Related quadrics and systems of a rational quartic curve. By Miss H. G. TELLING, Newnham College. (Communicated by Mr F. P. WHITE.)

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1. *Introduction.*

1·1. The points, tangents, osculating planes, ..., osculating primes of a curve may be said to form a *system* which is characterised by the number of these elements which are incident with a prime, ..., line, point, respectively. For the normal rational quartic curve the system is (4, 6, 6, 4); projection of this system from a general point gives the system (4, 6, 6) in [3]; section by a general prime gives the system (6, 6, 4). These two systems in [3], which are the systems with which we are concerned in this paper*, are duals of one another, and will be called systems of the first and second kinds respectively.

1·2. The points of a system of either kind lie on a quadric and the planes touch another quadric. The system is therefore said to be *inscribed* in the former quadric and *circumscribed* to the latter. The lines of a system touch both quadrics, touching the latter quadric at points of a system of the opposite kind, which is the reciprocal system in regard to the quadric. In fact, by reciprocating the system with respect to either of the quadrics we obtain a system of the opposite kind; by continuing this process we obtain systems alternately of the first and second kinds and thus a chain of quadrics for each of which one system is inscribed and the next circumscribed.

1·3. The points where two non-consecutive lines of a system in [3] intersect are the points of its *nodal system*; the planes in which these lines lie are the planes of the *bitangent system*. The points of the nodal system thus form the nodal or double curve of the osculating developable, and the planes of the bitangent system form the bitangent developable of the original curve. For systems of the first and second kinds, the nodal and bitangent systems are each of the opposite kind to the original. It is clear that the nodal system of the bitangent system, and the bitangent system of the nodal system each coincide with the original system. The process of forming the bitangent system is thus in this sense the reverse of that of forming the nodal system. By continuing the process of forming

* An account, with references, of the rational quartic curve and developables associated therewith is given by Rohn u. Berzolari, *Encyklopädie der Math. Wiss.*, III C 9, 1373–1382.

the nodal system (or bitangent system) we obtain systems which are alternately of the first and second kinds and thus another chain of systems and quadrics.

1·4. Systems of the same kind formed by either of the processes of (1·2) and (1·3), or by any combination of these, are related by a collineation. By speaking of one plane as the plane at infinity we can describe all the quadrics introduced as concentric and coaxial, each being written in the form $a_1x^2 + a_2y^2 + a_3z^2 + a_4t^2 = 0$ and referred to as (a), while the system inscribed in this quadric is denoted by A. The collineation transforming any of the systems into the original is then shewn (5·3) to be a repetition of one primitive collineation together with a magnification from the centre.

2. *Base systems H and E.*

2·1. The system (4, 6, 6, 4) associated with the normal rational quartic curve admits of ∞^3 collineations into itself. The main forms which are invariant under these collineations are the quadric I for which the system is self-reciprocal; the cubic primal J of chords of the normal curve; and the reciprocal of J with respect to I, which is a quartic surface K (the projection of a Veronese surface), the locus of points of intersections of osculating planes of the curve. On projecting the system from a point F, and on taking its section by the polar prime Φ of F with respect to I, we obtain a (4, 6, 6) and a (6, 6, 4) system in Φ, and these are in general capable of only a finite number of collineations into themselves, namely those which leave F (and Φ) unchanged. These consist of three harmonic inversions transforming into themselves a set of four *fundamental* points and incident planes (f-points and f-planes) common to the two systems. Three of the axes, one belonging to each inversion, are concurrent at the *centre* of the systems and are called the *principal axes*, while the remaining three axes complete the edges of the principal tetrahedron. The plane containing these three axes will be spoken of as the plane at infinity. The systems are then symmetrical about the principal axes, which are therefore used as coordinate axes throughout.

The two systems are reciprocal with respect to the quadric (e) which is the section of I by Φ. The f-elements of the systems are given by the quartic equation (f-quartic) represented by the point F or the prime Φ with reference to the normal curve; the absolute invariant of the f-quartic is the one essential constant of either system. Each f-plane is stationary (flex-plane) in the (4, 6, 6) system, and each f-point is stationary (cusp) in the (6, 6, 4) system. Each f-line of the one system is conjugate in regard to (e) to the corresponding f-line of the other system; these are respectively the projection from F of the tangent line,

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and the section by Φ of the osculating plane of the normal curve at the corresponding f-point. The (6, 6, 4) system is inscribed in the quadric (e) and is thus denoted by E. The points of E form the complete intersection of (e) with the cubic surface (J, Φ), which has nodes at the f-points; they thus lie by threes on the generators of each system of (e), and may therefore be described as lying [3, 3] upon (e).

The Hessian of the f-quartic gives the second important set of incident points, lines, and planes (h-elements), in each system; in the (4, 6, 6) system the h-lines are the four lines which pass through a further point of the system (tangent-trisecants of the curve), while in the (6, 6, 4) system the h-lines are the four lines which lie on a further plane of the system (tangent-triaxes of the curve). These further points and further planes are given by the Steinerian of the f-quartic (s-points and s-planes). The h-lines of the (4, 6, 6) system determine a quadric (h); the system is inscribed in (h) and is thus denoted by H. The points of H lie [3, 1] upon (h); the planes of H are the common tangent planes of (e) and the Steiner quartic surface [K, F], whose tropes are the f-planes. The system H is thus inscribed and circumscribed to this quartic surface while the system E is inscribed and circumscribed to the reciprocal cubic surface (J, Φ).

Let the f-quartic be taken in the canonical form $\theta^4 + 6m\theta^2 + 1$, and let the equation of the quadric (e) be

$$(e): \qquad x^2 + y^2 + z^2 + t^2 = 0.$$

Any point of the normal curve being given by $(\theta^4, \theta^3, \theta^2, \theta, 1)$, the section of I by Φ is given by

$$x_0 x_4 - 4x_1 x_3 + 3x_2^2 = 0 = x_0 + x_4 + 6mx_2,$$

which may be written

$$\tfrac{1}{4}(x_0 + x_4)^2 - \tfrac{1}{4}(x_0 - x_4)^2 - (x_1 + x_3)^2 + (x_1 - x_3)^2 + 3x_2^2 = 0$$
$$= x_0 + x_4 + 6mx_2,$$

and reduces to the above equations for (e) when the reference system is changed according to the formulae

$$x_0 - x_4 = 2x, \; x_0 + x_4 = \frac{2it}{M} = -6mx_2, \; x_1 + x_3 = y, \; x_1 - x_3 = iz. \quad 2{\cdot}11$$

The points (x) and F lie on the same trisecant plane of the normal curve if

$$\begin{vmatrix} x_0 & x_1 & x_2 & x_3 \\ x_1 & x_2 & x_3 & x_4 \\ 1 & 0 & m & 0 \\ 0 & m & 0 & 1 \end{vmatrix} = 0;$$

this gives the equation to the cone whose section by Φ is (h). On changing the system of reference by means of the above formulae we obtain

$$(h): \qquad x^2 + p^4y^2 + q^4z^2 + M^2t^2 = 0,$$

where we write

$$p^2 = -\frac{m-1}{2m}, \quad q^2 = -\frac{m+1}{2m}, \quad M^2 = \frac{1+3m^2}{3m^2},$$

as throughout the paper.

2·2. The elimination of m between the equations giving p^2, q^2, M^2 gives the two equations $p^2 + q^2 + 1 = 0$, $p^4 + q^4 + 1 = \frac{3}{2}M^2$, and thus two relations satisfied by the coefficients of (h); and since these coefficients are the invariants of (e) and (h), the relations are the two invariant relations satisfied by two quadrics which have a $(4, 6, 6)$ system circumscribed to the one and inscribed in the other. The same relations hold between (h^{-1}) and (e) as between (e) and (h), i.e. between two quadrics which have a $(6, 6, 4)$ system circumscribed to the one and inscribed in the other. It therefore appears that, if two quadrics (a) and (b) are such that a system of one kind is inscribed in (a) and circumscribed to (b), there is a system of the other kind inscribed in (a) and circumscribed to (b); the latter system we call the *complementary system* of the former. The complementary system is equally obtained by reciprocating the quadric (a) into the quadric (b); or, equivalently, by using one of the eight collineations $x = \pm (b/a)^{\frac{1}{2}} x'$ to transform (a) into (b) after reciprocating with respect to (a). The remaining seven collineations either differ from the first by two signs, in which case they give the same system on account of the symmetries; or, by one sign, when they give a system which is the image of the former in regard to the centre. Hence if there is a system of one kind inscribed in a quadric (a) and circumscribed to a quadric (b) there is also the complementary system which is of the other kind, and two further systems which are images of these in regard to the centre.

2·3. By reciprocating (e), (h) with respect to one another, and then reciprocating in regard to the successive (inscribed and circumscribed) quadrics so obtained, we get an infinite set of quadrics with coefficients characterised by

$$\ldots h^{-2},\ h^{-1},\ 1,\ h,\ h^2, \ldots.$$

There is then a chain of systems (or rather, four chains of systems, according to (2·2)) *each inscribed in one of the quadrics and circumscribed to the preceding one*; the lines of a system touch this former quadric at points of the system and touch the latter at points of the preceding system. The trisecants of a system are the triaxes of the succeeding system.

3. *Nodal and bitangent systems of E and H.*

3·1. We proceed to obtain the nodal and bitangent systems by a method which brings out their relation to the normal curve. The central feature appearing by this method is the polar quadric of F in regard to J; this quadric is known to be that which contains the normal curve and its four tangents at its points on Φ.

3·2. The section of the surface K by Φ is a rational quartic curve which is the locus of points of the nodal system of E. This system is denoted by G, and the quadric in which it is inscribed by (g). The normal curve and the locus of points of G lie on a quadric which is touched by the osculating planes of the normal curve at its points on Φ, since each osculating plane contains a line of G and a tangent of the normal curve. The normal curve is thus "asymptotic" at these points on the quadric, which therefore contains its tangents there. Thus the quadric (g) is the section by the prime Φ of the polar quadric of F in regard to J.

3·3. The reciprocal of G in regard to (e) is the bitangent system of H, which we denote by C. The quadric (c) in which this bitangent system is inscribed is the shadow from F of this same polar quadric. For the planes of C are the projection from F of the bitangent primes of the normal curve which pass through F. Such a prime, touching the normal curve at consecutive points P_1, P_2 and at Q_1, Q_2, touches J along P_1Q_1, P_2Q_2, which are thus consecutive lines of the polar quadric of F with respect to J. The line through F meeting P_1Q_1 and P_2Q_2 ultimately touches the polar quadric, while it projects from F into a point of C. It follows that the quadric (c) is the projection from F upon Φ of the section of the polar quadric by the polar prime of F with respect to J, and is thus similar and similarly situated to (g).

3·4. The reciprocal of G in regard to (g) is now shewn to be the nodal system of H and is denoted by D. Any point D_1 of the system D is the section by Φ of the transversal through F of the tangents at P, Q, which lie in a prime through F. The osculating planes at P, Q meet in a point G_1 of G and, since these planes contain the consecutive positions of the tangents, it is seen that a consecutive point D_2 of D lies on D_1G_1, i.e. D_1G_1 is a line of D. Moreover, F, D_1, G_1 form an apolar triad with respect to J, since the polar quadric of G_1 with respect to J contains the tangents at P, Q, while F and D_1 are harmonic with respect to these tangents. It follows that D_1, G_1

are conjugate with respect to the polar quadric of F and thus with respect to (g). Hence the lines and therefore the planes of D touch (g) at points of G, and *D is the reciprocal of G with respect to (g).*

3·5. The reciprocal of D with respect to (e) is the bitangent system of E, which we denote by F. It follows from what has just been proved that this system F is the reciprocal of C with respect to the quadric (f); since (f) is the reciprocal of (g) with respect to (e), the coefficients of these quadrics satisfy the relation $fg = 1$. It may be observed that generators of one system of (f) are triaxes of C: the planes through one of these generators cut out upon the curve of H a quartic involution which is syzygetic, since it contains three members given by planes bitangent to the curve. This result might have been used to determine (f) from H.

3·6. We now have systems and quadrics which may be tabulated as follows:

$$\begin{array}{cccccccc} & F & (f) & C & (c) & & & \\ \ldots(h^{-1}) & E & (e) & H & (h) & H^2 & (h^2)\ldots. & \\ & G & (g) & D & (d) & & & \end{array} \qquad 3·61$$

Any row may be extended by reciprocation as in the central row, and then if X is the reciprocal of D with respect to (d) it is the nodal system of H^2, since X and D are systems of the first and second kinds related by (d) in the same way as H and E are related by (e), these latter being the nodal systems of C and F. Thus *the extension of the rows by reciprocation is equivalent to extending the columns by forming the nodal (or bitangent) systems successively.*

3·7. The equations of the quadrics of this scheme may now be found. With the canonical form of the f-quartic, the polar quadric of $F \equiv (1, 0, m, 0, 1)$ with respect to

$$J \equiv \begin{vmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \\ x_2 & x_3 & x_4 \end{vmatrix} = 0$$

is given by

$$x_0x_2 - x_1^2 + m(x_0x_4 + 2x_1x_3 - 3x_2^2) + x_2x_4 - x_3^2 = 0.$$

The points of K represent quartic forms which are squared quadratics and hence are given parametrically by

$$[u^2, \ uv, \ \tfrac{1}{3}m(uw + 2v^2), \ vw, \ w^2]. \qquad 3·71$$

The points of K which are on the quadric are given by

$$(uw - v^2)[u^2 + 2m(uw + 2v^2) + w^2] = 0, \qquad 3·72$$

which verifies the statement that the quadric contains the points of K upon the normal curve and upon Φ. From the section of the polar quadric by Φ, subject to the transformation 2·11, the equation of (g) is found to be

$$(g): \qquad x^2+p^2y^2+q^2z^2+4\frac{p^2q^2}{M^2}t^2=0;$$

and consequently that of (f) is

$$(f): \qquad x^2+\frac{1}{p^2}y^2+\frac{1}{q^2}z^2+\frac{M^2}{4p^2q^2}t^2=0.$$

The quadric (c) is obtained by writing the equation of the envelop-ing cone of the polar quadric Q in the form

$$Q=\lambda\,(x_0+x_4+6\xi_2x_2)^2,$$

the polar prime of F with respect to J being given by

$$x_0+x_4+6\xi_2x_2=0,$$

where $6m\xi_2=1-3m^2$, and λ being determined by the vertex F. The section by Φ is given by

$$x^2+p^2y^2+q^2z^2+4p^2q^2\frac{t^2}{M^2}=\frac{4}{3(1-m^2)}\left(1-\frac{\xi_2}{m}\right)^2\frac{t^2}{M^2},$$

which reduces to

$$(c): \qquad x^2+p^2y^2+q^2z^2+\frac{M^4}{4p^2q^2}t^2=0;$$

and consequently the quadric (d), being the reciprocal of (c^{-1}) with respect to (g), is given by

$$(d): \qquad x^2+p^6y^2+q^6z^2+\frac{4p^2q^2}{M^4}t^2=0.$$

It is to be observed that (g) and (c) are homothetic with respect to the centre, the constant being $\frac{M^6}{16p^4q^4}=\frac{i^3}{27j^2}$, i.e. the absolute invariant of the f-quartic.

4. *The f- and h-elements of the systems.*

4·1. We next shew how the systems are linked by their f- and h-elements. Every line l of a system meets two other lines of the system. These two other lines coincide, and coincide with l, when l is an f-line of the system; they coincide, but remain distinct from l, when l is an h-line of the system, since two consecutive lines meet the h-line either in the s-point or on the s-plane. Since the trisecants of the system H^{-1} are the triaxes of E, the h-lines of H^{-1} and E

are the same. An h-plane of E, being the reciprocal with respect to (h^{-1}) of an h-point of H^{-1}, is the tangent plane of (h^{-1}) at this point and thus is the h-plane of H^{-1}. Thus all the h-elements of H^{-1} and E are the same; similarly for the systems H and H^2, etc. On the other hand we have already seen that the f-points and f-planes, but not the f-lines, of E and H are the same; similarly for the systems H^2 and H^3, etc. Thus *consecutive systems of a row are connected alternately by their h-elements, and by their f-points and f-planes, while the f-lines of two systems connected by f-elements are conjugate with respect to the quadric to which the system of the first kind is circumscribed.*

4·2. To shew how consecutive systems of a column are connected, take the nodal system G of E. The tangent-secants of the curve of G are the sections of the osculating planes at the four points of Φ on the normal curve, so that the h-points and h-lines of G are the f-points and f-lines of E. To find the stationary elements of G we note that through every point of G pass two osculating planes of the normal curve, while every osculating plane contains a conic of K and thus has two points of G upon it. The branch and double points of this correspondence on the two curves are related projectively. At a branch point of G the osculating plane of the normal curve, and its conic, touches the curve of G, and thus the osculating prime which meets K in this conic counted twice has for its section by Φ an f-plane of G, the branch point itself being an f-point. Hence the branch and double points of G are its four f-points and four h-points. On the normal curve therefore the double points are given by the Hessian of the f-quartic; thus on returning to the section by Φ we see that the f-lines and f-planes of G are the h-lines and h-planes of E*. Accordingly, in any column:

the f-points and f-lines of a system of the second kind are the h-points and h-lines of its nodal system,

and

the f-planes and f-lines of the nodal system are the h-planes and h-lines of the original system.

By using the connections of the rows as well as of the columns further results may be read off from the scheme: for example, the f-lines of H, being the h-lines of C, determine the quadric to which the system C is circumscribed, viz. (f), just as the h-lines of H determine (h). It also appears that each of the elements considered above belongs to four systems; in two of the systems it is an f-element and in the other two it is an h-element.

* If the points of K are represented by points (u, v, w) of a plane by means of the formula 3·71, the (2, 2) correspondence appears in a familiar form on two apolar conics given by the equation 3·72.

5. *Collineations of the systems.*

5·1. From the (2, 2) correspondence above (§ 4·2) follows the important result that the cross ratio of the f-points of G is equal to the cross ratio of the four points of Φ on the normal curve: hence the systems G and C and thus D and F have the same invariant as H and E. The f-quartics of G and H may be taken to be the same, and thus a (1, 1) correspondence is set up between the systems such that a coplanar set of points of G corresponds to a coplanar set of points of H: *thus the systems G and H are collinearly related.* Similarly the reciprocal systems C and E are collinearly related.

A collineation transforming G into H, as indeed every collineation referred to in this section, is symmetrical with respect to the axes, that is to say an affine transformation leaving the axes unaltered, and is thus determined by the motion of one point. It changes the h-points of G (which are f-points of H) into the h-points of H. On the other hand a collineation transforming E into C carries the f-points of E (which are f-points of H) into the f-points of C (which are h-points of H) and thus may well be identical with the former. In fact, any of these collineations transform the quadrics (e) into (c) and (g) into (h), being of the form $x = \pm\sqrt{c}x'$ (where c denotes as before the coefficients of the quadric) and $c = h/g = hf$ (3·7). Arguing from this, and using reciprocation with respect to (e) and (g), we see that the bitangent and nodal systems of H lie on the scheme between the quadrics (f) and (hf), (g) and (hg), respectively, while those of E lie between ($h^{-1}f$) and (f), ($h^{-1}g$) and (g). Hence *the bitangent and nodal systems of a system of either kind lying in the scheme between two consecutive quadrics (a), (ah) lie respectively between the quadrics (af) and (ahf), and the quadrics (ag) and (ahg)*; and the scheme, as far as its quadrics are concerned, may then be written:

$$\begin{array}{ccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & h^{-2}f^2 & h^{-1}f^2 & f^2 & hf^2 & h^2f^2 & \cdots \\ \cdots & h^{-2}f & h^{-1}f & f & hf & h^2f & \cdots \\ \cdots & h^{-2} & h^{-1} & 1 & h & h^2 & \cdots \\ \cdots & h^{-2}f^{-1} & h^{-1}f^{-1} & f^{-1} & hf^{-1} & h^2f^{-1} & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

It may be observed that any two quadrics whose positions in the scheme are images with respect to that of a third quadric are reciprocal with respect to that quadric. The collineations

$$x = \pm\sqrt{c}x'$$

change any quadric in the scheme into the one obtained by a diagonal step similar to that from 1 to hf.

5·2. We now determine the particular collineations of the set $x = \pm\sqrt{c}x'$ which transform G into H. There are four such collineations (since the whole figure is symmetrical about the axes), and these also transform the complementary of G into the complementary of H; the remaining four transform G into the image of H, and the complementary of G into the complementary of the image of H. The former collineations transform the f-points of H into the h-points of H and may be so determined. For if by a collineation $x = x'$, $y = \alpha y'$, $z = \beta z'$, $t = \gamma t'$ a point of H with parameter θ is transformed into ϕ, we require

$$\sigma(\theta^2 - \theta^{-2}) = \phi^2 - \phi^{-2}, \qquad \text{(i)}$$

$$\sigma(\theta + \theta^{-1}) = \alpha(\phi + \phi^{-1}), \qquad \text{(ii)}$$

$$\sigma(\theta - \theta^{-1}) = \beta(\phi - \phi^{-1}), \qquad \text{(iii)}$$

$$\sigma(\theta^2 + \theta^{-2} + 6\mu) = \gamma(\phi^2 + \phi^{-2} + 6\mu), \qquad \text{(iv)}$$

where $3m\mu = -1$.

Then, from (ii), (iii) and (i), we have

$$\sigma^2(\theta^2 - \theta^{-2}) = \alpha\beta(\phi^2 - \phi^{-2}) = \sigma\alpha\beta(\theta^2 - \theta^{-2}),$$

giving $\alpha\beta = \sigma$.

Thus
$$\beta(\theta + \theta^{-1}) = \phi + \phi^{-1},$$
$$\alpha(\theta - \theta^{-1}) = \phi - \phi^{-1},$$

whence
$$\phi^2 + \phi^{-2} = \tfrac{1}{2}(\alpha^2 + \beta^2)(\theta^2 + \theta^{-2}) + \beta^2 - \alpha^2,$$

so that the symmetrical set of four points (θ_i) is given by

$$\theta^2 + \theta^{-2} = \frac{2 - \alpha^2 - \beta^2}{2^{-1}(\beta^2 - \alpha^2)},$$

and the corresponding set (ϕ_i) by

$$\phi^2 + \phi^{-2} = \frac{2 - \alpha^{-2} - \beta^{-2}}{2^{-1}(\beta^{-2} - \alpha^{-2})}.$$

The sets of points given by (θ_i) and (ϕ_i) being known, we have two equations from which to determine α^2 and β^2; and when the signs of α and β are chosen, γ is determined, without ambiguity of sign, from

$$\alpha\beta(\theta^2 + \theta^{-2} + 6\mu) = \gamma(\phi^2 + \phi^{-2} + 6\mu).$$

The ambiguities of sign of α, β correspond to collineations in which the points θ_j of the set (θ_i) are transformed to different points ϕ_k of the set (ϕ_i). The collineations changing the f-points into the h-points of H are found to be given when

$$\alpha^2 = p^2, \quad \beta^2 = q^2, \quad \gamma = -\frac{M^2}{2pq}.$$

These four collineations transform any system (or quadric) of the scheme 3·61 into the consecutive system (or quadric) placed diagonally to it as E to C (or (e) to (c)).

5·3. Since the system D is the reciprocal with respect to (g) of the reciprocal with respect to (e) of C, the system C is transformed into D by the collineation $x = gx'$, and this transforms equally any system (or quadric) of the scheme into the system (or quadric) next but one below it, as C to D. By the method above (§ 5·2) this collineation appears as one transforming the h-points of H into the s-points of H, and, since the h-points of H are the f-points of C, the s-points of H are the f-points of D: a fact which is easily verified by observing the construction of the nodal system D. The combination of the above two collineations gives four of the collineations $x = \pm g\sqrt{c}x' = \pm\sqrt{d}x'$, which transform any system (or quadric) of the scheme into one placed diagonally, as E to D. Thus any system A is changed into the bitangent system of the reciprocal of A with respect to (a) by a collineation $x = \pm\sqrt{c}x'$ with the signs as determined; while any system A is changed into the nodal system of the reciprocal of A with respect to (a) by a collineation $x = \pm\sqrt{d}x'$. Thus *any system of the scheme can be found from the base systems H, or E, by combined use of these two primitive collineations.*

5·4. The description of the systems and their quadrics may be further simplified by the observation (3·7) that the quadrics (c) and (g) are homothetic. Writing $\rho_1 = \rho_2 = \rho_3 = 1$, $\rho_4 = 16p^4q^4M^{-6}$, we have $h = \rho c^2$, $f = \rho^{-1}c^{-1}$, and the scheme of quadrics becomes

$$\begin{array}{ccccccc} & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & \rho^{-4}c^{-6} & \rho^{-3}c^{-4} & \rho^{-2}c^{-2} & \rho^{-1} & c^2 & \cdots \\ \cdots & \rho^{-3}c^{-5} & \rho^{-2}c^{-3} & \rho^{-1}c^{-1} & c & \rho c^3 & \cdots \\ \cdots & \rho^{-2}c^{-4} & \rho^{-1}c^{-2} & 1 & \rho c^2 & \rho^2c^4 & \cdots \\ \cdots & \rho^{-1}c^{-3} & c^{-1} & \rho c & \rho^2c^3 & \rho^3c^5 & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \end{array}$$

Any two quadrics placed at "knight's move" in the scheme, as 1 and ρ^{-1}, are homothetic, and *any system is found from any other of the same kind by an appropriate use of the collineation $x = gx' = \rho cx'$ and of the homothety $x = \rho x'$, with a change of sign for the image system. Systems of opposite kind are found by first reciprocating with respect to (e).* For example, the nodal system of the nodal system of a quartic curve is found by transforming it by means of $x = \rho cx'$: it is homothetic with the complementary system of H^2.

5·5. The system might have been described in terms of reciprocation instead of collineation. For example, the bitangent system C is the reciprocal of the system H with respect to a certain quadric: for, if we combine a collineation $x = \sqrt{c}x'$ with the proposed reciprocation, we obtain a transformation which is equivalent to reciprocation in regard to (e), and the required reciprocating quadric (r) is given by

$$(r): \qquad x^2 + py^2 + qz^2 - \frac{M^2t^2}{2pq} = 0.$$

Similarly, a quadric (s) which reciprocates H into its nodal system D is given by

$$(s): \qquad x^2 + p^3y^2 + q^3z^2 - 2pqt^2 = 0.$$

The quadric (r) in fact reciprocates the quadric (h) into (f), and the quadric (e) into (c); while the quadric (s) reciprocates the quadric (h) into (g), and (e) into (d).

6. *Pencils and ranges among quadrics of the scheme.*

6·1. The four quadrics (e), (h), (f), (c), each touch the four f-lines of H, the former three at the f-points, thus forming a pencil, and the latter three on the f-planes, forming a range.

Hence $\qquad f = \lambda h + \mu,$

or, using $c = hf$, $\qquad h^{-1} = \lambda f^{-1} + \mu c^{-1}.$

It also follows that $\qquad 1 = \lambda d + \mu g$

and $\qquad d^{-1} = \lambda + \mu h^{-1},$

which correspond to the fact that the four quadrics (h), (c), (d), (g), touch the four h-lines of H (f-lines of C), the former three on the h-planes and the latter three at the h-points. From the equations of the quadrics it is found that

$$\lambda = p^{-2}q^{-2}, \quad \mu = -\tfrac{3}{4}M^2p^{-2}q^{-2}.$$

If A, B are two consecutive systems of the scheme between quadrics (α), (β), (γ), (δ), placed as

$$\begin{matrix} (\delta) & A & (\alpha) \\ (\gamma) & B & (\beta) \end{matrix}$$

they are systems of opposite kinds; denoting the system of the first kind by I and that of the second kind by II, we find that

(β), (γ), (δ) belong to a pencil touching the h-lines of II at the f-points of I,

(δ), (α), (β) belong to a range touching the h-lines of I at the f-points of II.

When A is the system II, the h-lines of either system are the f-lines of the other. When A is the system I, all four quadrics pass through the f-points of A (which are h-points of B) and touch the f-planes of B (which are h-planes of A). Moreover, in this case A is inscribed and circumscribed to a Steiner quartic surface in which also the system Γ is inscribed. The tropes of the surface are the f-planes of A (which are h-planes of Γ) and the trope-conics lie on a quadric which belongs to the pencil of (β), (γ), (δ), above.