ON A FINITE GROUP OF ORDER 576

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Introduction

Bagnera (1) shows that when a certain primitive finite group of order 576, G_{576}, is represented as a group of collineations in three-dimensional space, it contains twenty-four harmonic inversions with respect to points and planes, and he shows how to calculate the coordinates of the points and planes of the inversions. The purpose of this note is to discuss the configuration of these points, which we shall call *vertices*, to show that the whole group is generated by the twenty-four harmonic inversions, termed *projections* after Prof. Baker (2) and Dr J. A. Todd (4), and to enumerate the conjugate sets of the group.

1. The configuration

From Bagnera we find that the vertices are

$(1, 0, 0, 0)$	$(1, 1, 1, 1)$	$(-1, 1, 1, 1)$	$(1, 1, 0, 0)$	$(1, 0, 1, 0)$	$(1, 0, 0, 1)$
$(0, 1, 0, 0)$	$(-1, -1, 1, 1)$	$(1, -1, 1, 1)$	$(1, -1, 0, 0)$	$(1, 0, -1, 0)$	$(1, 0, 0, -1)$
$(0, 0, 1, 0)$	$(-1, 1, -1, 1)$	$(1, 1, -1, 1)$	$(0, 0, 1, 1)$	$(0, 1, 0, 1)$	$(0, 1, 1, 0)$
$(0, 0, 0, 1)$	$(-1, 1, 1, -1)$	$(1, 1, 1, -1)$	$(0, 0, 1, -1)$	$(0, 1, 0, -1)$	$(0, 1, -1, 0)$

The four vertices in each column form a tetrahedron, giving six tetrahedra in all. Each vertex of a tetrahedron and its opposite face are found to be the fixed point and plane for one of the twenty-four harmonic inversions of the group; we refer to such a vertex and its corresponding plane as a pole and its polar plane.

The six tetrahedra fall into two systems of three: the first three tetrahedra we call A-tetrahedra, and their vertices A-vertices, and the last three tetrahedra we call B-tetrahedra, with B-vertices. The A-tetrahedra and the B-tetrahedra each form a desmic system, the two systems being *associated* in that the eighteen edges of the three A-tetrahedra are also the edges of the B-tetrahedra (see Hudson (3)).

By examination of the coordinates, the configuration is seen to contain three different types of lines:

(i) There are 18 lines, *a-lines*, which are the edges of the tetrahedra, each containing four vertices, two of each system, forming a harmonic range. Evidently three of these lines pass through each vertex. The polar planes of the four vertices on an a-line meet in another a-line which is the opposite edge of each of the tetrahedra having the original a-line as edge. Two such a-lines are said to be *polar a-lines*.

(ii) There are 72 lines, called *e-lines*, joining two vertices of opposite systems, and containing no further vertices. Six of these lines pass through each vertex. The polar planes of the vertices of an e-line meet in a second e-line which is said to be the *polar e-line* of the first. If UV and $U'V'$ are polar e-lines, then the notation can be arranged so that U and U' come from one tetrahedron, V and V' from another, and then UU', VV' are polar a-lines.

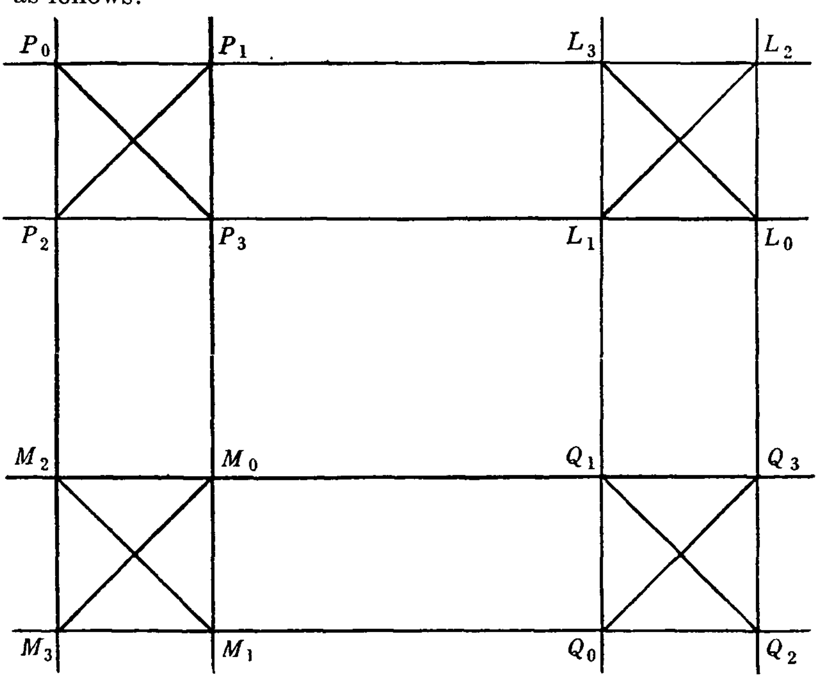
(iii) There are 32 lines, called *κ-lines*, each of which contains three vertices, one vertex from each of the three tetrahedra of a desmic system. These lines fall into two systems, according as they join A-vertices or B-vertices. The polar planes of the three vertices on a κ-line meet in a κ-line of the opposite system; two κ-lines so related are said to be *polar κ-lines*. The nine lines joining the vertices of a κ-line to those of its polar κ-line are all e-lines.

The configuration contains just two different types of planes through three or more non-collinear vertices:

(i) There are 24 planes, called *n-planes*, which are the faces of the tetrahedra and the polar planes of the vertices. Each contains nine vertices, three vertices of the tetrahedron whose face it is, and six vertices of the opposite system. These six vertices are the vertices of a complete quadrilateral of κ-lines whose diagonal triangle is formed by the three edges of the tetrahedron to which the plane belongs. Thus each n-plane contains four κ-lines, three a-lines and six e-lines. The joins of the pole of the plane to the vertices of the diagonal triangle are a-lines, and to the other six vertices of the plane are e-lines. These n-planes form two systems since their poles form two systems.

(ii) There are 96 planes, called *d-planes*, which are the planes joining a vertex to a κ-line in its polar plane. Each contains four points, three of one system on a κ-line, and one of the other system, the joins being e-lines. These d-planes form two systems since the κ-lines form two systems.

If we call the vertices of the tetrahedra of one system L_i, M_i, N_i $(i = 0, 1, 2, 3)$ and of the other system P_i, Q_i, R_i $(i = 0, 1, 2, 3)$, we can represent the configuration diagrammatically as follows:



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Here the vertices R_i are at the intersections of L_0L_3 and L_1L_2 with M_0M_3 and M_1M_2, while the vertices N_i are at the intersections of P_0P_3 and P_1P_2 with Q_0Q_3 and Q_1Q_2. The notation is chosen in such a way that the triads of points on κ-lines and the polar pairs of κ-lines are as follows:

		$L_0M_0N_0$		
		$P_0Q_0R_0$		
$L_0M_1N_1$	$L_1M_0N_1$	$L_1M_1N_0$	$L_1M_2N_3$	$L_1M_3N_2$
$P_0Q_1R_1$	$P_0Q_2R_2$	$P_0Q_3R_3$	$P_1Q_2R_3$	$P_1Q_3R_2$
$L_0M_2N_2$	$L_2M_0N_2$	$L_2M_2N_0$	$L_3M_1N_2$	$L_2M_1N_3$
$P_1Q_0R_1$	$P_2Q_0R_2$	$P_3Q_0R_3$	$P_2Q_3R_1$	$P_2Q_1R_3$
$L_0M_3N_3$	$L_3M_0N_3$	$L_3M_3N_0$	$L_2M_3N_1$	$L_3M_2N_1$
$P_1Q_1R_0$	$P_2Q_2R_0$	$P_3Q_3R_0$	$P_3Q_1R_2$	$P_3Q_2R_1$

2. The projections

We now consider the effect on the configuration of a harmonic inversion with respect to a vertex X and its polar plane. This collineation, leaving fixed all the points of the polar plane, leaves fixed the four vertices of the tetrahedron containing X, and all the vertices joined to X by e-lines. Since the four vertices on an a-line form a harmonic range, the second pair of vertices on each a-line through X is interchanged. Also we see from a simple example that the other two vertices on a κ-line through X are interchanged. This harmonic inversion we call the projection from X and denote by $p(X)$.

We note that $p(X)$ permutes the vertices among themselves, thus leaving the configuration invariant. Also $p(X)$ interchanges the other two tetrahedra of its own system and leaves each tetrahedron of the opposite system invariant. Each of the two systems of 12 vertices is therefore left invariant under a projection $p(X)$. It is easily verified that the group G generated by the projections is transitive on the vertices of each system, and also on the lines and planes of the various types and systems.

Since G_{576} contains the 24 projections $p(X)$, the group G generated by these projections is a subgroup of G_{576}. We shall show that in fact G is equal to G_{576}; we shall also express the operations of G as products of four or fewer projections, and find their periods and their distribution in conjugate sets.

We first show that the 24 projections form two complete sets of conjugate operations in G. Let T be any operation of G, and for any vertex S of the configuration define $T[S] = S^*$. Then if X, X', Y, Y' are the two pairs of vertices on an a-line, C and D lie on a κ-line through X, and XF is an e-line, since T is an automorphism of the configuration we have

$$\begin{aligned}
Tp(X)\,T^{-1}[X^*] &= Tp(X)\,[X] = T[X] = X^*,\\
Tp(X)\,T^{-1}[X'^*] &= Tp(X)\,[X'] = T[X'] = X'^*,\\
Tp(X)\,T^{-1}[Y^*] &= Tp(X)\,[Y] = T[Y'] = Y'^*,\\
Tp(X)\,T^{-1}[C^*] &= Tp(X)\,[C] = T[D] = D^*,\\
Tp(X)\,T^{-1}[F^*] &= Tp(X)\,[F] = T[F] = F^*.
\end{aligned}$$

Thus $Tp(X)T^{-1}$ has the same effect on the vertices as $p(X^*)$, and must be equal to $p(X^*)$, proving that an operation conjugate to a projection is a projection. Conversely, projections from vertices of the same system are conjugate, since G is transitive on the vertices of a system. Projections from vertices of opposite systems cannot be conjugate, since G leaves each system invariant. Thus the 24 projections from two complete conjugate sets in G.

We next obtain some fundamental relations between projections.

(i) If U and V are two vertices on an e-line,

$$p(U)p(V) = p(V)p(U), \tag{A}$$

for each has the same effect on the vertices of the polar planes of U and V.

(ii) If L, M and N are vertices on a κ-line,

$$p(L)p(M) = p(M)p(N) = p(N)p(L), \tag{B}$$

for each has the same effect on the vertices of n-planes through LMN.

(iii) If Y, Y', Z, Z' are the vertices on an a-line,

$$p(Y)p(Y') = p(Y')p(Y) = p(Z)p(Z') = p(Z')p(Z), \tag{C}$$

and

$$p(Y)p(Z) = p(Z)p(Y') = p(Y')p(Z') = p(Z')p(Y), \tag{D}$$

for in each case the four expressions have the same effect on the vertices of n-planes through YZ.

(iv) If U_i $(i = 0, 1, 2, 3)$ are the vertices of a tetrahedron, and $(jklm)$ is a permutation of (0123),

$$p(U_j)p(U_k)p(U_l)p(U_m) = E, \tag{E}$$

for it leaves fixed all the vertices on each face of the tetrahedron.

In the work which follows it will often be necessary to transform products of projections by means of these relations. One very common technique is that of 'moving a factor to the right (or left)'. From our relations (A)–(D) we see that, for each type of line UV, a product of the form $(pU)p(V)$ can also be written in either of the forms $p(V)p(U')$, $p(V')p(U)$ for some U' and V'. By using this result, any factor in a product of projections may be moved one or more places in either direction without altering the number of factors in the product. Thus if a product contains a repeated factor $p(X)$ it is always possible to move one of the factors $p(X)$ until it is adjacent to the other, and the two can then be cancelled.

We can now commence the enumeration of the operations of G. We have already two types of operation, namely,

I. The identical operation.

II. The 24 projections, each of period 2, forming two equal conjugate sets.

3. Products of projections from collinear sets of vertices

3·1. If UV is an e-line, relation (A) shows that $p(U)$ and $p(V)$ commute. Thus each e-line gives a single new product $p(U)p(V)$, and this is of period 2. Since it is easy to see that $p(U)p(V)$ leaves fixed U and V, and also the vertices U' and V' of the polar e-line, but no other vertices, this operation can arise from at most two distinct e-lines.

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But if U'' and U''' are the remaining vertices of the tetrahedron containing U and U', by relations (C) and (E) we have

$$p(U)\,p(V)\,.\,p(V')\,p(U') = p(U)\,.\,p(U'')\,p(U''')\,.\,p(U') = E.$$

Each operation does, therefore, arise from two distinct e-lines. Since G is transitive on e-lines, all the operations of this type belong to a single conjugate set. Hence, since there are 72 e-lines, G contains

III. 36 operations of period 2, products of projections from the two vertices of an e-line, forming a single conjugate set.

3·2. If LMN is a κ-line, from (B)

$$[p(L)\,p(M)]^3 = p(L)\,p(M)\,.\,p(M)\,p(N)\,.\,p(N)\,p(L) = E,$$

so that $p(L)\,p(M)$ is of period 3. Its square (or inverse) is $p(M)\,p(L)$, and these are the only two new operations arising from the κ-line. Since $p(L)\,p(M)$ keeps fixed the vertices of the polar κ-line and no others, distinct κ-lines give distinct operations. The two operations arising from a κ-line are conjugate, for $p(L)$ transforms one into the other; also G is transitive on κ-lines in each system. Hence, since there are 32 κ-lines in two systems, G contains

IV. 64 operations of period 3, each the product of projections from two vertices of a κ-line, forming two conjugate sets of 32 operations each.

3·3. From an a-line $YY'ZZ'$ we have two different types of operation. Firstly, there are operations such as $p(Y)\,p(Y')$. By relation (C) there is only one operation of this type from an a-line, and it is equal to its inverse and so is of period two. By (E), since the polar a-line is the opposite edge of the tetrahedron containing Y, the polar line gives the same operation. Now $p(Y)\,p(Y')$ leaves fixed the vertices on YY' and those on the polar a-line, but no others, so each operation arises from just two a-lines, and this operation is distinct from those of types II and III. Since G is transitive on the 18 a-lines we have that G contains

V. 9 operations of period 2, each the product of projections from two vertices of the same tetrahedron, forming a single conjugate set.

Secondly, we have operations such as $p(Y)\,p(Z)$ arising from an a-line. Now by (D),

$$[p(Y)\,p(Z)]^2 = p(Y)\,p(Z)\,.\,p(Z)\,p(Y') = p(Y)\,p(Y').$$

Thus the operation is of period 4, its square being of type V. Its inverse is $p(Z)\,p(Y)$ and, from (D), these are the two operations of this type arising from the a-line. Since the only vertices left fixed by the operation are those on the polar a-line, distinct a-lines yield distinct operations. The two operations from an a-line are conjugate, for $p(Z)$ transforms one into the other. G is transitive on the 18 a-lines, and so contains

VI. 36 operations of period 4 whose squares are of type V, each the product of projections from two vertices of an a-line, forming a single conjugate set.

It is easily seen that no further operations arise from an a-line, as any product of projections from three points of such a line can be transformed to one of the types which has already been considered.

4. Products of projections from coplanar sets of vertices

4·1. If $XLMN$ is a d-plane, with LMN a κ-line, $p(X)$ commutes with each of $p(L)$, $p(M)$, $p(N)$. We have, therefore, just two new operations $p(X)\,p(L)\,p(M)$ and its inverse $p(X)\,p(M)\,p(L)$ arising from the plane. Now

$$[p(X)\,p(L)\,p(M)]^2 = [p(X)]^2\,[p(L)\,p(M)]^2 = p(M)\,p(L)$$

and $$[p(X)\,p(L)\,p(M)]^3 = p(X)\,p(L)\,p(M).p(M)\,p(L) = p(X).$$

Thus $p(X)\,p(L)\,p(M)$ has period 6, and, from the nature of its square and cube, distinct d-planes give rise to distinct operations of this type. Now $p(L)$ transforms each of the operations arising from the plane $XLMN$ into the other, and G is transitive on the d-planes of each system. Hence, since there are 96 d-planes, G contains

VII. 192 operations of period 6, whose squares are of type IV and whose cubes are of type II, each being the product of projections from three vertices of a d-plane, forming two equal conjugate sets.

We have now obtained 362 operations of G. Since $362 > \frac{1}{2} \times 576$, we have thus shown that $G = G_{576}$.

4·2. We next consider the operations derived from vertices in an n-plane, the polar plane of the vertex P_0. Let the four κ-lines in the plane be $L'MN$, $LM'N$, LMN', $L'M'N'$, and the three a-lines be P_2P_3LL', P_3P_1MM', P_1P_2NN'. We can find at once six new operations, the results of permuting L, M and N in the product $p(L)\,p(M)\,p(N)$. Now, by bringing the two factors $p(L)$ together, we see that

$$\begin{aligned}[p(L)\,p(M)\,p(N)]^2 &= p(L)\,p(M)\,p(N).p(L)\,p(M)\,p(N)\\ &= p(N')\,p(M').p(M)\,p(N)\\ &= p(M')\,p(M),\end{aligned}$$

which is of type V. Thus $p(L)\,p(M)\,p(N)$ has period 4, and we see that the six operations are distinct. The only vertices which an operation of this type can leave fixed are P_0 and vertices in the plane LMN; $p(L)\,p(M)\,p(N)$ leaves fixed in fact just the vertices P_0 and P_2, and so can arise from just two n-planes, the polar planes of these two points. Now

$$p(L)\,p(M)\,p(N) = p(L)\,p(L')\,p(M) = p(P_2)\,p(P_3)\,p(M) = p(P_0)\,p(P_1)\,p(M),$$

and this last product is written in terms of projections from vertices in the polar plane of P_2. From this we see that $p(L)\,p(M)\,p(N)$ can be written as an operation of the same type in the polar plane of P_2, and does arise from two n-planes. The six operations from a plane are all conjugate in G, for $p(L)$ and $p(L')$ transform $p(L)\,p(M)\,p(N)$ into $p(M)\,p(N)\,p(L)$ and $p(L)\,p(N)\,p(M)$ respectively. G is transitive on each of the two systems of n-planes, and hence contains

VIII. 72 operations of period 4, whose squares are of type V, each the product of projections from three vertices of an n-plane, forming two conjugate sets of 36 operations each.

Since there are twelve n-planes of each system permuted transitively by G, the order of the sub-group leaving an n-plane fixed is $\frac{1}{12} \times 576 = 48$. We have already at least 40

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of these operations, namely (i) the identity, (ii) the projection from the pole of the plane, (iii) 9 projections from the vertices of the plane, (iv) 6 operations of type III, (v) 8 operations of type IV, (vi) 3 operations of type V, (vii) 6 operations of type VI, and (viii) 6 operations of type VIII. Also the 8 operations of type VII from the four d-planes joining the pole of the plane to the four κ-lines in the plane leave fixed the pole of the plane. Thus we have all the operations which leave fixed an n-plane, and therefore all the operations arising from projections from vertices in an n-plane.

5. Products of projections from non-coplanar sets of vertices

We consider first the products of projections from four vertices of the same system. Evidently the only distribution of the vertices among the tetrahedra of the system giving a product which cannot be reduced to a product of less than four projections, is to have two vertices from each of two tetrahedra of the system, the projections from two vertices of one of the tetrahedra being adjacent factors. If the vertices of the three tetrahedra are P_i, Q_i, R_i $(i = 0, 1, 2, 3)$ we can write every such product in the form $T = p(P_i)\,p(P_j)\,p(Q_k)\,p(Q_l)$ $(i, j, k, l = 0, 1, 2, 3;\ i \neq j,\ k \neq l)$. For a product in the form $p(Q_m)\,p(Q_n)\,p(P_q)\,p(P_r)$ can be written in the form of T by moving the product $p(Q_m)\,p(Q_n)$ to the right. Also a product in the form $p(P_m)\,p(P_n)\,p(R_q)\,p(R_r)$ can, by moving the product $p(R_q)\,p(R_r)$ one place to the left, be put in the form

$$p(P_m)\,p(R_q)\,p(R_r)\,p(P_s)$$

and this can be altered to the required form by moving $p(P_s)$ two places to the left. Further, since products of type V from opposite edges of a tetrahedron are the same, any such product can be written in the form $\overline{T} = p(P_0)\,p(P_i)\,p(Q_j)\,p(Q_0)$, where $i, j = 1, 2, 3$, but $P_0 P_i Q_j Q_0$ are not coplanar. Reference to the diagram on p. 27 shows that this implies $i \neq j$. As $P_i Q_j$ is a κ-line and distinct κ-lines give distinct operations of type IV we have just 6 distinct operations of the type $\overline{T}$. Also the two systems give the same operations, since $p(P_0)\,p(P_i)$ and $p(Q_j)\,p(Q_0)$, being of type V, can be expressed as similar products in terms of vertices of the other system. Now we have

$$\begin{aligned}[p(P_0)\,p(P_i)\,p(Q_j)\,p(Q_0)]^2 &= p(P_0)\,p(P_i)\,p(Q_j)\,p(Q_0)\cdot p(P_0)\,p(P_i)\,p(Q_j)\,p(Q_0)\\ &= p(P_0)\,p(P_i)\,p(Q_j)\cdot p(R_0)\,p(R_i)\,p(Q_j)\cdot p(Q_0)\,p(Q_0)\\ &= p(P_0)\,p(P_i)\,p(P_j)\,p(P_k)\\ &= E,\end{aligned}$$

where (ijk) is a permutation of (123). Since these operations leave each tetrahedron invariant, but no vertices invariant, they are distinct from the operations of period 2 so far obtained. These six operations form a single conjugate set, for

$$\begin{aligned}p(P_0)\,[p(P_0)\,p(P_i)\,p(Q_j)\,p(Q_0)]\,[p(P_0)]^{-1} &= p(P_i)\,p(P_j)\,p(Q_j)\,p(Q_0)\\ &= p(P_0)\,p(P_k)\,p(Q_j)\,p(Q_0),\end{aligned}$$

and

$$\begin{aligned}p(Q_0)\,[p(P_0)\,p(P_i)\,p(Q_j)\,P(Q_0)]\,[p(Q_0)]^{-1} &= p(P_0)\,p(P_i)\,p(Q_i)\,p(Q_j)\\ &= p(P_0)\,p(P_i)\,p(Q_k)\,p(Q_0).\end{aligned}$$

Thus G contains

IX. Six operations of period 2, each the product of projections from four vertices of the same system, forming a single conjugate set.

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5·2. We now consider the products of projections from three vertices of one system and one vertex of the opposite system. The product can always be written so that the three vertices from the same system come two from one tetrahedron and one from another, the two from the same tetrahedron occurring as adjacent factors. These two factors then form a product of type V, which can be replaced by a similar product in terms of vertices of the opposite system. Thus operations involving three A-vertices and one B-vertex are the same as those involving three B-vertices and one A-vertex. Moreover, the product can be arranged so that its first factor is the projection from a B-vertex and the other three factors are projections from A-vertices. These A-vertices must lie in an n-plane, so their product is an operation of type VIII. If the B-vertex belongs to the tetrahedron which contains the pole of the n-plane, the operation leaves that pole invariant, and so has already been considered. Thus for a new operation the B-vertex must belong to a tetrahedron which does not contain the pole of the n-plane. A typical operation of this form is $T = p(M_0)p(P_0)p(Q_0)p(R_1) = p(M_0)p(P_0)p(P_1)p(Q_0)$.

$$\begin{aligned} T^2 &= p(M_0).p(P_0)p(P_1)p(Q_0).p(M_0)p(P_0)p(P_1)p(Q_0) \\ &= p(P_0)p(P_3)p(Q_0).p(P_0)p(P_1)p(Q_0) \\ &= p(P_0)p(P_3)p(P_0)p(P_1)p(Q_1)p(Q_0) \\ &= p(P_0)p(P_2)p(Q_1)p(Q_0), \end{aligned}$$

which is of type IX, showing that T is of period 4, and is a new operation. Now there are 36 operations of type VIII from the n-planes of one system, and 8 possible vertices of the opposite system with each, giving 36×8 operations of this kind. At least 36×2 of these are distinct, as operations with the single vertex belonging to different tetrahedra have different effects on the tetrahedra. We find that each operation can be written in four ways, with the single vertex being each in turn of the four vertices of a tetrahedron. For example

$$\begin{aligned} p(M_0)p(P_0)p(P_1)p(Q_0) &= p(M_1)p(P_2)\,p(P_1)p(Q_0), \\ p(M_0)p(P_0)p(P_1)p(Q_0) &= p(M_0)p(Q_1)p(P_0)p(P_1) \\ &= p(M_2)p(Q_3)p(P_0)p(P_1) \\ &= p(M_2)p(P_0)\,p(P_1)p(Q_2), \end{aligned}$$

and $$p(M_0)p(P_0)p(P_1)p(Q_0) = p(M_3)p(P_2)\,p(P_1)p(Q_2)$$

since $$p(M_3)p(P_2) = p(M_2)p(P_0),$$

by (D). Thus we have just 72 distinct operations of this type. By considering the effect of operations such as $p(P_0)$, $p(Q_0)$ it is easily seen that T is conjugate to

$$p(M_0)T_i \quad (i = 1, 2, 3, 4, 5),$$

where T_i are the remaining five operations of type VIII from the plane $P_0P_1Q_0$; also $p(L_0)$ transforms T to $p(N_0)p(P_0)p(P_1)p(Q_0)$. Hence, since G is transitive on the n-planes of a system, G contains

X. 72 operations of period 4, whose squares are of type IX, each the product of projections from four vertices, three of one system and one of the other, forming a single conjugate set.

5·3. There now remain to be considered the products of projections from four vertices when two of the vertices belong to each system. If the product cannot be transformed to an operation already considered, the vertices must all belong to different tetrahedra, so that the product can be written as the product of two operations of type IV from κ-lines of different systems. Three cases arise, according as one κ-line (i) is the polar line of the other, (ii) meets the polar line of the other, or (iii) is skew to the polar line of the other.

The case (iii) can be transformed to one of the cases (i) or (ii). For, if the operation is $p(P)\,p(Q)\,p(L)\,p(M)$ where PQR and LMN are κ-lines, then QLM is an n-plane and one of QL, QM, QN is an e-line; without loss of generality, by (B), we can suppose that QL is an e-line. Then $p(Q)\,p(L)$ can be replaced by the operation of the same type from the polar e-line $Q'L'$, where QQ' is an a-line. But then Q' is the pole of the plane QLM, and so is on the polar line of each κ-line in that plane. In particular it is on the polar line of $L'M$. Thus we have the operation written in the form $p(P)\,p(Q')\,p(L')\,p(M)$ where Q' is on the polar line of $L'M$.

We now consider the case (i), of which a typical operation is

$$T = p(P_0)\,p(Q_0)\,p(L_0)\,p(M_0).$$

Since each of $p(P_0)$, $p(Q_0)$ commutes with each of $p(L_0)$, $p(M_0)$,

$$T^2 = [p(L_0)\,p(M_0)]^2\,[p(P_0)\,p(Q_0)]^2 = p(M_0)\,p(L_0)\,p(Q_0)\,p(P_0) = T^{-1}$$

and T is of period 3. By its effect on the tetrahedra, this operation can be seen to be distinct from the operations of type IV, the only other operations of G with period 3. There are four operations like this from each pair of polar κ-lines, and 16 such pairs, giving 64 operations in all. We shall see that each distinct operation arises from just four pairs of polar κ-lines. In the first place, any operation which is the same as T can be written in the form $p(P_i)\,p(Q_j)\,p(L_k)\,p(M_l)$ (by considering the effect of the various possibilities on the tetrahedra). Since $Q_jL_kM_l$ is a d-plane, and distinct d-planes give distinct operations, no two different expressions for the same operation can involve the same vertex P_i. Hence each operation can arise from at most four pairs of polar κ-lines. Secondly,

$$\begin{aligned} p(P_0)\,p(Q_0)\,p(L_0)\,p(M_0) &= p(P_0)\,p(L_0)\,p(Q_0)\,p(M_0) \\ &= p(P_1)\,p(L_1)\,p(Q_2)\,p(M_2) \\ &= p(P_1)\,p(Q_2)\,p(L_1)\,p(M_2), \end{aligned}$$

$$\begin{aligned} p(P_0)\,p(Q_0)\,p(L_0)\,p(M_0) &= p(P_0)\,p(M_0)\,p(Q_0)\,p(N_0) \\ &= p(P_2)\,p(M_1)\,p(Q_0)\,p(N_0) \\ &= p(P_2)\,p(Q_2)\,p(L_1)\,p(M_1) \\ &= p(P_2)\,p(Q_3)\,p(L_3)\,p(M_1), \end{aligned}$$

and

$$\begin{aligned} p(P_0)\,p(Q_0)\,p(L_0)\,p(M_0) &= p(P_2)\,p(Q_3)\,p(L_3)\,p(M_1) \\ &= p(P_2)\,p(L_3)\,p(Q_3)\,p(M_1) \\ &= p(P_3)\,p(L_2)\,p(Q_1)\,p(M_3) \\ &= p(P_3)\,p(Q_1)\,p(L_2)\,p(M_3). \end{aligned}$$

Hence each operation does arise from four pairs of polar κ-lines, and we have 16 distinct operations of this type. The four operations arising from one pair of polar κ-lines are evidently conjugate, and G is transitive on the κ-lines of a system. We have, therefore, that G contains

XI. 16 operations of period 3, each the product of two operations of type IV from polar κ-lines, and thus expressible as a product of four projections, forming a single conjugate set.

If two κ-lines are such that one meets the polar κ-line of the other, it is evident that the relation is symmetric. This means that the operations arising in case (ii) can be written in the form $p(U)p(V)p(X)p(Y)$ where UV and XY are κ-lines, and V is on the polar line of XY, X on the polar line of UV. We shall consider only products in this form, of which $T_1 = p(Q_0)p(P_2)p(M_0)p(L_3)$ is typical. Now

$$\begin{aligned} T_1^2 &= p(Q_0)p(P_2)p(M_0)p(L_3).p(Q_0)p(P_2)p(M_0)p(L_3) \\ &= p(Q_0)p(P_2).p(Q_3)p(P_2).[p(M_0)p(L_3)]^2 \\ &= p(Q_0)p(R_1)p(L_3)p(M_0) \\ &= p(P_0)p(Q_0)p(L_0)p(M_0) \end{aligned}$$

since R_1L_3, R_0L_0 are polar e-lines. Hence

$$\begin{aligned} T_1^3 &= p(P_0)p(Q_0)p(L_0)p(M_0).p(Q_0)p(P_2)p(M_0)p(L_3) \\ &= p(P_0)p(L_0)p(M_0).p(P_2)p(M_0)p(L_3) \\ &= p(P_0)p(L_0)p(P_2)p(L_3) \\ &= p(P_0)p(P_2)p(Q_1)p(Q_0). \end{aligned}$$

Thus T_1^2 is of type XI and T_1^3 of type IX, showing that T_1 has period 6 and is a new operation. We find that each operation can be written in twelve ways, with a projection from each of the twelve vertices of a system as the first factor. For example:

$$\begin{aligned} T_1 = p(Q_0)p(P_2)p(M_0)p(L_3) &= p(Q_0)p(L_3)p(P_2)p(N_3) \\ &= p(Q_1)p(L_1)p(P_2)p(N_3) \\ &= p(Q_1)p(P_3)p(M_2)p(L_1) \\ &= p(Q_1)p(P_1)p(M_3)p(L_1), \end{aligned}$$

and similarly $$T_1 = p(Q_2)p(P_3)p(M_2)p(L_2),$$

and $$T_1 = p(Q_3)p(P_0)p(M_1)p(L_0).$$

Also $$\begin{aligned} T_1 &= p(Q_0)p(P_2)p(L_3)p(N_3) \\ &= p(Q_0)p(P_3)p(L_2)p(N_3) \\ &= p(P_3)p(R_3)p(L_2)p(N_3), \end{aligned}$$

and similarly $$T_1 = p(R_2)p(Q_2)p(N_1)p(M_2).$$

There are 16 κ-lines of one system, each of which has nine κ-lines of the opposite system meeting its polar κ-line, and each pair of suitable κ-lines yields four operations of this type, giving $16 \times 9 \times 4$ products. Since each operation can be written in twelve ways at least, not more than 48 of these products are distinct. As there are 16 distinct

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operations of type XI which are the squares of the operations at present under consideration, we have at most three distinct products with the same square.

Now $T_2 = p(R_0)p(Q_2)p(M_0)p(L_1)$ and $T_3 = p(P_0)p(R_2)p(M_0)p(L_2)$ each have the same square as T_1, and the three operations T_1, T_2, T_3 are distinct, for they transform M_0 to L_1, L_2, L_3 respectively. We have, therefore, three distinct operations of this type with the same square. Now T_1, T_2, T_3 are conjugate, for T_1 is transformed to T_2 and T_3 by $p(P_0)p(Q_0)$ and $p(Q_0)p(P_0)$ respectively. Since the operations of type XI form a single conjugate set, it follows that the operations of the present type form a single conjugate set. Hence G contains

XII. 48 operations of period 6, whose squares are of type XI and whose cubes are of type IX, each the product of projections from four vertices, two from each system, forming a single conjugate set.

We have now obtained all the 576 operations of the group G generated by our 24 projections, expressing each as a product of four, or fewer, projections, and have shown how they are distributed in conjugate sets. The results are summarized in the following table, in which n is the least number of projections whose product is of the given type, and N is the number of operations of that type.

Operations of the group G_{576}

Type	N	Period	Types of the powers of the operations	n	Number of conjugate sets
I	1	1	—	—	1
II	24	2	—	1	2
III	36	2	—	2	1
IV	64	3	—	2	2
V	9	2	—	2	1
VI	36	4	Squares, V	2	1
VII	192	6	Squares, IV; cubes, II	3	2
VIII	72	4	Squares, V	3	2
IX	6	2	—	4	1
X	72	4	Squares, IX	4	1
XI	16	3	—	4	1
XII	48	6	Squares, XI; cubes, IX	4	1

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