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Periodic solutions of certain nonlinear autonomous wave equations

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1. *Introduction.* The hyperbolic boundary value problem

$$\left.\begin{aligned} &\partial_t^2 u - \partial_x^2 u + \epsilon u = \delta f(u) \quad (t,x)\in\mathbb{R}\times(0,\pi) \quad (\epsilon > 0),\\ &u|_{x=0} = u|_{x=\pi} = 0. \end{aligned}\right\} \tag{1}$$

has, for $\delta = 0$ and each positive integer k, t-periodic solutions of period $\tau_k(\epsilon) = 2\pi/(k^2+\epsilon)^{\frac{1}{2}}$. For a reasonably large class of functions, f, and, for each k, certain values of ϵ, one at least of these solutions persists for small δ.

For a given positive integer, k, consider the following conditions on ϵ and f.

(I) $\exists$ a positive constant c such that

$$\min\{|(n^2+\epsilon)^{\frac{1}{2}}/(k^2+\epsilon)^{\frac{1}{2}} - j|; j\in\mathbb{N}\} > c/n \quad \forall n\in\mathbb{N}, n \neq k.$$

(II) $f\colon \mathbb{R}\to\mathbb{R}$ is a C^2 function with $f(0) = 0$, and the function

$$\mathbb{R}\ni r \to \int_0^{2\pi}\int_0^{\pi} \sin kx \,.\, \sin s f(r\sin kx \,.\, \sin s)\, dx\, ds$$

takes both positive and negative values for positive r. Let $C(\mathbb{R}\times[0,\pi])$ be the space of continuous real-valued functions on $\mathbb{R}\times[0,\pi]$.

THEOREM 1. *Under conditions* (I) *and* (II), *there exists* $\delta_0 > 0$ *and a map*

$$(-\delta_0, \delta_0)\ni\delta\to u(\delta)\in C(\mathbb{R}\times[0,\pi]),$$

such that $u = u(\delta)$ *satisfies* (1), *in the sense of distributions, is t-periodic of period* $\tau_k(\epsilon)$ *and not identically zero.*

The proof of this theorem occupies sections three to six, but first, in section two, some remarks on conditions (I) and (II) are made.

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2. *Remarks on conditions (I) and (II).* First, it will be shown that, for each (positive) integer k, the set of ϵ satisfying (I) has 0 as a point of accumulation (and is therefore at least countably infinite). To see this, define, for $q = 1, 2, \ldots$

$$\epsilon_q = 2k/q + 1/q^2.$$

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Then, $(k^2+\epsilon_q)^{\frac{1}{2}} = k+1/q$ and, for any n,

$$\omega(n) \equiv \left(\frac{n^2+\epsilon_q}{k^2+\epsilon_q}\right)^{\frac{1}{2}} = \left(\frac{qn}{kq+1}\right)\left(1+\frac{\epsilon_q}{n^2}\right)^{\frac{1}{2}}.$$

Now, the ϵ_q decreases as q increases so it follows from Taylor's theorem that there exists q_0 such that, $\forall\, n \in \mathbb{N}$ and $q \geqslant q_0$,

$$\left(1+\frac{\epsilon_q}{n^2}\right)^{\frac{1}{2}} = 1+\frac{\epsilon_q}{2n^2}(1+\mu), \quad |\mu| < \tfrac{1}{2}.$$

Thus,

$$\omega(n) = \frac{qn}{(kq+1)}+\frac{\epsilon_q q}{2n(kq+1)}(1+\mu).$$

If $n \geqslant \epsilon_q q$, it follows that

$$0 < \frac{\epsilon_q q}{2n(kq+1)}(1+\mu) < \frac{1}{kq+1},$$

and therefore that $\omega(n)$ is not an integer. $\epsilon_q q = 2k+1/q$, so assuming $q \geqslant q_0 \geqslant 1$, this means that $\omega(n)$ is not an integer if $n > 2k$,

If $n \leqslant 2k$, then $n^2+\epsilon_q < 4k^2+4\epsilon_q$ so $\omega(n) < 2$, and therefore (I) is true if it is true for large n. However, for large n,

$$\omega(n) = \frac{qn}{kq+1}+\frac{\epsilon_q q}{2n(kq+1)}+O\left(\frac{1}{n^2}\right)$$

so condition (I) holds for $\epsilon = \epsilon_q$, if $q \geqslant q_0$.

Secondly, note that if $f(s)$ is odd, negative for small positive s and $f(s) \to +\infty$ as $s \to \infty$ then (II) holds, so it is not unduly restrictive. Moreover, if $\hat{f}(s) = f(-s)$ satisfies condition (II) then the theorem remains true.

Finally, it is not difficult to show the continuity and local uniqueness of the map in Theorem 1 under a suitable monotonicity condition on the function occurring in (II).

3. *Existence of solutions to the abstract Cauchy problem.* The existence and uniqueness of solutions to the initial-boundary value problem

$$\left.\begin{aligned} &\partial_t^2 u - \partial_x^2 u + \epsilon u = \delta f(u) \quad (t,x) \in \mathbb{R}\times(0,\pi) \\ &u|_{x=0} = u|_{x=\pi} = 0, \quad u|_{t=0} = u_1, \quad \partial_t u|_{t=0} = u_2 \end{aligned}\right\} \tag{2}$$

is classical. However, a proof, similar in spirit to that of Theorem 1, will be given.

Let $H_s = H_s(0,\pi)$, $\mathring{H}_s = \mathring{H}_s(0,\pi)\,(H_0 = L_2)$ be, for $s \in \mathbb{Z}$, the usual (real) Sobolev spaces. For any Banach space B, let $C([0,T];B)$ be the Banach space of continuous functions $[0,T] \to B$.

Theorem 2. *If f is C^1, then for each $T > 0\,\exists$ a neighbourhood $U(T) \subset \mathring{H}_1 \oplus L_2 \oplus \mathbb{R}$ of the hyperplane $\mathring{H}_1 \oplus L_2 \oplus \{0\}$ such that, for each $(u_1,u_2,\delta) \in U(T)$, the problem* (2) *has a unique solution $u \in C([0,T];\mathring{H}_1)$ with $\partial_t u \in C([0,T];L_2)$. The map so defined*

$$S: \mathring{H}_1 \oplus L_2 \oplus \mathbb{R} \supset U(T) \ni (u_1,u_2,\delta) \mapsto (u,\partial_t u) \in C([0,T];\mathring{H}_1) \oplus C([0,T];L_2)$$

is C^1. If F is a primitive of f, then the energy,

$$E(u) = \int_0^\pi [(\partial_t u)^2 + (\partial_x u)^2 + \epsilon u^2 - 2\delta F(u)]\,dx$$

is independent of t.

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Proof. Put $A = \{u \in C([0,T]; \mathring{H}_1);\ \partial_t u \in C([0,T]; L_2)\}$, with the obvious Banach topology. The linear map

$$Q: A \ni u \mapsto (u|_{t=0}, \partial_t u|_{t=0}, \partial_t^2 u - \partial_x^2 u + \epsilon u) \in \mathring{H}_1 \oplus L_2 \oplus \mathscr{D}'((0,T) \times (0,\pi))$$

is $1-1$ and so defines an isomorphism onto its range, B, with the Banach topology given by $\|Qu\|_B = \|u\|_A$. The solution map for the problem

$$\left.\begin{aligned} &\partial_t^2 u - \partial_x^2 u + \epsilon u = g, \\ &u|_{x=0} = u|_{x=\pi} = 0, \quad u|_{t=0} = u_1, \partial_t u|_{t=0} = u_2 \end{aligned}\right\} \tag{3}$$

is a continuous linear map

$$\mathring{H}_1 \oplus L_2 \oplus C([0,T]; L_2) \ni (u_1, u_2, g) \mapsto (u, \partial_t u) \in C([0,T]; \mathring{H}_1) \oplus C([0,T]; L_2).$$

This map, followed by Q is the identity, that is the natural embedding of

$$\mathring{H}_1 \oplus L_2 \oplus C([0,T]; L_2) \quad \text{into} \quad \mathring{H}_1 \oplus L_2 \oplus \mathscr{D}'((0,T) \times (0,\pi));$$

$$\text{so} \quad \mathring{H}_1 \oplus L_2 \oplus C([0,T]; L_2)$$

is continuously embedded in B.

Now, consider the nonlinear term. When f is continuous, the fact that $\mathring{H}_1$ is continuously embedded in $C([0,T]; \mathbb{R})$ implies that $f(u) \in C([0,T]; L_2)$, if $u \in C([0,T]; \mathring{H}_1)$. The map

$$\mathscr{F}: C([0,T]; \mathring{H}_1) \ni u \mapsto f(u) \in C([0,T]; L_2)$$

is then continuous, and C^1 if f is C^1. If $u \in A$, (2) is equivalent to

$$Qu = (u_1, u_2, \delta \mathscr{F} u) \in B,$$

i.e. if P is the map

$$P: \mathring{H}_1 \oplus L_2 \oplus \mathbb{R} \oplus A \ni (u_1, u_2, \delta, u) \mapsto Qu - (u_1, u_2, \delta \mathscr{F} u) \in B,$$

then (2) is equivalent to $P(u_1, u_2, \delta, u) = 0$. P is clearly C^1 and its partial derivative with respect to the variables in A, evaluated at the point $(u_1, u_2, 0, \bar{u}), \bar{u} = Q^{-1}(u_1, u_2, 0)$, is the map $Q \in \mathscr{L}(A, B)$. As this is an isomorphism, the implicit function theorem (Dieudonné (1), Theorem 10·2·1) can be applied and easily gives the theorem as stated, except for the constancy of the energy functional. This can be proved by integration by parts, in the usual way.

4. *Periodicity modulo a subspace.* For each $t \in [0,T]$, let $\gamma(t)$ be the map

$$\gamma(t): C([0,T]; \mathring{H}_1) \oplus C([0,T]; L_2) \ni (u, v) \mapsto (u(t), v(t)) \in \mathring{H}_1 \oplus L_2.$$

Thus,

$$\gamma(\tau_k) \circ S: \mathring{H}_1 \oplus L_2 \oplus \mathbb{R} \supset U(\tau_k) \mapsto \mathring{H}_1 \oplus L_2.$$

The partial derivative of this map, with respect to the variables in $\mathring{H}_1 \oplus L_2$ evaluated at $\delta = 0$, is easily seen to be the constant map $\mathring{H}_1 \oplus L_2 \to \mathscr{L}(\mathring{H}_1 \oplus L_2)$, whose image is $\gamma(\tau_k) \circ (S|_{\delta=0})$, which will be denoted by S_k.

If $(u_1, u_2) \in \mathring{H}_1 \oplus L_2$, then there are unique expansions

$$u_1(x) = \sum_{n=1}^{\infty} \alpha_n \sin nx, \quad u_2(x) = \sum_{n=1}^{\infty} \beta_n \sin nx$$

and

$$\sum_n n^2 \alpha_n^2 + \beta_n^2 < \infty.$$

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It is easily verified that

$$\left.\begin{aligned} S_k(u_1,u_2) = \Big(&\sum_n \Big(\alpha_n \cos (n^2+\epsilon)^{\frac{1}{2}}\tau_k + \frac{\beta_n}{(n^2+\epsilon)^{\frac{1}{2}}}\sin (n^2+\epsilon)^{\frac{1}{2}}\tau_k\Big)\sin nx,\\ &\sum_n (-\alpha_n (n^2+\epsilon)^{\frac{1}{2}} \sin (n^2+\epsilon)^{\frac{1}{2}}\tau_k + \beta_n \cos (n^2+\epsilon)^{\frac{1}{2}}\tau_k)\sin nx\Big).\end{aligned}\right\} \tag{4}$$

Condition (I) implies that $(n^2+\epsilon)^{\frac{1}{2}}\tau_k = 2\pi(n^2+\epsilon)^{\frac{1}{2}}/(k^2+\epsilon)^{\frac{1}{2}}$ is an integral multiple of 2π only for $n = k$. It follows from this and (4) that the kernel of $S_k - I$ is the two-dimensional subspace, V_k, of $\mathring{H}_1 \oplus L_2$; spanned by $(\sin (kx), 0)$ and $(0, \sin (kx))$.

Define, for $n \neq k$, $\gamma_n^{-1} = \sin^2[(n^2+\epsilon)^{\frac{1}{2}}\frac{1}{2}\tau_k]$ and $\gamma_k = 1$. Denote by W the subspace of $\mathring{H}_1 \oplus L_2$ consisting of those pairs $\phi = (\phi_1, \phi_2)$ which have sine expansions

$$\phi_1(x) = \sum_{n=1}^{\infty} \sigma_n \sin nx, \quad \phi_2(x) = \sum_{n=1}^{\infty} \mu_n \sin nx$$

such that

$$\|\phi\|_w^2 = \sum_n \gamma_n[(n^2+\epsilon)\sigma_n^2 + \mu_n^2] < \infty.$$

With this norm, W is clearly a Hilbert space. Let W_k be the orthogonal complement of V_k in W, and let π_W be the projection onto W_k. The asymptotic information in condition (I) has the following consequence.

LEMMA 3. *If condition* (I) *holds,* $f: \mathbb{R} \to \mathbb{R}$ *is* C^2 *and* $f(0) = 0$, *then the map*

$$\gamma(\tau_k) \circ S - I : \mathring{H}_1 \oplus L_2 \oplus \mathbb{R} \supset U(\tau_k) \to \mathring{H}_1 \oplus L_2,$$

has range contained in W *and the induced map*

$$R: \mathring{H}_1 \oplus L_2 \oplus \mathbb{R} \supset U(\tau_k) \to W$$

is C^1.

Proof. First consider the initial boundary-value problem (3) with $u_1 = u_2 = 0$. Standard theory shows that the solution of this defines, for each $t \in [0, T]$, a continuous linear map

$$P(t): C([0,T]; \mathring{H}_1) \ni g \mapsto (u(t), \partial_t u(t)) \in (H_2 \cap \mathring{H}_1) \oplus \mathring{H}_1,$$

where $(H_2 \cap \mathring{H}_1)$ is treated as a subspace of H_2.

Now, the map R can be written

$$R = (S_k - I) + \delta P(\tau_k) \circ \mathscr{F}_1 \circ S, \tag{5}$$

where $\mathscr{F}_1$ is the map $C([0,\tau_k]; \mathring{H}_1) \ni u \mapsto f(u) \in C([0,\tau_k]; \mathring{H}_1)$, which is C^1 if f is C^2 and $f(0) = 0$. So the last term in (5) is a C^1 map into $(H_2 \cap \mathring{H}_1) \oplus \mathring{H}_1$. To prove the lemma, it only needs to be shown that $(H_2 \cap \mathring{H}_1) \oplus \mathring{H}_1$ is continuously embedded in W and that $S_k - I$ maps $U(\tau_k)$ continuously into W. The first of these results follows from the definition of W and the fact that

$$\gamma_n = \sin^{-2}[(n^2+\epsilon)^{\frac{1}{2}}\tfrac{1}{2}\tau_k] < C'n^2$$

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for some constant C'. The second follows from the expansion (4), since

$$\|(S_k - I)(u_1, u_2)\|_w^2 = \sum_n \gamma_n \left\{(n^2+\epsilon)\left(\alpha_n(\cos(n^2+\epsilon)^{\frac{1}{2}}\tau_k - 1) + \frac{\beta_n}{(n^2+\epsilon)^{\frac{1}{2}}}\sin(n^2+\epsilon)^{\frac{1}{2}}\tau_k\right)^2 + (-\alpha_n(n^2+\epsilon)^{\frac{1}{2}}\sin(n^2+\epsilon)^{\frac{1}{2}}\tau_k + \beta_n(\cos(n^2+\epsilon)^{\frac{1}{2}}\tau_k - 1))^2\right\}$$

$$= 2\sum_n \gamma_n((n^2+\epsilon)\alpha_n^2 + \beta_n^2)(1 - \cos[(n^2+\epsilon)^{\frac{1}{2}}\tau_k])$$

$$\leqslant 4\sum_n [(n^2+\epsilon)\alpha_n^2 + \beta_n^2]$$

$$\leqslant C''\|(u_1, u_2)\|^2_{\mathring{H}_1 \oplus L_2}.$$

So the lemma is proved. It leads to

LEMMA 4. *Under the conditions of the previous lemma* $\exists$ *a neighbourhood* Φ *of* $V_k \oplus \{0\}$ *in* $V_k \oplus \mathbb{R}$ *and a* C^1 *map*

$$v: V_k \oplus \mathbb{R} \supset \Phi \to V_k^{\perp} \subset \mathring{H}_1 \oplus L_2$$

such that

$$\gamma(\tau_k) \circ S(\phi + v(\phi, \delta), \delta) = v(\phi, \delta) + \delta\bar{\phi}(\phi, \delta) + \phi,$$

where $\bar{\phi}: \Phi \to V_k$ is continuous.

Proof. Consider the map $\pi_W \circ R: U(\tau_k) \to W_k$, which is C^1. Splitting the domain variables $U(\tau_k) \subset (V_k \oplus \mathbb{R}) \oplus V_k^{\perp}$, the implicit function theorem can be applied at each point $(\phi, 0, 0) \in (V_k \oplus \mathbb{R}) \oplus V_k^{\perp}$ (clearly $\pi_W \circ R(\phi, 0, 0) = 0$) provided that the partial derivative with respect to the variables in $V_k^{\perp}$, evaluated at $(\phi, 0, 0)$, is an isomorphism of $V_k^{\perp}$ onto W_k. This derivative is just $\pi_W \circ S_k - I$, restricted to $V_k^{\perp}$, and from the proof of the last lemma is clearly continuous and 1–1.

Suppose that $(\psi_1, \psi_2) \in W_k$, so

$$\psi_1 = \sum_{n \neq k} \sigma_n \sin nx, \quad \psi_2 = \sum_{n \neq k} \mu_n \sin nx$$

with

$$\sum_n \gamma_n [(n^2+\epsilon)\sigma_n^2 + \mu_n^2] = \|(\psi_1, \psi_2)\|_w^2 < \infty.$$

Put

$$\alpha_n = \tfrac{1}{4}\gamma_n \left[\sigma_n(\cos(n^2+\epsilon)^{\frac{1}{2}}\tau_k - 1) - \frac{\mu_n}{(n^2+\epsilon)^{\frac{1}{2}}}\sin(n^2+\epsilon)^{\frac{1}{2}}\tau_k\right],$$

$$\beta_n = \tfrac{1}{4}\gamma_n[\sigma_n(n^2+\epsilon)^{\frac{1}{2}}\sin(n^2+\epsilon)^{\frac{1}{2}}\tau_k + \mu_n(\cos(n^2+\epsilon)^{\frac{1}{2}}\tau_k - 1)].$$

If $u_1 = \sum_n \alpha_n \sin(nx)$, $u_2 = \sum_n \beta_n \sin(nx)$, it is easy to show that $(u_1, u_2) \in \mathring{H}_1 \oplus L_2$, $\|(u_1, u_2)\|_{\mathring{H}_1 \oplus L_2} \leqslant C''\|(\psi_1, \psi_2)\|_w$ for some constant C'', and

$$\pi_W \circ (S_k - I)(u_1, u_2) = (\psi_1, \psi_2).$$

Thus, $\pi_W \circ (S_k - I)$ is an isomorphism, and the implicit function theorem gives, for each $\phi \in V_k$, a neighbourhood U_ϕ of $(\phi, 0) \in V_k \oplus R$ and a unique, C^1, mapping $v_\phi: U_\phi \to V_k^{\perp}$ such that $R(\psi + v_\phi(\psi, \delta), \delta) \in V_k\ \forall\ (\psi, \delta) \in U_\phi$ and $v_\phi(\psi, 0) = 0$. As they are locally unique, these maps piece together to give v on some neighbourhood Φ of $V_k \oplus \{0\} \subset V_k \oplus \mathbb{R}$, with the required properties, since

$$\eta(\phi, \delta) = \gamma(\tau_k) \circ S(\phi + v(\phi, \delta), \delta) - \phi - v(\phi, \delta)$$

defines a C^1 map $\eta: \Phi \to V_k$, with $\eta(\phi, 0) = 0$, so $\eta = \delta\bar{\phi}$ with $\bar{\phi}: \Phi \to V_k$ continuous.

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5. *The projected equation.*

Put

$$g(y) = \frac{2}{\pi}\int_0^\pi \sin kx f(y \sin kx)\,dx.$$

LEMMA 5. *If condition* (II) *holds then* $\exists\,\delta_0 > 0$, $C > 0$ *and* $r_1, r_2 > 0$ *such that the solutions* $y_i, i = 1, 2$, *to*

$$\left.\begin{aligned} y_i'' + (k^2+\epsilon)\,y_i &= \delta g(y_i),\\ y_i(0) = 0, y_i'(0) &= r_i \end{aligned}\right\} \tag{6}$$

satisfy $y_1(\tau_k(\epsilon)) = C\delta + O(\delta^2)$, $y_2(\tau_k(\epsilon)) = -C\delta + O(\delta^2)$, $\forall |\delta| < \delta_0$.

Proof. Standard theory shows that

$$\left| y_i(t) - \frac{r_i}{(k^2+\epsilon)^{\frac{1}{2}}} \sin (k^2+\epsilon)^{\frac{1}{2}} t \right| = O(|\delta|) \quad \text{as} \quad \delta \to 0$$

uniformly for (t, r) in any compact subset of $\mathbb{R}^2$. So, from the integral equation

$$y_i(t) = \frac{r_i}{(k^2+\epsilon)^{\frac{1}{2}}} \sin (k^2+\epsilon)^{\frac{1}{2}} t + \frac{\delta}{(k^2+\epsilon)^{\frac{1}{2}}} \int_0^t \sin (k^2+\epsilon)^{\frac{1}{2}} (t-s)\, g(y_i(s))\,ds$$

and Taylor's theorem applied to g, it follows that

$$y_i(\tau_k) = \frac{\delta}{(k^2+\epsilon)^{\frac{1}{2}}} \int_0^{\tau_k} \sin (k^2+\epsilon)^{\frac{1}{2}} (\tau_k - s)\, g\left(\frac{r_i}{(k^2+\epsilon)^{\frac{1}{2}}} \sin (k^2+\epsilon)^{\frac{1}{2}} s\right) ds + O(\delta^2)$$

uniformly for r_i in any compact set. Thus,

$$y_i(\tau_k(\epsilon)) = -\frac{2\delta}{\pi(k^2+\epsilon)^{\frac{1}{2}}} \int_0^{2\pi}\int_0^{\pi} \sin kx \sin s\, f\left(\frac{r_i}{(k^2+\epsilon)^{\frac{1}{2}}} \sin kx \sin s\right) dx\,ds + O(\delta^2).$$

By condition (II) it is possible to pick $r_1, r_2 > 0$ such that the integral is negative for $i = 1$ and positive for $i = 2$, and this implies the validity of the lemma.

6. *Existence of periodic solutions.* Put

$$u(\phi, \delta) = S(\phi + v(\phi, \delta), \delta) \quad \text{so} \quad u(\phi, \delta) \in C([0, \tau_k]; \mathring{H}_1), \quad \partial_t u \in C([0, \tau_k]; L_2),$$

for $(\phi, \delta) \in \Phi$. Thus, putting $\phi = (\lambda_1 \sin(kx), \lambda_2 \sin(kx))$, $u(\phi, \delta)$ is the solution of the initial boundary value problem with initial conditions

$$\phi + v(\phi, \delta) = (\lambda_1 \sin(kx) + v_1(\phi, \delta), \lambda_2 \sin(kx) + v_2(\phi, \delta)), v \in V_k^\perp.$$

As S is C^1 and $u(\phi, 0) \in C([0, \tau_k]; V_k)$ with $\partial_t u \in C([0, \tau_k]; V_k)$, there exists a C^1 map $\alpha: \Phi \to C^1([0, \tau_k]; \mathbb{R})$ and a continuous map $w: \Phi \to C([0, \tau_k]; \mathring{H}_1)$ such that

$$w(\phi, \delta; t) \in \mathring{H}_1$$

is orthogonal to $\sin kx$ for all $(\phi, \delta; t) \in \Phi \times [0, \tau_k]$ and

$$u = \alpha(t) \sin kx + \delta w.$$

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Equation (2) holds in $C([0,\tau_k]; H_{-1})$, so clearly,

$$\alpha'' + (k^2+\epsilon)\,\alpha = \frac{2}{\pi}\,\delta\int_0^\pi \sin kx f(u(\phi,\delta))\,dx$$

$$= \frac{2}{\pi}\,\delta\int_0^\pi \sin kx f(\alpha\sin kx)\,dx + \delta^2 h(t),$$

where $h\in C([0,\tau_k];\mathbb{R})$ is uniformly bounded for ϕ in any compact subset of V_k and $|\delta|$ sufficiently small.

Now, put $\phi_i = (0, r_i\sin(kx))$, i.e. $\lambda_1 = 0$, $\lambda_2 = r_i$, it follows that $\alpha = y_i + O(\delta^2)$, as $|\delta|\to 0$, where y_i is discussed in Lemma 5, and the error term is uniform for $|r_i| \leqslant R$. So, choosing r_1, r_2 as in Lemma 5, and $\delta_0 > 0$ sufficiently small, it follows that for each δ with $0 < |\delta| < \delta_0$, $\delta\alpha(\tau_k) > 0$ if $\phi = (0, r_1\sin(kx))$ and $\delta\alpha(\tau_k) < 0$ if

$$\phi = (0, r_2\sin(kx)).$$

Thus, for each δ with $0 < |\delta| < \delta_0$, there exists $r(\delta)$ between r_1 and r_2 (and therefore positive), such that if $\phi = (0, r(\delta)\sin(kx))$, $\alpha(\tau_k) = 0$.

From Lemma 4, $u(0) = u(\tau_k)$ and $\partial_t u(0) - \alpha'(0)\sin(kx) = \partial_t u(\tau_k) - \alpha'(\tau_k)\sin(kx)$ is orthogonal to $\sin(kx)$ in L_2. So, the energy equation (Theorem 2) becomes simply

$$(\alpha'(0))^2 = (\alpha'(\tau_k))^2$$

and since $\alpha'(\tau_k) = -\alpha'(0)$ is not possible for small δ (by continuity at $\delta = 0$) the theorem is proved.

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