0 + 2E - C', \qquad V = A_0 + E - C'.$$

The apparatus records the time NT (§ 4) occupied by a complete revolution, relative to the earth, of the frame about the vertical shaft. When we integrate (27) from 0 to NT, the terms involving U, V disappear, and thus

$$2\pi C\omega = NT\,(Mgh - C\omega\Theta\sin\lambda).$$

If S be the sidereal day of 86,164 secs., $\Theta = 2\pi/S$, and thus

$$Mgh = \frac{2\pi C\omega}{NT}\left\{1 + \frac{NT\sin\lambda}{S}\right\}. \quad \text{.....................(28)}$$

The angular velocity of the wheel relative to the frame is η, where $\eta = \omega - \theta_3 = \omega + \Theta \cos\lambda \cos\psi$. In time NT the wheel makes n revolutions relative to the frame. Hence

$$2\pi n = \omega NT + \Theta \cos\lambda \int_0^{2\pi} \cos\psi \, (dt/d\psi)\, d\psi = \omega NT + \Theta \cos\lambda \,.\, J.$$

By (27), $\cos\psi\,(dt/d\psi)$ is of the form $L\cos\psi\,(1 + a\cos\psi)/(1 - \beta\cos\psi)$, where, in our case, $|\beta| < 1$. On integration, we obtain

$$J = L \times \int_0^{2\pi} \frac{1 + a\cos\psi}{1 - \beta\cos\psi} \cos\psi d\psi = \frac{2\pi L(a+\beta)}{\beta^2} \left\{ \frac{1}{\sqrt{(1-\beta^2)}} - 1 \right\}$$

$$= 2\pi L\,(a+\beta)\,(\tfrac{1}{2} + \tfrac{3}{4} \,.\, \tfrac{1}{2}\beta^2 + \ldots). \quad \ldots\ldots(29)$$

Since $|\beta| < 1$, the integrand, when expanded in powers of β, is uniformly convergent for all ranges of values of ψ, and may therefore be integrated term by term. The result agrees with (29).

If we retain only the first power of Θ in J, we have

$$2\pi n = \omega NT + \pi U \Theta^2 \cos^2\lambda / Mgh.$$

Using the approximation $Mgh = 2\pi C\omega/NT$, we have

$$2\pi n = \omega NT \left\{ 1 + \frac{U \cos^2\lambda}{2C} \frac{\Theta^2}{\omega^2} \right\}.$$

Neglecting the term involving Θ^2/ω^2, we have $2\pi n = \omega NT$, or $\omega = 2\pi n/(NT)$. Thus, by (28), in place of (17), we have

$$Mgh = \frac{4\pi^2 Cn}{N^2 T^2} \left\{ 1 + \frac{NT \sin\lambda}{S} \right\}. \quad \ldots\ldots(30)$$

The greatest value of NT recorded in § 8 is $11{\cdot}5 \times 0{\cdot}712 = 8{\cdot}19$ secs., $S = 86{,}164$ secs., and at Cambridge $\lambda = 52°\,13'$. Hence $NT \sin\lambda/S = 7{\cdot}5 \times 10^{-5}$. The correction due to the earth's rotation, as given by (30), is therefore inappreciable.

In § 8, the rotation of the frame about the vertical was opposite to the component of the earth's rotation about the same line. In this case the correcting factor is $1 - NT\sin\lambda/S$.

This investigation has been added at the suggestion of Mr R. W. Duncan, of the Royal Aircraft Establishment. The precession of the Cavendish Laboratory gyroscope is so rapid in comparison with the earth's rotation that the latter has little effect. But in many important applications it is precisely the earth's rotation which makes the gyrostatic mechanism effective for its purpose.