THE ASYMPTOTIC FORM OF THE TITCHMARSH–WEYL m-FUNCTION ASSOCIATED WITH A NON-DEFINITE, LINEAR, SECOND ORDER DIFFERENTIAL EQUATION

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§1. *Introduction.* We consider the differential equation

$$-y''-(\lambda w-q)y=0 \qquad \text{for} \qquad 0\leqslant x\leqslant\infty, \tag{1.1}$$

where $w(x)=x^\alpha$ for $\alpha>-1$, q is a real-valued member of $L^1_{loc}(0, \infty)$ and λ is a complex number with

$$0<\varepsilon<\arg(\lambda)<\pi-\varepsilon. \tag{1.2}$$

We are concerned here with the Titchmarsh–Weyl m-function associated with (1.1) which may be defined as follows.

Let Θ and φ be the solutions of (1.1) with

$$\Theta(0,\lambda)=0, \qquad \Theta'(0,\lambda)=1, \qquad \varphi(0,\lambda)=-1, \qquad \varphi'(0,\lambda)=0.$$

The Weyl disc $D(X, \lambda)$ is defined to be the closed interior of the circle which is the image of the real line under the map

$$\xi\to\frac{\{\Theta(X,\lambda)-\xi\Theta'(X,\lambda)\}}{\{\varphi(X,\lambda)-\xi\varphi'(X,\lambda)\}}, \qquad 0<X<\infty.$$

It is known, see **[2]**, **[3]**, **[6]**, **[7]**, **[10]** and the references listed therein, that as X increases the discs, $D(X, \lambda)$, nest and as $X\to\infty$ converge to either a limit point or a limit circle. In the limit point case we define $m(\lambda)$ to be the limit point. In the limit circle case we fix $m(\lambda)$ as a point on the limit circle.

An equivalent, and more constructive definition of $D(X, \lambda)$ due to Atkinson **[3]** is as follows.

Definition 1. *The Weyl disc, $D(X,\lambda)$ consists of those $m\in\mathbb{C}$ which are such that the equation*

$$v'=-1-(\lambda w-q)v^2, \tag{1.3}$$

with $v(0,\lambda)=m$ has a solution, $v(x,\lambda)$, for $0\leqslant x\leqslant X$, with

$$\operatorname{Im}\{v(X,\lambda)\}\geqslant 0. \tag{1.4}$$

It is our concern here to find the asymptotic form of the $m(\lambda)$ function defined above as $|\lambda|\to\infty$ in a sector of the form (1.2). This is a topic which has received considerable attention in recent years following the innovative approach of **[2]**. The case $\alpha=0$ has been particularly well explored, see for example **[2]**, **[3]**, **[7]**, **[8]** and **[10]**.

The case where $w(x)$ is not, at least asymptotically, a constant has received less attention. The articles [3] and [9] obtain order of magnitude results for a general w while [5] derives exact results for an equation of the form

$$-(x^\beta y')' - \lambda x^\alpha y = 0.$$

The first term in the asymptotic expansion of the m-function associated with (1.1) for $w(x) = x^\alpha (\alpha > -1)$ and a general q was found in [6]. We derive the full asymptotic expansion in this case.

§2. *The results.* We define $v = 1/(\alpha+2)$ and $k = (\alpha+2)/2$ and note, since $\alpha > -1$, that $0 < v < 1$ and $\frac{1}{2} < k < \infty$. Let $X(\lambda) = X(|\lambda|)$ be a function which satisfies

$$X(\lambda) \to 0 \quad \text{as} \quad |\lambda| \to \infty, \tag{2.1}$$

$$|\lambda|^v X(\lambda)/(\log|\lambda|)^{2v} \to \infty \quad \text{as} \quad |\lambda| \to \infty. \tag{2.2}$$

It follows from (2.2) that $|\lambda|^v X(\lambda) \to \infty$ as $|\lambda| \to \infty$.

The conditions (2.1) and (2.2) will be satisfied if, for example, we set $X(\lambda) = |\lambda|^{-\gamma}$ where $0 < \gamma < v$.

By hypothesis $q \in L^1[0, \delta)$ for $\delta > 0$ and so, by (2.1), there exists a function $\eta(\lambda)$ such that

$$\int_0^{X(\lambda)} |q(t)|\, dt = \eta(\lambda), \tag{2.3}$$

and $\eta(\lambda) \to 0$ as $|\lambda| \to \infty$. We further assume that there exists a K with $|\lambda|^{-K} \leqslant \eta(\lambda)$.

We define a sequence of functions $\{r_j(x, \lambda)\}$ for $j = 0, \ldots, N$, where N is arbitrary but fixed, for $x \in [0, X(\lambda)]$ and λ satisfying (1.2) with $|\lambda| > \lambda_0$, as follows.

Let

$$r_0(x, \lambda) = \lambda^{1/2} x^{k-1} \frac{H^{(1)}_{v-1}(k^{-1} x^k \lambda^{1/2})}{H^{(1)}_v(k^{-1} x^k \lambda^{1/2})}, \tag{2.4}$$

where $H^{(1)}_v$ denotes the Hankel–Bessel function of the first type of order v.

Further define

$$r_1(x, \lambda) = -\int_x^{X(\lambda)} q(t) \exp\left(2\int_x^t r_0(s, \lambda)\, ds\right)$$

and for $j = 1, \ldots, N-1$,

$$r_{j+1}(x, \lambda) = \int_x^{X(\lambda)} r_j(t, \lambda)^2 \exp\left(2\sum_{k=0}^{j} \int_x^t r_k(s, \lambda)\, ds\right) dt.$$

We note that r_0, and hence the subsequent r_j, are defined on $(0, X(\lambda))$ rather than $[0, X(\lambda)]$ but we show below that the domain may be extended by continuity.

THEOREM 1. *As* $|\lambda|\to\infty$ *subject to* (1.2)

$$m(\lambda) = -\left(\sum_{j=0}^{N} r_j(0,\lambda)\right)^{-1} + O(\eta(\lambda)^{2^N}|\lambda|^{-2v}).$$

In the case where $m(\lambda)$ *is not uniquely defined, but is a point on the limit circle; Thereom* 1 *applies to all such functions.*

In the case where we impose more restrictions on q it is possible to give a more explicit form of Theorem 1.

THEOREM 2. *If* q *is continuous on a right neighbourhood of* 0 *then*

$$m(\lambda) = \left[\lambda^v e^{v\pi i} v^{2v-1}\frac{\Gamma(1-v)}{\Gamma(v)} - \frac{\pi^2|\lambda|^{-v}e^{iv\theta}q(0)}{2^{4v-1}v^{2v-1}\Gamma(v)^2}\int_0^\infty z^{4v-1}(H_v^{(1)}(ze^{i\theta/2}))^2 dz\right]^{-1} + o(|\lambda|^{-2v})$$

as $|\lambda|\to\infty$ *subject to* (1.2) *where* $\theta = \arg(\lambda)$.

In the light of Theorem 2 it would seem likely that, if q were sufficiently smooth, there exists an asymptotic expansion of $m(\lambda)$ in decreasing powers of λ^v which would enable the derivation of an inverse result along the lines of the main theorem of **[8]**.

Our proof of Theorem 1 proceeds along the following lines. Firstly we derive a bound for the radius of the disc $D(X, \lambda)$. Such a bound will of course depend on both X and λ where X is constrained by (2.1)–(2.3). We suppose in the sequel that $X = X(\lambda)$ satisfies the constraints of (2.1)-(2.3). We next derive the asymptotic form of a member of $D(X, \lambda)$. By the nesting circles principle $m(\lambda)$ has the same asymptotic form to within the radius of $D(X, \lambda)$ which is shown to be exponentially small. This approach was first used by Atkinson in **[2]**.

§3. *The radius of* $D(X, \lambda)$. It is convenient now, and in the sequel to write

$$\lambda^{1/2} = \mu + i\beta, \tag{3.1}$$

where μ, $\beta > 0$, thus fixing the branch of the square root. We note from (1.2) that

$$|\lambda|^{1/2}\sin(\varepsilon/2) \leqslant \beta \leqslant |\lambda|^{1/2}\cos(\varepsilon/2). \tag{3.2}$$

LEMMA 1.

$$\text{radius}(D(X,\lambda)) \leqslant c|\lambda|^{-v}e^{-4v\beta X^{1/(2v)}}. \tag{3.3}$$

Proof. It is known, see [**2**], [**3**] and [**6**] that

$$\text{radius }(D(X,\lambda)) = 1/(|[\varphi,\varphi](X,\lambda)|) = 1/(|\text{Im }(\lambda)| \int_0^X t^\alpha |\varphi(t,\lambda)|^2 dt),$$

where $\varphi(t, \lambda)$ is the solution of (1.1) with $\varphi(0, \lambda) = 1$ and $\varphi'(0, \lambda) = 0$. From [**6**] we also have that

$$|\text{Im }(\lambda)| \int_0^X t^\alpha |\varphi(t,\lambda)|^2 dt \geqslant c|\lambda|^v e^{4v\beta X^{1/(2v)}}$$

from which the result follows.

§4. *The approximation of a member of* $D(X, \lambda)$. We characterize a member of $D(X, \lambda)$ by the construction of a $\tau(0, \lambda)$ where $\tau(x, \lambda)$ satisfies the requirements of Definition 1. If τ is a solution of (1.3), we set $u = -1/\tau$ so that $\text{Im }\{u(X, \lambda)\} \geqslant 0$ if, and only if, $\text{Im }\{\tau(X, \lambda)\} \geqslant 0$. We also see that

$$u' = -(\lambda x^\alpha - q) - u^2, \tag{4.1}$$

so that, if u exists and is non-zero in $[0, X]$,

$$-1/u(0, \lambda) \in D(X, \lambda). \tag{4.2}$$

We seek now to approximate u, which through (4.2), furnishes our approximation to $m(\lambda)$.

Let $r = r(x, \lambda)$ be a function, to be chosen below in such a way that it is differentiable on $[0, X]$ and

$$\text{Im }\{r(X, \lambda)\} \geqslant 0. \tag{4.3}$$

Now let u be the solution of (4.1) with

$$u(X, \lambda) = r(X, \lambda). \tag{4.4}$$

Strictly we should write $u_r(x, \lambda)$ but no ambiguity will arise if we omit the r.

We set $\sigma(x, \lambda) = u(x, \lambda) - r(x, \lambda)$ so that

$$\sigma(X, \lambda) = 0, \tag{4.5}$$

and

$$\sigma' = Q - 2r\sigma - \sigma^2, \tag{4.6}$$

where

$$Q(x, \lambda) = -\{\lambda x^\alpha - q + r' + r^2\}. \tag{4.7}$$

We now define

$$A(\lambda)=\sup_{0\leqslant x\leqslant X}\left|\int_x^X Q(t)\exp\left(2\int_x^t r(s,\lambda)ds\right)dt\right|$$

and

$$B(\lambda)=\sup_{0\leqslant x\leqslant X}\int_x^X\left|\exp\left(2\int_x^r r(s,\lambda)ds\right)\right|dt$$

LEMMA 2. *If* $4A(\lambda)B(\lambda)<1$ *for* $|\lambda|>\lambda_0$ *then*

$$|\sigma(x,\lambda)|<2A(\lambda)\qquad \textit{for}\qquad |\lambda|>\lambda_0\qquad \textit{and}\quad \textit{all}\qquad x\in[0,X].$$

Proof. Our proof is essentially the same as that of the corresponding result in [**2**], [**7**] and [**8**], and is omitted.

We examine now the means of selecting a function $r(x, \lambda)$ which fulfills the requirements of (4.3), makes Q small for large values of λ, and ensures that the hypotheses of Lemma 2 are satisfied.

For integral $N(\geqslant 1)$ we set

$$r(x,\lambda)=\sum_{n=0}^{N} r_n(x,\lambda) \tag{4.8}$$

and observe that

$$-Q=\lambda x^{\alpha}-q+r_0'+r_0^2+\sum_{n=1}^{N} r_n'+\sum_{n=1}^{N}\sum_{m=0}^{N} r_m+r_0\sum_{m=1}^{N} r_m.$$

We choose $r_0(x, \lambda)$ to satisfy

$$\lambda x^{\alpha}+r_0'+r_0^2=0. \tag{4.9}$$

Then

$$-Q=-q+r_1'+2r_0r_1+r_1^2+\sum_{n=2}^{N} r_n'+\sum_{n=2}^{N} r_n\sum_{m=0}^{N} r_m+\sum_{n=0}^{1} r_n\sum_{m=2}^{N} r_m.$$

We choose r_1 so that

$$r_1+2r_0r_1=q,\qquad r_1(X,\lambda)=0,$$

whence

$$r_1(x,\lambda)=-\int_x^X q(t)\exp\left(2\int_x^t r_0(s,\lambda)ds\right)dt. \tag{4.10}$$

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Proceeding iteratively in this way we have

$$r_{j+1}(x, \lambda) := \int_x^X r_j(t, \lambda)^2 \exp\left(2 \sum_{k=0}^{j} \int_x^t r_k(s, \lambda)ds\right)dt \tag{4.11}$$

for $j = 1, \ldots, N-1$ and

$$Q(x, \lambda) = -r_N(x, \lambda)^2. \tag{4.12}$$

We note from (4.8), (4.10) and (4.11) that

$$r(X, \lambda) = r_0(X, \lambda). \tag{4.13}$$

§5. *The choice of* r_0. We set

$$r_0(x, \lambda) = \lambda^{1/2} x^{k-1} \frac{H_{v-1}^{(1)}(k^{-1}x^k\lambda^{1/2})}{H_v^{(1)}(k^{-1}x^k\lambda^{1/2})}$$

and it may be verified from ([1], 9.1.27) that (4.9) is satisfied.

LEMMA 3. Im $\{r_0(X, \lambda)\} \geqslant 0$ *for* $|\lambda| > \lambda_0$.

Proof. We note from (2.2) that

$$|k^{-1}X^k\lambda^{1/2}| = k^{-1}(X(\lambda)|\lambda|^v)^k \to \infty \quad \text{as} \quad |\lambda| \to \infty.$$

Writing z for $k^{-1}X^k\lambda^{1/2}$ we have from ([1], 9.2.3) that

$$H_v^{(1)}(z) \sim \sqrt{2/(\pi z)}\, e^{-i(z - \frac{1}{2}v\pi - \frac{1}{4}\pi)} \quad \text{as} \quad |z| \to \infty$$

and

$$H_{v-1}^{(1)}(z)/H_v^{(1)}(z) \to e^{-i(v-1)\pi/2}/e^{-iv\pi/2} = i \quad \text{as} \quad |z| \to \infty.$$

Thus, Im $\{r_0(X, \lambda)\} \sim \mu X^{k-1} > 0$ as $|z| \to \infty$ where μ is defined in (3.1). It follows from Lemma 3 and (4.13) that Im $\{r(X, \lambda)\} \geqslant 0$ for $|\lambda|$ sufficiently large subject to (1.2).

We now derive bounds for $r_0(x, \lambda)$ for $x \in [0, X]$. It is convenient to decompose $[0, X(\lambda)]$ into three regions as follows.

$$\left.\begin{aligned} &A|\lambda|^{-v} \leqslant x \leqslant X, \\ &0 \leqslant x \leqslant B|\lambda|^{-v}, \\ &B|\lambda|^{-v} \leqslant x \leqslant A|\lambda|^{-v}, \end{aligned}\right\} \tag{5.1}$$

where A and B are positive constants independent of x, and λ which will be chosen later. We derive bounds for $r_0(x, \lambda)$ on each of the intervals of (5.1).

LEMMA 4. *There exists a constant* $A = A(\alpha)$ *such that for* $A|\lambda|^{-v} \leqslant x \leqslant X$ *and* $|\lambda| > \lambda_0$ *subject to* (1.2) *we have*

$$-\tfrac{17}{16}\beta x^{k-1} \leqslant \operatorname{Re}\{r_0(x, \lambda)\} \leqslant -\tfrac{15}{16}\beta x^{k-1},$$

and

$$\tfrac{15}{16}\mu x^{k-1} \leqslant \operatorname{Im}\{r_0(x, \lambda)\} \leqslant \tfrac{17}{16}\mu x^{k-1}.$$

Proof. We again write $z = k^{-1}x^k\lambda^{1/2}$. It is shown in ([**11**] 7.2) that there exist constants c_1 and c_2 depending only on v (and hence only on α) such that

$$H_v^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{1/2} e^{i(z - v\pi/2 - \pi/4)}[1 + R(v, z)], \tag{5.2}$$

where

$$|R(v, z)| \leqslant c_1/|z| \qquad \text{for all} \qquad |z| \geqslant c_2. \tag{5.3}$$

We set $\delta = (1/256)\sin(\varepsilon/2)$ where ε is defined in (1.2).

By hypothesis $|z| \geqslant k^{-1}A^k$ so, by (5.3), we may choose A so large that

$$|R(v, z)| \leqslant \delta \qquad \text{and} \qquad |R(v-1, z)| \leqslant \delta, \tag{5.4}$$

for $|z| \geqslant k^{-1}A^k$. Thus, by (5.2)

$$\frac{H_{v-1}^{(1)}(z)}{H_v^{(1)}(z)} = i\left(1 + \frac{R(v-1, z) - R(v, z)}{1 + R(v, z)}\right),$$

and, by (5.4)

$$\left|\frac{R(v-1, z) - R(v, z)}{1 + R(v, z)}\right| \leqslant \frac{2\delta}{1-\delta} \leqslant 4\delta. \tag{5.5}$$

Writing $H_{v-1}^{(1)}(z)/H_v^{(1)}(z) = t + i\rho$, where t and ρ are real, we have from (3.1) and (5.5) that, for $A|\lambda|^{-v} \leqslant x \leqslant X$,

$$r_0(x, \lambda) = x^k([\mu t - \beta\rho] + i[\mu\rho + \beta t]),$$

and so, by (5.5),

$$x^k[-\beta(1 + 4\delta) - 4\mu\delta] \leqslant \operatorname{Re}(r_0(x, \lambda)) \leqslant x^k[-\beta(1 - 4\delta) + 4\mu\delta],$$

so that

$$-x^k\beta\left[1 - \frac{\sin(\varepsilon/2)}{64} - \frac{1}{64}\right] \geqslant \operatorname{Re}(r_0(x, \lambda)) \geqslant -x^k\beta\left[1 + \frac{\sin(\varepsilon/2)}{64} + \frac{1}{64}\right],$$

and the first part of the lemma follows. The second part may be proved in a similar way.

LEMMA 5. *There exists a constant* $B = B(\alpha) > 0$ *such that if* λ *satisfies* (1.2), $|\lambda| > \lambda_0$ *and* $0 \leqslant x \leqslant B|\lambda|^{-v}$ *then*

$$0 < k_1|\lambda|^v \leqslant |r_0(x, \lambda)| \leqslant k_2|\lambda|^v,$$

for positive constants k_1 *and* k_2 *which are independent of* x *and* λ.

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Proof. We use the inequality $\Gamma(t)>\frac{1}{2}$ for $0<t$ and observe that, in the notation of the proof of Lemma 4, the hypotheses imply that $0\leqslant|z|\leqslant k^{-1}B^{k}$. We use the representation of ([1], 9.1.14) and start from the relation

$$H_v^{(1)}(z)=i\csc(v\pi)\{e^{-v\pi i}J_v(z)-J_{-v}(z)\}. \tag{5.6}$$

From ([1], 9.1.10) we have that if B is sufficiently small

$$J_v(z)=\left(\frac{z}{2}\right)^v\sum_{l=0}^{\infty}\frac{(-\frac{1}{4}z^2)^l}{l!\Gamma(v+l+1)},$$

and

$$\begin{aligned}|J_v(z)|&\leqslant\left|\frac{z}{2}\right|^v\left\{\frac{1}{\Gamma(v+1)}+\sum_{l=1}^{\infty}\frac{1}{4}\times\frac{1}{l!}\times\frac{1}{\Gamma(v+1)}\right\}\\&\leqslant\frac{|z|^v}{2^{v-2}}=\left(\frac{2}{|z|}\right)^v\left(\frac{|z|^{2v}}{2^{2v-2}}\right)\\&\leqslant\left(\frac{2}{|z|}\right)^v\left(\frac{k^{-2v}B}{2^{2v-2}}\right).\end{aligned} \tag{5.7}$$

Also, if $|z|\neq0$, and B is sufficiently small

$$J_{-v}(z)=\left(\frac{2}{z}\right)^v\sum_{l=0}^{\infty}\frac{(-z^2/2)^l}{l!\Gamma(1+l-v)}=\left(\frac{2}{z}\right)^v\left\{\frac{1}{\Gamma(1-v)}+\sum_{l=1}^{\infty}\frac{(-z^2/2)^l}{l!\Gamma(1+l-v)}\right\}.$$

Now,

$$\left|\sum_{l=1}^{\infty}\frac{(-z^2/4)}{l!\Gamma(1+l-v)}\right|\leqslant\frac{|z|^2}{4}\sum_{l=1}^{\infty}\frac{1}{l!\Gamma(1+l-v)}\leqslant\frac{|z|^2}{2}\sum_{l=1}^{\infty}\frac{1}{l!}\leqslant2|z|^2.$$

Thus, from (5.7)

$$\left.\begin{aligned}J_{-v}(z)&=(2/z)^v\{(1/\Gamma(1-v))+E_1\},\\J_v(z)&=E_2,\end{aligned}\right\} \tag{5.8}$$

where $|E_1|\leqslant2k^{-2}B^{2B}$ and $|E_2|\leqslant k^{-2v}B/2^{2v-2}$. It now follows from (5.8) that

$$H_v^{(1)}(z)=-\frac{\csc(v\pi)}{\Gamma(1-v)}\left(\frac{2}{z}\right)^v(1+\sigma_1), \tag{5.9}$$

where $|\sigma_1(z)|\to0$ as $|z|\to0$. In particular we choose B so small that $|\sigma_1(z)|\leqslant1/256$.

We now consider $H_{v-1}^{(1)}(z)$. If follows as before that

$$J_{v-1}(z)=(\tfrac{1}{2}z)^{v-1}\left\{\frac{1}{\Gamma(v)}+\sum_{l=1}^{\infty}\frac{(-\frac{1}{4}z^2)^l}{l!\Gamma(v+l)}\right\}.$$

We note that

$$\left|\sum_{l=1}^{\infty}\frac{(-\frac{1}{4}z^2)^l}{l!\Gamma(v+l)}\right|\leqslant\frac{|z^2|}{4}\sum_{l=1}^{\infty}\frac{1}{l!\Gamma(v+l)}\leqslant 2|z|^2,$$

whence

$$J_{v-1}(z)=(\tfrac{1}{2}z)^{v-1}\left\{\frac{1}{\Gamma(v)}+E_3\right\}\qquad\text{where}\qquad |E_3|\leqslant 2|z|^2, \tag{5.10}$$

also

$$J_{1-v}(z)=(\tfrac{1}{2}z)^{v-1}(\tfrac{1}{2})^{2-2v}\left\{\frac{1}{\Gamma(2-v)}+\sum_{l=1}^{\infty}\frac{(-\frac{1}{4}z^2)^l}{l!\Gamma(2+l-v)}\right\},$$

and

$$J_{1-v}(z)=(\tfrac{1}{2}z)^{v-2}E_4, \tag{5.11}$$

where $|E_4|\leqslant 2^{4-2v}|z|^{2-2v}$. On combining (5.10) and (5.11) we see that

$$\begin{aligned}H_{v-1}^{(1)}(z)&=i\csc((v-1)\pi)\left\{\{e^{-(v-1)\pi i}(\tfrac{1}{2}z)^{v-1}\left[\frac{1}{\Gamma(v)}+E_3\right]+(\tfrac{1}{2}z)^{v-1}E_4\right\}\\&=\frac{i\csc((v-1)\pi)}{\Gamma(v)}e^{-(v-1)\pi i}(\tfrac{1}{2}z)^{v-1}\{1+\Gamma(v)E_3+e^{(v-1)\pi i}\Gamma(v)E_4\}\\&=\frac{i\csc((v-1)\pi)}{\Gamma(v)}e^{-(v-1)\pi i}(\tfrac{1}{2}z)^{v-1}\{1+\sigma_2(z)\},\end{aligned} \tag{5.12}$$

where $\sigma_2(z)\to 0$ as $|z|\to 0$. In particular we may choose B so that $|\sigma_2(z)|\leqslant 1/256$ for $|\lambda|\leqslant\lambda_0$ and $0\leqslant x\leqslant B|\lambda|^{-v}$. It follows now from (5.9) and (5.12) that, under these circumstances

$$\begin{aligned}r_0(x,\lambda)&=\lambda^{1/2}x^{k-1}\frac{H_{v-1}^{(1)}(z)}{H_v^{(1)}(z)}\\&=-\lambda^{1/2}x^{k-1}(\tfrac{1}{2}z)^{2v-1}\frac{\Gamma(1-v)}{\Gamma(v)}e^{-(v-1)\pi i}\frac{\csc((v-1)\pi)}{\csc(v\pi)}\frac{(1+\sigma_2(z))}{(1+\sigma_1(z))}\\&=-\lambda^v e^{-v\pi i}v^{2v-1}\frac{\Gamma(1-v)}{\Gamma(v)}\frac{(1+\sigma_2(z))}{(1+\sigma_1(z))}.\end{aligned} \tag{5.13}$$

The result now follows from the bounds imposed on σ_1 and σ_2.

We notice in particular from (5.13) that

$$\lim_{x\to 0+}r_0(x,\lambda)=-\lambda^v e^{-v\pi i}v^{2v-1}\frac{\Gamma(1-v)}{\Gamma(v)}. \tag{5.14}$$

LEMMA 6. *For $B|\lambda|^{-v}\leqslant x\leqslant A|\lambda|$ where A and B are the constants of Lemmas* 4 *and* 5, *and λ satisfies* (1.2) *with $|\lambda|>\lambda_0$, there exist constants c_4 and c_5 with*

$$0<c_3|\lambda|^v\leqslant|r_0(x,\lambda)|\leqslant c_4|\lambda|^v.$$

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Proof. We again set $z=k^{-1}x^k\lambda^{1/2}$ and note from (1.2) and the hypotheses that

$$0<\delta<\arg(z)<\pi/2, \tag{5.15}$$

and

$$k^{-1}B^k \leqslant |z| \leqslant k^{-1}A^k. \tag{5.16}$$

It is known, ([**4**], 7.9), that $H_v^{(1)}(z)$ is an analytic function with no zeros in $0<\arg(z)<\pi$. It follows from (5.15) and (5.16) that there exist constants c_1 and c_2 with

$$0<c_1<|H_v^{(1)}(z)|<c_2. \tag{5.17}$$

We write $H_{v-1}^{(1)}(z)=H_{-(1-v)}^{(1)}(z)=e^{(1-v)\pi i}H_{1-v}^{(1)}(z)$ from ([**1**], 9.1.6) and, since $0<v<1$, see that there exist c_5 and c_6 with

$$0<c_5<|H_{v-1}^{(1)}(z)|<c_6. \tag{5.18}$$

From (5.17) and (5.18) we see that there exist constants M^- and M^+ with

$$0<M^- \leqslant \left|\frac{H_{v-1}^{(1)}(z)}{H_v^{(1)}(z)}\right| \leqslant M^+ < \infty$$

from which the result follows.

We summarize the main results of Lemmas 4, 5 and 6, with a slight change of notation, as follows. If $|\lambda|>\lambda_0$, subject to (1.2) then

$$-\tfrac{17}{16}\beta x^{k-1} \leqslant \operatorname{Re}\{r_0(x,\lambda)\} \leqslant -\tfrac{15}{16}\beta x^{k-1} \quad \text{for} \quad A|\lambda|^{-v} \leqslant x < X, \tag{5.19}$$

$$|r_0(x,\lambda)| \leqslant c|\lambda|^v \quad \text{for} \quad 0 \leqslant x \leqslant A|\lambda|^{-v}. \tag{5.20}$$

§6. *Bounds for* $r(x,\lambda)$. We recall that

$$r_1(x,\lambda)=-\int_x^X q(t)\exp\left(2\int_x^t r_0(s,\lambda)\,ds\right)dt,$$

and

$$r_{j+1}(x,\lambda)=\int_x^X r_j(t,\lambda)^2 \exp\left(2\sum_{l=0}^{j}\int_x^t r_l(s,\lambda)\,ds\right) r_j(t,\lambda)^2\,dt.$$

It follows from (2.3), (5.19) and (5.20) that there exists a constant c with

$$|r_1(x,\lambda)| \leqslant c\eta(\lambda) \quad \text{and} \quad \eta(\lambda)\to 0 \quad \text{as} \quad |\lambda|\to\infty. \tag{6.1}$$

LEMMA 7. *For λ satisfying the conditions of* (1.2), $0 \leqslant x \leqslant X$ *and* $|\lambda| > \lambda_0$

$$|r_j(x, \lambda)| \leqslant c\eta(\lambda)^{2^{j-1}},$$

for $j = 1, \ldots, N$.

Proof. We use induction on j. When $j = 1$ this reduces to (6.1). Suppose now that the result holds for $j = J$. By increasing λ_0 if necessary it may be shown from (5.19), (5.20) and the induction hypothesis that for $|\lambda| > |\lambda_0|$, subject to (1.2).

$$\operatorname{Re}\left\{\sum_{l=0}^{J} r_l(s, \lambda)\right\} \leqslant -\tfrac{7}{8}\beta s^{k-1} \qquad \text{for} \qquad A|\lambda|^{-v} \leqslant s \leqslant X,$$

and

$$\left|\operatorname{Re}\left\{\sum_{l=0}^{J} r_l(s, \lambda)\right\}\right| \leqslant c|\lambda|^{v} \qquad \text{for} \qquad 0 \leqslant s \leqslant A|\lambda|^{-v}.$$

In either case we have that

$$2\operatorname{Re}\left\{\int_x^t \sum_{l=0}^{J} r_c(s, \lambda)\,ds\right\} \leqslant c.$$

It follows that

$$|r_{J+1}(x, \lambda)| \leqslant c\int_x^X |r_J(t, \lambda)|^2 dt \leqslant c\int_x^X \eta(\lambda)^{2^J} dt,$$

from which the result follows.

We note in passing that the bounds of Lemma 7 are quite crude in that, aside from (2.4), they make no use of the fact that $X(\lambda) \to 0$ as $|\lambda| \to \infty$. The result also does not use the fact that the exponent in the integrand is negative for most of the range of integration.

§7. *Proof of Theorem* 1. We show now that the $r(x, \lambda)$ chosen above satisfies the conditions of Section 4. From (4.12) and Lemma 7 we see that

$$|Q(x, \lambda)| \leqslant c\eta(\lambda)^{2^N}, \tag{7.1}$$

from (4.11) and Lemma 7 that

$$2\int_x^t r(s, \lambda)\,ds \leqslant c. \tag{7.2}$$

It follows from (7.1) and (7.2) that, in the notation of Section 4, $4A(\lambda)B(\lambda) < 1$ for λ subject to (1.2) with $|\lambda| > \lambda_0$ where λ_0 is sufficiently large. Thus, by

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Lemma 2,

$$|\sigma(x,\lambda)|\leqslant c\eta(\lambda)^{2^N} \quad \text{for} \quad x\in[0,X].$$

In the notation of Section 4 we then have that

$$u(0,\lambda)=\sum_{j=0}^{N} r_j(0,\lambda)+O(\eta(\lambda)^{2^N}),$$

and, from Lemma 5,

$$-u(0,\lambda)^{-1}=-\left(\sum_{j=0}^{N} r_j(0,\lambda)\right)^{-1}+O(\eta(\lambda)^{2^N}|\lambda|^{-2v}). \tag{7.3}$$

It remains to show that rad $(D(X,\lambda))=O(\eta(\lambda)^{2^N}|\lambda|^{-2v})$. From Lemma 1 we need to show that

$$|\lambda|^{-v}e^{-4v\sin(\varepsilon/2)(|\lambda|^v X(\lambda))^{1/(2v)}}=O(\eta(\lambda)^{2^N}|\lambda|^{-2v}).$$

But this follows from (2.2) and (2.4).

§8. *Proof of Theorem* 2. We consider the case $N=1$ of Theorem 1. Since q is continuous on $[0,X(\lambda)]$ if $|\lambda|$ is sufficiently large we have from (2.4) that $\eta(\lambda)=o(1)$ so the error term in Theorem 1 is $o(|\lambda|^{-2v})$.

We recall that

$$r_0(x,\lambda)=\lambda^{1/2}x^{k-1}\frac{H_{v-1}^{(1)}(k^{-1}x^k\lambda^{1/2})}{H_v^{(1)}(k^{-1}x^k\lambda^{1/2})}=\frac{(x^{1/2}H_v^{(1)}(k^{-1}x^k\lambda^{1/2}))'}{x^{1/2}H_v^{(1)}(k^{-1}x^k\lambda^{1/2})}.$$

It may be shown, ([**1**], 9.1.9) that

$$\lim_{x\to 0} x^{1/2}H_v^{(1)}(k^{-1}x^k\lambda^{1/2})=-\frac{i\Gamma(v)}{\pi v^v\lambda^{v/2}}.$$

Thus, evaluated as an improper integral,

$$\exp\left(2\int_0^t r_0(s,\lambda)ds\right)=-\frac{\pi^2v^{2v}\lambda^v t(H_v^{(1)}(k^{-k}t^k\lambda^{1/2}))^2}{\Gamma(v)^2},$$

and

$$r_1(0,\lambda)=-\int_0^X q(t)\exp\left(2\int_0^t r_0(s,\lambda)ds\right)dt$$

$$=\frac{\pi^2v^{2v}\lambda^{1/2}}{\Gamma(v)^2}\int_0^X t(H_v^{(1)}(k^{-1}t^k\lambda^{1/2}))^2q(t)dt. \tag{8.1}$$

We write $\lambda=|\lambda|e^{i\theta}$ and make the change of variable $z=k^{-1}t^{k}|\lambda|^{1/2}$ in (8.1) to yield,

$$r_1(0,\lambda)=\frac{\pi^2|\lambda|^{-v}e^{iv\theta}}{2^{4v-1}v^{2v-1}\Gamma(v)^2}\times\int_0^{k^{-1}(X|\lambda|^v)^{1/(2v)}} z^{4v-1}(H_v^{(1)}(ze^{i\theta/2}))^2q((\tfrac{1}{2}v^{-1})^{2v}|\lambda|^vz^{2v})dz. \tag{8.2}$$

We write the integral in (8.2) as I_1+I_2 where

$$I_1=\int_0^{k^{-1}(X|\lambda|^v)^{1/2v}} z^{4v-1}(H_v^{(1)}(ze^{i\theta/2}))^2q(0)dz,$$

and

$$I_2=\int_0^{k^{-1}(X|\lambda|^v)^{1/(2v)}} z^{4v-1}(H_v^{(1)}(ze^{i\theta/2}))^2[q((\tfrac{1}{2}v^{-1})^{2v}|\lambda|^vz^{2v})-q(0)]dz.$$

For $0\leqslant z\leqslant k^{-1}(X|\lambda|^v)^{1/2v}$ we have that $0\leqslant|\lambda|^{-v}z^{2v}\leqslant cX(\lambda)\to 0$ as $|\lambda|\to\infty$, so the term [. . .] of I_2 is $o(1)$ by hypothesis. Further, from ([**1**], 9.2.3)

$$H_v^{(1)}(y)\sim\sqrt{\frac{2}{\pi y}}\,e^{i(y-v\pi/2-\pi/4)}\qquad\text{as}\qquad |y|\to\infty\qquad\text{if}\qquad 0<\arg(y)<\pi/2,$$

so

$$|H_v^{(1)}(ze^{i\theta/2})|=O(z^{-1/2}e^{z\sin(\theta/2)}).$$

It follows that I_2 is $o(1)$ as $|\lambda|\to\infty$. We now write

$$I_1=I_{11}+I_{12},$$

where

$$I_{11}=q(0)\int_0^\infty z^{4v-1}(H_v^{(1)}(ze^{i\theta/2}))^2dz,$$

and

$$I_{12}=-q(0)\int_{k^{-1}(X|\lambda|^v)^{1/2v}}^\infty z^{4v-1}(H_v^{(1)}(ze^{i\theta/2}))^2dz.$$

Both integrals are convergent, by (8.3) and, since $X(\lambda)|\lambda|^v\to\infty$ as $|\lambda|\to\infty$, I_{12} is $o(1)$ as $|\lambda|\to\infty$. We thus have from (8.2) that

$$r_1(0,\lambda)=\frac{\pi^2|\lambda|^{-v}e^{iv\theta}q(0)}{2^{4v-1}v^{2v-1}\Gamma(v)^2}\int_0^\infty z^{4v-1}(H_v^{(1)}(ze^{i\theta/2}))^2dz+o(|\lambda|)^{-v}). \tag{8.4}$$

It remains only to observe that

$$\lim_{x\to 0} r_0(x,\lambda) = -\lambda^v e^{-v\pi i} v^{2v-1}\frac{\Gamma(1-v)}{\Gamma(v)}.$$

It is reassuring to observe that in the case $\alpha=0$ so that $v=\frac{1}{2}$, the asymptotic form from Theorem 2 coincides with that of [7].

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