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On the existence of a symplectic desingularization of some moduli spaces of sheaves on a K3 surface

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Abstract

Let M_c be the moduli space of semistable torsion-free sheaves of rank 2 with Chern classes $c_1 = 0$ and $c_2 = c$ over a K3 surface with generic polarization. When $c = 2n \geqslant 4$ is even, M_c is a singular projective variety which admits a symplectic form, called the Mukai form, on the smooth part. A natural question raised by O'Grady asks if there exists a desingularization on which the Mukai form extends everywhere nondegenerately. In this paper we show that such a desingularization does not exist for many even integers c by computing the stringy Euler numbers.

1. Introduction

Let X be a projective K3 surface with generic polarization $\mathcal{O}_X(1)$ and let $M_c = M(2, 0, c)$ be the moduli space of semistable torsion-free sheaves on X of rank 2, with Chern classes $c_1 = 0$ and $c_2 = c$. When $c = 2n \geqslant 4$ is even, M_c is a singular projective variety. Recently O'Grady raised the following question [Ogr99, (0.1)].

Question 1.1. Does there exist a symplectic desingularization of M_{2n}?

In [Ogr99], he analyzed Kirwan's desingularization $\widehat{M_c}$ of M_c and proved that $\widehat{M_c}$ can be blown down twice and that as a result he obtained a symplectic desingularization $\widetilde{M_c}$ of M_c in the case when $c = 4$. This turns out to be a new irreducible symplectic variety.

When $c \geqslant 6$, O'Grady conjectures that there is no smooth symplectic model of M_c (see [Ogr99, p. 50]). The purpose of this paper is to provide a partial answer to Question 1.1.

Theorem 1.2. *There is no symplectic desingularization of M_{2n} if $na_n/(2n-3)$ is not an integer where a_n is the Euler number of the Hilbert scheme $X^{[n]}$ of n points in X.*

It is well known that a_n is given by the equation

$$\sum_{n=0}^{\infty} a_n q^n = \prod_{n=1}^{\infty} 1/(1-q^n)^{24}.$$

By direct computation, one can check that $na_n/(2n-3)$ is not an integer for $n = 5, 6, 8, 11, 12, 13, 15, 16, 17, 18, 19, 20, \ldots$.

The idea of the proof is to use properties of the stringy Euler numbers. If there is an irreducible symplectic desingularization $\widetilde{M_c}$ of M_c, then the stringy Euler number of M_c is equal to the ordinary Euler number of $\widetilde{M_c}$ because the canonical divisors of both $\widetilde{M_c}$ and M_c are trivial (Theorem 2.2).

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In particular, we deduce that the stringy Euler number $e_{\mathrm{st}}(M_c)$ must be an integer. Therefore, Theorem 1.2 is a consequence of the following.

Proposition 1.3. *The stringy Euler number* $e_{\mathrm{st}}(M_{2n})$ *is of the form*

$$\frac{na_n}{2n-3} + \textit{integer}.$$

We prove this proposition in § 3 after a brief review of preliminaries.

One motivation for Question 1.1 is to find a mathematical interpretation of Vafa–Witten's formula [VW94, (4.17)] which says that the 'Euler characteristic' of M_{2n} is

$$e^{\mathrm{VW}}(M_{2n}) = a_{4n-3} + \tfrac{1}{4}a_n.$$

Because $k/4 \neq l/(2n-3)$ for $1 \leqslant k \leqslant 3$, $1 \leqslant l < 2n-3$, we deduce the following from Proposition 1.3.

Corollary 1.4. *The stringy Euler number* $e_{\mathrm{st}}(M_{2n})$ *is not Vafa–Witten's Euler characteristic* $e^{\mathrm{VW}}(M_{2n})$ *in general.*

Independently, Kaledin and Lehn [KL04a] proved that there is no symplectic desingularization of M_{2n} for any $n \geqslant 3$ by a very different method.

2. Preliminaries

In this section, we recall the definition and basic facts about stringy Euler numbers. The references are [Bat98, DL99].

Let W be a variety with at worst *log-terminal* singularities, i.e.:

- W is $\mathbb{Q}$-Gorenstein;
- for a resolution of singularities $\rho : V \to W$ such that the exceptional locus of ρ is a divisor D whose irreducible components $D_1, \dots, D_r$ are smooth divisors with only normal crossings, we have

$$K_V = \rho^* K_W + \sum_{i=1}^{r} a_i D_i$$

  with $a_i > -1$ for all i, where D_i runs over all irreducible components of D. The divisor $\sum_{i=1}^{r} a_i D_i$ is called the *discrepancy divisor*.

For each subset $J \subset I = \{1, 2, \dots, r\}$, define $D_J = \bigcap_{j \in J} D_j$, $D_\emptyset = Y$ and $D_J^0 = D_J - \bigcup_{j \in I-J} D_j$. Then the stringy E-function of W is defined by

$$E_{\mathrm{st}}(W; u, v) = \sum_{J \subset I} E(D_J^0; u, v) \prod_{j \in J} \frac{uv - 1}{(uv)^{a_j+1} - 1} \tag{2.1}$$

where

$$E(Z; u, v) = \sum_{p,q} \sum_{k \geqslant 0} (-1)^k h^{p,q}(H_c^k(Z; \mathbb{C})) u^p v^q$$

is the Hodge–Deligne polynomial for a variety Z. Note that the Hodge–Deligne polynomials have:

- the additive property: $E(Z; u, v) = E(U; u, v) + E(Z - U; u, v)$ if U is a smooth open subvariety of Z;
- the multiplicative property: $E(Z; u, v) = E(B; u, v)E(F; u, v)$ if Z is a locally trivial F-bundle over B.

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DEFINITION 2.1. The stringy Euler number is defined as

$$e_{\mathrm{st}}(W) = \lim_{u,v\to 1} E_{\mathrm{st}}(W; u, v) = \sum_{J\subset I} e(D_J^0) \prod_{j\in J} \frac{1}{a_j + 1} \tag{2.2}$$

where $e(D_J^0) = E(D_J^0; 1, 1)$.

The 'change of variable formula' (Theorem 6.27 in [Bat98], Lemma 3.3 in [DL99]) implies that the function E_{st} is independent of the choice of a resolution and the following holds.

THEOREM 2.2 [Bat98, Theorem 3.12]. *Suppose W is a $\mathbb{Q}$-Gorenstein algebraic variety with at worst log-terminal singularities. If $\rho : V \to W$ is a crepant desingularization (i.e. $\rho^* K_W = K_V$) then $E_{\mathrm{st}}(W; u, v) = E(V; u, v)$. In particular, $e_{\mathrm{st}}(W) = e(V)$ is an integer.*

3. Proof of Proposition 1.3

We fix a generic polarization of X as in [Ogr99, p. 50]. The moduli space M_{2n} has a stratification

$$M_{2n} = M_{2n}^s \sqcup (\Sigma - \Omega) \sqcup \Omega$$

where M_{2n}^s is the locus of stable sheaves and

$$\Sigma \cong (X^{[n]} \times X^{[n]})/\text{involution}$$

is the locus of sheaves of the form $I_Z \oplus I_{Z'}$ ($[Z], [Z'] \in X^{[n]}$) while

$$\Omega \cong X^{[n]}$$

is the locus of sheaves $I_Z \oplus I_Z$. Kirwan's desingularization $\rho : \widehat{M}_{2n} \to M_{2n}$ is obtained by blowing up M_c first along the deepest stratum Ω, next along the proper transform of the middle stratum Σ and finally along the proper transform of a subvariety Δ in the exceptional divisor of the first blow-up which is the locus of $\mathbb{Z}_2$ quotient singularities [Kir85]. This is indeed a desingularization by [Ogr99, Proposition 1.8.3].

Let $D_1 = \widehat{\Omega}$, $D_2 = \widehat{\Sigma}$, $D_3 = \widehat{\Delta}$ be the (proper transforms of the) exceptional divisors of the three blow-ups. Then they are smooth divisors with only normal crossings and the discrepancy divisor of $\rho : \widehat{M}_{2n} \to M_{2n}$ is [Ogr99, (6.1)]

$$(6n - 7)D_1 + (2n - 4)D_2 + (4n - 6)D_3.$$

Therefore the singularities are terminal for $n \geqslant 2$ and from (2.2) the stringy Euler number of M_{2n} is given by

$$\begin{aligned} e(M_{2n}^s) + e(D_1^0)\frac{1}{6n-6} + e(D_2^0)\frac{1}{2n-3} + e(D_3^0)\frac{1}{4n-5} + e(D_{12}^0)\frac{1}{6n-6}\frac{1}{2n-3} \\ + e(D_{23}^0)\frac{1}{2n-3}\frac{1}{4n-5} + e(D_{13}^0)\frac{1}{6n-6}\frac{1}{4n-5} + e(D_{123}^0)\frac{1}{6n-6}\frac{1}{2n-3}\frac{1}{4n-5}. \end{aligned} \tag{3.1}$$

We need to compute the (virtual) Euler numbers of D_J^0 for $J \subset \{1, 2, 3\}$. Let (E, ω) be a symplectic vector space of dimension $c = 2n$. Let $\mathrm{Gr}^\omega(k, c)$ be the Grassmannian of k-dimensional subspaces of E isotropic with respect to the symplectic form ω (i.e. the restriction of ω to the subspace is zero).

LEMMA 3.1. *For $k \leqslant n$, the Euler number of $\mathrm{Gr}^\omega(k, 2n)$ is $2^k\binom{n}{k}$.*

Proof. Consider the incidence variety

$$\{(a, b) \in \mathrm{Gr}^\omega(k - 1, 2n) \times \mathrm{Gr}^\omega(k, 2n) \mid a \subset b\}.$$

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This is a $\mathbb{P}^{2n-2k+1}$-bundle over $\mathrm{Gr}^\omega(k-1,2n)$ and a $\mathbb{P}^{k-1}$-bundle over $\mathrm{Gr}^\omega(k,2n)$. The formula follows from an induction on k. □

Let $\hat{\mathbb{P}}^5$ be the blow-up of $\mathbb{P}^5$ (projectivization of the space of 3×3 symmetric matrices) along $\mathbb{P}^2$ (the locus of rank 1 matrices). We have the following from [Ogr99, § 6] and [Ogr97, § 3].

PROPOSITION 3.2.

(1) *D_1 is a $\hat{\mathbb{P}}^5$-bundle over a $\mathrm{Gr}^\omega(3,2n)$-bundle over $X^{[n]}$.*
(2) *D_2^0 is a $\mathbb{P}^{2n-4}$-bundle over a $\mathbb{P}^{2n-3}$-bundle over $(X^{[n]}\times X^{[n]}-X^{[n]})/$involution.*
(3) *D_3 is a $\mathbb{P}^{2n-4}\times\mathbb{P}^2$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*
(4) *$D_1\cap D_2$ is a $\mathbb{P}^2\times\mathbb{P}^2$-bundle over $\mathrm{Gr}^\omega(3,2n)$-bundle over $X^{[n]}$.*
(5) *$D_2\cap D_3$ is a $\mathbb{P}^{2n-4}\times\mathbb{P}^1$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*
(6) *$D_1\cap D_3$ is a $\mathbb{P}^2\times\mathbb{P}^{2n-5}$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*
(7) *$D_1\cap D_2\cap D_3$ is a $\mathbb{P}^1\times\mathbb{P}^{2n-5}$-bundle over a $\mathrm{Gr}^\omega(2,2n)$-bundle over $X^{[n]}$.*

For instance, (1) is just Proposition 6.2 of [Ogr99] and (2) is Proposition 3.3.2 of [Ogr97], while (3) is Lemma 3.5.4 in [Ogr97].

From Proposition 3.2 and Lemma 3.1, we have the following by the additive and multiplicative properties of the (virtual) Euler numbers:

$$\begin{aligned}
e(D_1^0)&=0, & e(D_2^0)&=(2n-3)(2n-2)\tfrac{1}{2}(a_n^2-a_n),\\
e(D_3^0)&=2^2\binom{n}{2}a_n, & e(D_{12}^0)&=3\cdot 2^3\binom{n}{3}a_n\\
e(D_{23}^0)&=2\cdot 2^2\binom{n}{2}a_n, & e(D_{13}^0)&=(2n-4)2^2\binom{n}{2}a_n\\
e(D_{123}^0)&=2(2n-4)2^2\binom{n}{2}a_n.
\end{aligned}$$

Hence from the formula (3.1), the stringy Euler number of M_{2n} is given by

$$e_{\mathrm{st}}(M_{2n})=e(M_{2n}^s)+(n-1)(a_n^2-a_n)+n\frac{2n-2}{2n-3}a_n=\frac{n\,a_n}{2n-3}+\text{integer}$$

since $e(M_{2n}^s)$ is an integer. So we have proved Proposition 1.3.

Remark 3.3. For the moduli space of rank 2 bundles over a smooth projective curve, the stringy E-function and the stringy Euler number are computed in [Kie03] and [KL04b].

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REFERENCES

Bat98 V. Batyrev, *Stringy Hodge numbers of varieties with Gorenstein canonical singularities*, in *Integrable systems and algebraic geometry* (Kobe/Kyoto, 1997) (World Scientific, River Edge, NJ, 1998), 1–32.

DL99 J. Denef and F. Loeser, *Germs of arcs on singular varieties and motivic integration*, Invent. Math. **135** (1999), 201–232.

Kie03 Y.-H. Kiem, *The stringy E-function of the moduli space of rank* 2 *bundles over a Riemann surface of genus* 3, Trans. Amer. Math. Soc. **355** (2003), 1843–1856.

Kir85 F. Kirwan, *Partial desingularisations of quotients of nonsingular varieties and their Betti numbers*, Ann. of Math. (2) **122** (1985), 41–85.

KL04a D. Kaledin and M. Lehn, *Local structure of hyperkaehler singularities in O'Grady's examples*, Preprint (2004), math.AG/0405575.

KL04b Y.-H. Kiem and J. Li, *Desingularizations of the moduli space of rank* 2 *bundles over a curve*, Math. Ann. **330** (2004), 491–518.

Ogr97 K. G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, Preprint (1997), alg-geom/9708009.

Ogr99 K. G. O'Grady, *Desingularized moduli spaces of sheaves on a K3*, J. reine angew. Math. **512** (1999), 49–117.

VW94 C. Vafa and E. Witten, *A strong coupling test of S-duality*, Nucl. Phys. B **431** (1994), 3–77.

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