GENERALIZED RIEMANN SPACES

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ABSTRACT. The recent attempt at a physical interpretation of non-Riemannian spaces by Einstein (1, 2) has stimulated a study of these spaces (3–8). The usual definition of a non-Riemannian space is one of n dimensions with which is associated an asymmetric fundamental tensor, an asymmetric linear affine connexion and a generalized curvature tensor. We can also consider an n-dimensional space with which is associated a complex symmetric fundamental tensor, a complex symmetric affine connexion and a generalized curvature tensor based on these. Some aspects of this space can be compared with those of a Riemann space endowed with two metrics (9). In the following the fundamental properties of this non-Riemannian manifold will be developed, so that the relation between the geometry and physical theory may be studied.

A generalized Riemann space of real coordinates x_μ ($\mu = 1, \ldots, n$) is defined as one with which is associated a complex symmetric tensor $g_{\mu\nu}$. We have

$$g_{\mu\nu} = s_{\mu\nu} + a_{\mu\nu}, \tag{1}$$

where $s_{\mu\nu}$ and $a_{\mu\nu}$ denote real and imaginary symmetric tensors respectively.

It is assumed that the determinant $g = \operatorname{Det}(g_{\mu\nu}) \neq 0$. We associate uniquely the contravariant tensor $g^{\mu\nu}$ with $g_{\mu\nu}$ by the relation (normalized cofactors)

$$g^{\mu\nu} g_{\mu\sigma} = \delta^\nu_\sigma, \tag{2}$$

where the summation convention of a repeated index is implied, and δ^ν_σ is the Kronecker tensor. Furthermore, it is assumed that the determinant $s = \operatorname{Det}(s_{\mu\nu}) \neq 0$. We define uniquely the real tensor $*s^{\mu\nu}$ by the relation

$$*s^{\mu\nu} s_{\mu\sigma} = \delta^\nu_\sigma. \tag{3}$$

We shall use $g^{\mu\nu}$ and $g_{\mu\nu}$ to raise and lower indices. However, we shall also have occasion to use $*s^{\mu\nu}$ and $s_{\mu\nu}$ and we have $*s^{\mu\nu} \neq g^{\mu\rho} g^{\nu\sigma} s_{\rho\sigma}$.

With the space is also related a complex symmetric affine connexion $\Gamma^\lambda_{\mu\nu}$:

$$\Gamma^\lambda_{\mu\nu} = *\Gamma^\lambda_{\mu\nu} + \hat{\Gamma}^\lambda_{\mu\nu}, \tag{4}$$

where $*\Gamma^\lambda_{\mu\nu}$ and $\hat{\Gamma}^\lambda_{\mu\nu}$ denote the real and imaginary parts of $\Gamma^\lambda_{\mu\nu}$ respectively. The affine connexions $\Gamma'^\lambda_{\mu\nu}$ and $\Gamma^\sigma_{\alpha\beta}$ in coordinate systems x'_μ and x_α are related by the equations

$$\Gamma'^\lambda_{\mu\nu} \frac{\partial x_\rho}{\partial x'_\lambda} = \Gamma^\rho_{\alpha\beta} \frac{\partial x_\alpha}{\partial x'_\mu} \frac{\partial x_\beta}{\partial x'_\nu} + \frac{\partial^2 x_\rho}{\partial x'_\mu \partial x'_\nu}. \tag{5}$$

From this it follows that the real $*\Gamma^\lambda_{\mu\nu}$ transform as an affinity and the imaginary $\hat{\Gamma}^\lambda_{\mu\nu}$ as a tensor.

We can introduce covariant differentiation with respect to $\Gamma^\lambda_{\mu\nu}$ as in Riemannian geometry:

$$\left.\begin{aligned} \lambda^\mu{}_{;\sigma} &= \lambda^\mu{}_{,\sigma} + \lambda^\alpha \Gamma^\mu_{\alpha\sigma}, \\ \lambda_{\mu;\sigma} &= \lambda_{\mu,\sigma} - \lambda_\alpha \Gamma^\alpha_{\mu\sigma}, \end{aligned}\right\} \tag{6}$$

where $\lambda^\mu{}_{,\sigma} = \partial \lambda^\mu / \partial x_\sigma$.

Let us determine the $\Gamma^\lambda_{\mu\nu}$ according to the $g_{\mu\nu}$ by the equations

$$g_{\mu\nu;\sigma} = g_{\mu\nu,\sigma} - g_{\tau\nu} \Gamma^\tau_{\mu\sigma} - g_{\mu\tau} \Gamma^\tau_{\nu\sigma} = 0. \tag{7}$$

Permuting the indices μ, ν, σ twice in (7), adding the resulting two equations and subtracting (7), we get

$$s_{\sigma\tau}\Gamma^{\tau}_{\mu\nu} = \tfrac{1}{2}(g_{\nu\sigma,\mu} + g_{\sigma\mu,\nu} - g_{\mu\nu,\sigma}) - a_{\sigma\tau}\Gamma^{\tau}_{\mu\nu}. \tag{8}$$

Using the real tensor $^*s^{\lambda\sigma}$, defined by (3), we find the implicit solution for $\Gamma^{\lambda}_{\mu\nu}$:

$$\Gamma^{\lambda}_{\mu\nu} = \begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix} + \begin{bmatrix}\lambda\\ \mu\nu\end{bmatrix} - {}^*s^{\lambda\sigma}a_{\sigma\tau}\Gamma^{\tau}_{\mu\nu}. \tag{9}$$

Here we write

$$\left.\begin{aligned}\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix} &= \tfrac{1}{2}{}^*s^{\lambda\sigma}(s_{\nu\sigma,\mu} + s_{\sigma\mu,\nu} - s_{\mu\nu,\sigma}),\\ \begin{bmatrix}\lambda\\ \mu\nu\end{bmatrix} &= \tfrac{1}{2}{}^*s^{\lambda\sigma}(a_{\nu\sigma,\mu} + a_{\sigma\mu,\nu} - a_{\mu\nu,\sigma}),\end{aligned}\right\} \tag{10}$$

where $\begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix}$ are the real Christoffel symbols of the second kind. Separating (9) into real and imaginary parts, we get

$$\left.\begin{aligned}{}^*\Gamma^{\lambda}_{\mu\nu} &= \begin{Bmatrix}\lambda\\ \mu\nu\end{Bmatrix} - {}^*s^{\lambda\sigma}a_{\sigma\tau}\hat{\Gamma}^{\tau}_{\mu\nu},\\ \hat{\Gamma}^{\lambda}_{\mu\nu} &= \begin{bmatrix}\lambda\\ \mu\nu\end{bmatrix} - {}^*s^{\lambda\sigma}a_{\sigma\tau}{}^*\Gamma^{\tau}_{\mu\nu}.\end{aligned}\right\} \tag{11}$$

If the condition of integrability

$$\frac{\partial}{\partial x'_{\sigma}}\left(\frac{\partial^2 x_{\rho}}{\partial x'_{\mu}\partial x'_{\nu}}\right) = \frac{\partial}{\partial x'_{\nu}}\left(\frac{\partial^2 x_{\rho}}{\partial x'_{\mu}\partial x'_{\sigma}}\right) \tag{12}$$

of equations (5) is reduced by means of equations of the form (5), we find

$$R^{\lambda}_{\mu\nu\sigma} = \Gamma^{\lambda}_{\mu\nu,\sigma} - \Gamma^{\lambda}_{\mu\sigma,\nu} - \Gamma^{\lambda}_{\tau\nu}\Gamma^{\tau}_{\mu\sigma} + \Gamma^{\lambda}_{\tau\sigma}\Gamma^{\tau}_{\mu\nu} \tag{13}$$

are the components of a tensor. This tensor is the generalized Riemann–Christoffel tensor of the space. From (13) we see that $R^{\lambda}_{\mu\nu\sigma}$ is skew-symmetric in the last two indices

$$R^{\lambda}_{\mu\nu\sigma} = -R^{\lambda}_{\mu\sigma\nu}. \tag{14}$$

Splitting (13) into real and imaginary parts, we get

$${}^*R^{\lambda}_{\mu\nu\sigma} = {}^*\Gamma^{\lambda}_{\mu\nu,\sigma} - {}^*\Gamma^{\lambda}_{\mu\sigma,\nu} - {}^*\Gamma^{\lambda}_{\tau\nu}{}^*\Gamma^{\tau}_{\mu\sigma} - \hat{\Gamma}^{\lambda}_{\tau\nu}\hat{\Gamma}^{\tau}_{\mu\sigma} + {}^*\Gamma^{\lambda}_{\tau\sigma}{}^*\Gamma^{\tau}_{\mu\nu} + \hat{\Gamma}^{\lambda}_{\tau\sigma}\hat{\Gamma}^{\tau}_{\mu\nu}, \tag{15}$$

and

$$\hat{R}^{\lambda}_{\mu\nu\sigma} = \hat{\Gamma}^{\lambda}_{\mu\nu,\sigma} - \hat{\Gamma}^{\lambda}_{\mu\sigma,\nu} - \hat{\Gamma}^{\lambda}_{\tau\nu}{}^*\Gamma^{\tau}_{\mu\sigma} - {}^*\Gamma^{\lambda}_{\tau\nu}\hat{\Gamma}^{\tau}_{\mu\sigma} + \hat{\Gamma}^{\lambda}_{\sigma\tau}{}^*\Gamma^{\tau}_{\mu\nu} + {}^*\Gamma^{\lambda}_{\sigma\tau}\hat{\Gamma}^{\tau}_{\mu\nu}. \tag{16}$$

Here we have denoted by $^*R^{\lambda}_{\mu\nu\sigma}$ and $\hat{R}^{\lambda}_{\mu\nu\sigma}$ the real and imaginary parts of $R^{\lambda}_{\mu\nu\sigma}$ respectively. We can write (15) and (16) as

$${}^*R^{\lambda}_{\mu\nu\sigma} = B^{\lambda}_{\mu\nu\sigma} - \hat{\Gamma}^{\lambda}_{\tau\nu}\hat{\Gamma}^{\tau}_{\mu\sigma} + \hat{\Gamma}^{\lambda}_{\tau\sigma}\hat{\Gamma}^{\tau}_{\mu\nu} \tag{17}$$

and

$$\hat{R}^{\lambda}_{\mu\nu\sigma} = \hat{\Gamma}^{\lambda}_{\mu\nu/\sigma} - \hat{\Gamma}^{\lambda}_{\mu\sigma/\nu}, \tag{18}$$

where $B^{\lambda}_{\mu\nu\sigma}$ is the tensor based on the $^*\Gamma^{\lambda}_{\mu\nu}$ and / means covariant differentiation with respect to $^*\Gamma^{\lambda}_{\mu\nu}$.

Contracting (13) on the indices λ and σ, we get

$$R_{\mu\nu} = R^{\sigma}_{\mu\nu\sigma} = \Gamma^{\sigma}_{\mu\nu,\sigma} - \Gamma^{\sigma}_{\mu\sigma,\nu} - \Gamma^{\tau}_{\mu\sigma}\Gamma^{\sigma}_{\tau\nu} + \Gamma^{\sigma}_{\mu\nu}\Gamma^{\tau}_{\sigma\tau}. \tag{19}$$

We call the tensor $R_{\mu\nu}$ the generalized Ricci tensor. Splitting (19) into real and imaginary parts, we have

$${}^*R_{\mu\nu} = P_{\mu\nu} + \Delta^{\sigma}_{\mu\nu,\sigma} - \Delta^{\sigma}_{\mu\sigma,\nu} - \Delta^{\tau}_{\mu\sigma}\Delta^{\sigma}_{\tau\nu} + \Delta^{\sigma}_{\mu\nu}\Delta^{\tau}_{\sigma\tau} - \hat{\Gamma}^{\tau}_{\mu\sigma}\hat{\Gamma}^{\sigma}_{\tau\nu} + \hat{\Gamma}^{\sigma}_{\mu\nu}\hat{\Gamma}^{\tau}_{\sigma\tau}, \tag{20}$$

and
$$\hat{R}_{\mu\nu} = \hat{\Gamma}^{\sigma}_{\mu\nu/\sigma} - \hat{\Gamma}^{\sigma}_{\mu\sigma/\nu}, \tag{21}$$

where $P_{\mu\nu}$ is the real Ricci tensor based on the $\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}$:

$$P_{\mu\nu} = \begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix}_{,\sigma} - \begin{Bmatrix} \sigma \\ \mu\sigma \end{Bmatrix}_{,\nu} - \begin{Bmatrix} \tau \\ \mu\sigma \end{Bmatrix}\begin{Bmatrix} \sigma \\ \tau\nu \end{Bmatrix} + \begin{Bmatrix} \sigma \\ \mu\nu \end{Bmatrix}\begin{Bmatrix} \tau \\ \sigma\tau \end{Bmatrix}. \tag{22}$$

Moreover, the quantities $\Delta^{\lambda}_{\mu\nu}$ are determined by

$$\Delta^{\lambda}_{\mu\nu} = {}^*\Gamma^{\lambda}_{\mu\nu} - \begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}. \tag{23}$$

A straightforward calculation shows that the tensor $R^{\lambda}_{\mu\nu\sigma}$ satisfies the differential identities because of (14):

$$R^{\lambda}_{\boldsymbol{\mu\nu\sigma};\tau} = R^{\lambda}_{\boldsymbol{\mu\nu\sigma},\tau} + R^{\rho}_{\boldsymbol{\mu\nu\sigma}}\Gamma^{\lambda}_{\rho\tau} - R^{\lambda}_{\boldsymbol{\rho\nu\sigma}}\Gamma^{\rho}_{\mu\tau} = 0, \tag{24}$$

where we have used the notation

$$A_{\boldsymbol{\mu\nu\sigma}} = A_{\mu\nu\sigma} + A_{\nu\sigma\mu} + A_{\sigma\mu\nu}. \tag{25}$$

By means of (23) we can write (24) as

$$R^{\lambda}_{\boldsymbol{\mu\nu\sigma}\underset{\circ}{;}\tau} + R^{\rho}_{\boldsymbol{\mu\nu\sigma}}\Delta^{\lambda}_{\rho\tau} - R^{\lambda}_{\boldsymbol{\rho\nu\sigma}}\Delta^{\rho}_{\mu\tau} + R^{\rho}_{\boldsymbol{\mu\nu\sigma}}\hat{\Gamma}^{\lambda}_{\rho\tau} - R^{\lambda}_{\boldsymbol{\rho\nu\sigma}}\hat{\Gamma}^{\rho}_{\mu\tau} = 0, \tag{26}$$

where $\underset{\circ}{;}$ means covariant differentiation with respect to $\begin{Bmatrix} \lambda \\ \mu\nu \end{Bmatrix}$.

The author has developed a generalization of Einstein's gravitational theory by using this non-Riemannian manifold in the case when $n = 4$. The correct equations of motion of charged particles in an electromagnetic field have been derived from the field equations adopted in the theory by using the method introduced by Infeld (**10**), in which the particles are described by singular world-lines.

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