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On the summability $|N, p_n|$ of a Fourier series at a point

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1. *Introduction.* 1·1. Let Σa_n be a given infinite series and $\{s_n\}$ the sequence of its partial sums. Let $\{p_n\}$ be a sequence of constants, real or complex, and let us write

$$P_n = p_0 + p_1 + \dots + p_n; \quad p_{-1} = p_{-2} = 0.$$

The sequence-to-sequence transformation

$$t_n = \sum_{\nu=0}^{n} p_{n-\nu} s_\nu / P_n = \sum_{\nu=0}^{n} P_{n-\nu} a_\nu / P_n \quad (P_n \neq 0) \tag{1·1·1}$$

defines the sequence $\{t_n\}$ of Nörlund means (9) of the sequence $\{s_n\}$, generated by the sequence of coefficients $\{p_n\}$. The series Σa_n is said to be summable (N, p_n) to the sum s if $\lim_{n\to\infty} t_n$ exists and is equal to s, and is said to be absolutely summable (N, p_n), or summable $|N, p_n|$, if the sequence $\{t_n\}$ is of bounded variation† (8) that is,

$$\sum_n |t_n - t_{n-1}| \leqslant K.‡$$

In the special case in which

$$p_n = \binom{n+\alpha-1}{\alpha-1} = \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\,\Gamma(\alpha)} \quad (\alpha > -1), \tag{1·1·2}$$

the Nörlund mean reduces to the familiar (C, α) mean ((6), section 4·1).

The regularity conditions for the (N, p_n) method are ((6), section 4·2)

$$\sum_{k=0}^{n} |p_k| \leqslant K|P_n|, \quad p_n/P_n \to 0. \tag{1·1·3}$$

1·2. Let $f(t)$ be a periodic function, with period 2π and integrable in the sense of Lebesgue over $(-\pi, \pi)$. We assume, as we may without any loss of generality, that the constant term in the Fourier series of $f(t)$ is zero, so that

$$\int_{-\pi}^{\pi} f(t)\,dt = 0$$

and

$$f(t) \sim \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt) = \sum_{n=1}^{\infty} A_n(t).$$

† Symbolically, $\{t_n\} \in BV$; similarly by '$f(x) \in BV(h, k)$', we shall mean that $f(x)$ is a function of bounded variation over the interval (h, k) and $\{\mu_n\} \in B$ means that $\{\mu_n\}$ is a bounded sequence.

‡ Throughout this paper K denotes a positive constant, not necessarily the same at each occurrence.

We write throughout:

$$\phi(t) = \tfrac{1}{2}\{f(x+t)+f(x-t)\};$$

$$\phi^*(t) = \{f(x+t)+f(x-t)-2f(x)\};$$

$$\Phi_\alpha(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-u)^{\alpha-1}\phi(u)\,du \quad (\alpha > 0);$$

$$\Phi_0(t) = \phi(t);$$

$$\phi_\alpha(t) = \Gamma(\alpha+1)\,t^{-\alpha}\,\Phi_\alpha(t) \quad (\alpha \geqslant 0) \quad \text{in particular,}$$

$$\Phi_1(t) = \int_0^t \phi(u)\,du, \quad \phi_1(t) = t^{-1}\Phi_1(t);$$

$$\Delta^j p_k = \Delta^{j-1}p_k - \Delta^{j-1}p_{k+1}, \text{ where } j \text{ is some positive integer;}$$

....

$$\Delta p_k = \Delta^1 p_k = p_k - p_{k+1};$$

$$R_n = (n+1)\,p_n/P_n;$$

$$T_n = (1/R_n) = P_n/\{(n+1)\,p_n\};$$

$$S_n = \sum_{\nu=0}^{n} P_\nu(\nu+1)^{-1}/P_n;$$

$$c_n = \sum_{\nu=n}^{\infty} \{P_\nu(\nu+1)\}^{-1};$$

$$\lambda_{n,k}(t) = \{\sin(n-k)\,t\}/(n-k);$$

$\tau = [\pi/t]$, that is, the greatest integer not greater than π/t.

A summability method is said to be K_α-effective ($\alpha > 0$), if it sums the Fourier series of $f(t)$ at every point $t = x$, at which

$$\int_0^t (t-u)^{\alpha-1}\phi^*(u)\,du = o(t^\alpha),$$

as $t \to 0$.

1·3. Concerning the $|C|$ summability of Fourier series, Bosanquet has established the following theorem.

Theorem A (2). *If $\phi_\alpha(t) \in BV(0,\pi)$, then the Fourier series of $f(t)$, at $t = x$, is summable $|C,\beta|$, for every $\beta > \alpha \geqslant 0$.*

In the special case in which $\alpha = 0$, Theorem A becomes a particular case of the following theorem of Pati.

Theorem B (11). *If $\phi(t) \in BV(0,\pi)$, and $\{p_n\}$ is a positive sequence and $\{R_n\} \in BV$ and $\{S_n\} \in BV$, then the Fourier series of $f(t)$, at $t = x$, is summable $|N,p_n|$.*†

Regarding the Nörlund summability of Fourier series, Astrachan proved the following theorem.

† Formerly Varshney (13), had obtained Theorem B, assuming $P_n c_n = O(1)$, instead of $\{S_n\} \in BV$. In (11), Pati has established the equivalence of these two results and has also proved Theorem B as such. An alternative proof of Varshney's theorem is contained in (12). The case '$\{p_n\}$ is monotonic non-decreasing' of Theorem B has been proved independently by a new and shorter technique by the present author in (3), while the case '$\{p_n\}$ is non-increasing' is disposed of by a new and shorter technique in (5) by the author.

THEOREM C (cf. (1), Theorem I). *A regular method (N,p_n) is K_α-effective* $(0 < \alpha \leqslant 1)$, *if the generating sequence $\{p_n\}$ satisfies the following conditions*:

$$n|p_n|/|P_n| \leqslant K, \tag{1·3·1}$$

$$\sum_{k=1}^{n} k|\Delta p_{k-1}|/|P_n| \leqslant K, \tag{1·3·2}$$

$$\sum_{k=1}^{n} k(n-k)|\Delta^2 p_{k-2}|/|P_n| \leqslant K, \tag{1·3·3}$$

$$n\sum_{k=1}^{n} k^{-2}|P_k|/|P_n| \leqslant K, \tag{1·3·4}$$

where $p_{-1} = p_{-2} = 0$.

Theorem C implies *inter alia* that a regular (N,p_n) method sums the Fourier series of $f(t)$ to $f(x)$, at every point $t = x$, at which

$$\lim_{t\to 0}\phi_\alpha(t) = f(x) \quad (0 < \alpha \leqslant 1),$$

provided the generating sequence $\{p_n\}$ satisfies the conditions (1·3·1)–(1·3·4).

It is well known that generally bounded variation is the property associated with absolute summability in the same sense in which continuity is the property associated with ordinary summability. Naturally, therefore, it is expected that corresponding to Theorem C of Astrachan one might get a theorem for the $|N,p_n|$ summability of Fourier series under the hypothesis: $\phi_\alpha(t)\in BV(0,\pi), 0 < \alpha \leqslant 1$. The object of our Theorem 1 is to prove such a theorem. Further, replacing the hypothesis (2·1·3) and (2·1·4) of Theorem 1 by the hypothesis $\{p_n\}$ is non-decreasing, $\{p_{n+1}-p_n\}$ is non-increasing and (2·1·5), we have obtained Theorem 2 which covers Theorem A when $\alpha = 1$, whereas Theorem 1 covers the case $\alpha = 1, \beta \geqslant 2$.

2. *The main results.* 2·1. We establish the following theorems.

THEOREM 1. *If $\phi_1(t)\in BV(0,\pi)$ and $\{p_n\}$ is a positive sequence such that*

$$\{R_n\}\in BV, \tag{2·1·1}$$

$$\{S_n\}\in BV, \tag{2·1·2}$$

$$\sum_{k=0}^{n} |\Delta p_{k-1}| = O(P_n/n) \tag{2·1·3}$$

and

$$\sum_{k=0}^{n} |\Delta^2 p_{k-2}| = O(P_n/n^2), \tag{2·1·4}$$

then the Fourier series of $f(t)$, at $t = x$, is summable $|N,p_n|$.

THEOREM 2. *If $\phi_1(t)\in BV(0,\pi)$ and $\{p_n\}$ is a positive monotonic non-decreasing sequence such that $\{p_{n+1}-p_n\}$ is non-increasing, $\{R_n\}\in BV$ and*

$$\sum_{k=n+1}^{\infty} \frac{1}{P_k} \leqslant K\,(n/P_n), \tag{2·1·5}$$

then the Fourier series of $f(t)$, at $t = x$, is summable $|N,p_n|$.

Remarks. It may be remarked that an equivalent form of our Theorem 1 is one where the hypothesis $\{S_n\} \in BV$ is replaced by $\{S_n\} \in B$. This follows when one refers to a recent paper by the present author (4), where it has been pointed out that the hypothesis $\{S_n\} \in BV$ is actually equivalent to the apparently lighter hypothesis $\{S_n\} \in B$, whenever $\{R_n\} \in BV$ and $\{p_n\}$ is a non-negative sequence. And as shown in (3), $\{R_n\} \in BV$ implies $\{S_n\} \in BV$, whenever $\{p_n\}$ is a positive non-decreasing sequence. It may also be observed here that our condition (2·1·4) is equivalent to the condition

$$\sum_{k=0}^{n} (n-k)|\Delta^2 p_{k-2}| = O(P_n/n), \tag{2·1·6}$$

if $\{p_n\}$ is non-negative and $\{R_n\} \in B$. That (2·1·4) implies (2·1·6) is obvious for any $\{p_n\}$. And that if $\{p_n\}$ is a non-negative sequence and $\{R_n\} \in B$, (2·1·6) implies (2·1·4) is apparent from the following reasoning, kindly suggested by the referee. If (2·1·6) holds, then

$$Kn^{-1} P_{2n} \geqslant \sum_{k=0}^{2n} (2n-k)|\Delta^2 p_{k-2}| \geqslant \sum_{k=0}^{n} (2n-k)|\Delta^2 p_{k-2}| \geqslant n \sum_{k=0}^{n} |\Delta^2 p_{k-2}|$$

and therefore

$$\sum_{k=0}^{n} |\Delta^2 p_{k-2}| = O(P_{2n}/n^2).$$

Now if $\{R_n\} \in B$ and $\{p_n\}$ is a non-negative sequence, then

$$\log(P_{2n}/P_n) = \sum_{k=n}^{2n-1} (\log P_{k+1} - \log P_k)$$

$$= \sum_{k=n}^{2n-1} \log\left(1 + \frac{p_{k+1}}{P_k}\right) \leqslant \sum_{k=n}^{2n-1} \frac{p_{k+1}}{P_k} \leqslant K \sum_{k=n}^{2n-1} \frac{1}{k+1} \leqslant K,$$

since by Lemma 6, $P_{k+1}/P_k = O(1)$. Thus $P_{2n}/P_n = O(1)$ and the hypothesis (2·1·4) follows.

The condition (2·1·6) would suggest itself in the context of Lemma 8·1 of Astrachan (1), wherein he erroneously infers the truth of (2·1·6) from some of the hypotheses of Theorem C. Thus in Astrachan's Theorem C the condition (2·1·6) needs to be explicitly stated.

2·2. We require the following lemmas for the proof of our theorems.

Lemma 1. *If $\{p_n\}$ is a positive sequence, then uniformly in $0 < t \leqslant \pi$,*

$$\left|\sum_{k=0}^{\nu} P_k \cos(n-k)t\right| \leqslant Kt^{-1} P_\nu.$$

Proof. Since in this case P_n is monotonic increasing, by Abel's Lemma, we have

$$\left|\sum_{k=0}^{\nu} P_k \cos(n-k)t\right| \leqslant P_\nu \max_{0 \leqslant \mu \leqslant \nu} \left|\sum_{k=\mu}^{\nu} \cos(n-k)t\right| \leqslant Kt^{-1} P_\nu.$$

Lemma 2. *If $\{p_n\}$ is a positive sequence, then uniformly in $0 < t \leqslant \pi$,*

$$\left|\sum_{k=0}^{\nu} P_k \lambda_{n,k}(t)\right| \leqslant KP_\nu,$$

for $0 \leqslant \nu < n$.

Proof. Applying Abel's Lemma as in the proof of Lemma 1 and observing that $\left|\sum_{k=\mu}^{\nu} \lambda_{n,k}(t)\right| \leqslant K$, we get the result of Lemma 2.

LEMMA 3. *Let* $x \neq 1$, *then*

$$\sum_{k=0}^{n} a_k x^k = (1-x)^{-2}\left[\sum_{k=0}^{n} \Delta^2 a_{k-2} x^k + \Delta a_{n-1} x^{n+1} - a_n x^{n+1}(1-x)\right],$$

where $a_{-1} = a_{-2} = 0$.

Proof. Let us write

$$\sum_{k=0}^{n} a_k x^k = (1-x)^{-2}\left[\sum_{k=0}^{n} a_k x^k - 2\sum_{k=0}^{n} a_k x^{k+1} + \sum_{k=0}^{n} a_k x^{k+2}\right]$$

$$= (1-x)^{-2}\left[\sum_{k=0}^{n} a_k x^k - 2\sum_{k=1}^{n+1} a_{k-1} x^k + \sum_{k=2}^{n+2} a_{k-2} x^k\right]$$

$$= (1-x)^{-2}\left[\sum_{k=0}^{n} \Delta^2 a_{k-2} x^k + \Delta a_{n-1} x^{n+1} - a_n x^{n+1}(1-x)\right],$$

since $\Delta^2 a_{k-2} = a_{k-2} - 2a_{k-1} + a_k$ and $a_{-1} = a_{-2} = 0$.

LEMMA 4. *If* $\{p_n\}$ *satisfies the conditions* (2·1·3) *and* (2·1·4) *and* $\{R_n\} \in B$, *then uniformly in* $0 < t \leqslant \pi$.

$$\left|\sum_{k=0}^{n} p_k(n-k)\, e^{ikt}\right| \leqslant Kt^{-2} P_n n^{-1}.$$

Proof (cf. (1) Lemma 9·2). Let us write

$$\mu_k = (n-k)\,p_k, \quad \mu_{-1} = \mu_{-2} = 0.$$

Then, for $0 \leqslant k \leqslant n$,

$$\Delta\mu_{k-1} = \mu_{k-1} - \mu_k = (n-k)\,\Delta p_{k-1} + p_{k-1},$$

$$\Delta^2\mu_{k-2} = \Delta\mu_{k-2} - \Delta\mu_{k-1} = (n-k)\,\Delta^2 p_{k-2} + 2\Delta p_{k-2}.$$

Also
$$\mu_n = 0, \quad \Delta\mu_{n-1} = p_{n-1}.$$

Now, by Lemma, 3, we have

$$\sum_{k=0}^{n} \mu_k e^{ikt} = (1-e^{it})^{-2}\left[\sum_{k=0}^{n} \Delta^2\mu_{k-2} e^{ikt} + \Delta\mu_{n-1} e^{i(n+1)t} - \mu_n e^{i(n+1)t}(1-e^{it})\right].$$

Thus, rewriting $\mu_k = p_k(n-k)$,

$$\left|\sum_{k=0}^{n} p_k(n-k)\, e^{ikt}\right| = |(1-e^{it})|^{-2}\left|\left\{\sum_{k=0}^{n} (n-k)\,(\Delta^2 p_{k-2})\, e^{ikt} + 2\sum_{k=0}^{n} \Delta p_{k-2} e^{ikt} + p_{n-1} e^{i(n+1)t}\right\}\right|$$

$$\leqslant Kt^{-2}\left\{n\sum_{k=0}^{n} |\Delta^2 p_{k-2}| + 2\sum_{k=0}^{n-1} |\Delta p_{k-1}| + p_{n-1}\right\}$$

$$\leqslant Kt^{-2} P_n n^{-1},$$

using the hypotheses (2·1·3), (2·1·4) and the fact that $\{R_n\} \in B$.

LEMMA 5 (11). *If* $\{p_n\}$ *is a positive sequence such that* $\{R_n\} \in BV$, *then the assertion*: $\{S_n\} \in BV$ *is equivalent to*: $P_n c_n = O(1)$, $n = 0, 1, 2, \ldots$.

LEMMA 6 (11). *If* $\{p_n\}$ *is a positive sequence such that* $\{R_n\} \in B$, *then* $P_n/P_{n-1} = O(1)$, *as* $n \to \infty$.

LEMMA 7. *If* $\{q_n\}$ *is non-negative and non-increasing, then for* $0 \leqslant a \leqslant b \leqslant \infty$ *and* $0 \leqslant t \leqslant \pi$,

$$\left|\sum_{k=a}^{b} q_k e^{ikt}\right| \leqslant KQ_\tau,$$

where $\tau = [1/t]$ *and* $Q_m = q_0 + q_1 + \ldots + q_m$.

This Lemma may be proved by following the technique of proof of Lemma 5·11 in McFadden(7).

LEMMA 8. *If* $\{p_n\}$ *is a positive, monotonic non-decreasing sequence such that* $\{R_n\} \in BV$, *then uniformly in* $0 < t \leqslant \pi$

$$\sum_n \frac{1}{P_n P_{n-1}} \left|\sum_{k=0}^{n-1} (P_n p_k - p_n P_k) \frac{\sin(n-k)t}{(n-k)}\right| \leqslant K.$$

Proof. If $\{p_n\}$ is positive and monotonic non-decreasing, then $(n+1)p_n \geqslant P_n$ and therefore $\{T_n\} \in B$. Hence $\{R_n\} \in BV$ implies $\{T_n\} \in BV$ and the result of Lemma 8 follows from ((3), proof of $\Sigma \leqslant K$ on p. 815).

3. *Proof of the main results.* 3·1. *Proof of Theorem* 1. We have

$$t_n - t_{n-1} = \sum_{\nu=0}^{n-1} \left(\frac{P_\nu}{P_n} - \frac{P_{\nu-1}}{P_{n-1}}\right) a_{n-\nu}$$
$$= \frac{1}{P_n P_{n-1}} \sum_{\nu=0}^{n-1} (P_n p_\nu - p_n P_\nu) a_{n-\nu}.$$

For the Fourier series of $f(t)$,

$$A_n(x) = \frac{2}{\pi}\int_0^\pi \phi(t)\cos nt\, dt,$$

so that
$$t_n - t_{n-1} = \frac{2}{\pi}\int_0^\pi \phi(t)\left\{\frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k)\cos(n-k)t\right\} dt.$$

Thus, in order to prove our theorem we have to show that

$$\sum_n \left|\int_0^\pi \phi(t)\, g(n,t)\, dt\right| \leqslant K,$$

uniformly in $0 < t \leqslant \pi$, where

$$g(n,t) = \frac{1}{P_n P_{n-1}} \sum_{k=0}^{n-1} (P_n p_k - p_n P_k)\cos(n-k)t.$$

Integrating by parts, we get

$$\int_0^\pi \phi(t)\, g(n,t)\, dt = \left[\Phi_1(t)\, g(n,t)\right]_0^\pi - \int_0^\pi \Phi_1(t)\left\{\frac{d}{dt} g(n,t)\right\} dt$$
$$= \left[t\phi_1(t)\, g(n,t)\right]_0^\pi - \int_0^\pi \phi_1(t)\left\{t\frac{d}{dt} g(n,t)\right\} dt$$

and $$\int_0^\pi \phi_1(t)\left\{t\frac{d}{dt}g(n,t)\right\}dt$$

$$= \left[\phi_1(t)\int_0^t u\left\{\frac{d}{du}g(n,u)\right\}du\right]_0^\pi - \int_0^\pi \left\{\int_0^t u\left(\frac{d}{du}g(n,u)\right)du\right\}d\phi_1(t).$$

But $$\int_0^t u\left\{\frac{d}{du}g(n,u)\right\}du = [ug(n,u)]_0^t - \int_0^t g(n,u)\,du$$

$$= tg(n,t) - \int_0^t g(n,u)\,du.$$

Thus, we have

$$\int_0^\pi \phi(t)\,g(n,t)\,dt = \left[\phi_1(t)\int_0^t g(n,u)\,du\right]_{t=\pi} + \int_0^\pi \left\{tg(n,t) - \int_0^t g(n,u)\,du\right\}d\phi_1(t).$$

And

$$\sum_n \left|\int_0^\pi \phi(t)\,g(n,t)\,dt\right| \leqslant \left[\phi_1(t)\sum_n\left|\int_0^t g(n,u)\,du\right|\right]_{t=\pi} + \int_0^\pi \{\sum_n |tg(n,t)|\}\,|d\phi_1(t)|$$
$$+ \int_0^\pi \left\{\sum_n \left|\int_0^t g(n,u)\,du\right|\right\}|d\phi_1(t)|.$$

Since by hypothesis, $\int_0^\pi |d\phi_1(t)| \leqslant K$, it suffices for our purpose to show that, uniformly in $0 < t \leqslant \pi$,

$$\sum_n \left|\int_0^t g(n,u)\,du\right| \leqslant K \tag{3·1·1}$$

and $$\sum_n |tg(n,t)| \leqslant K, \tag{3·1·2}$$

which is equivalent to showing that

$$\sum_n \frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k - p_nP_k)\frac{\sin(n-k)t}{(n-k)}\right| \leqslant K \tag{3·1·3}$$

and $$t\sum_n \frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k - p_nP_k)\cos(n-k)t\right| \leqslant K. \tag{3·1·4}$$

We first proceed to establish (3·1·4). We have

$$\begin{aligned}\Sigma &= t\sum_{n=1}^\infty \frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k - p_nP_k)\cos(n-k)t\right| \\
&= t\sum_{n=1}^\infty \frac{1}{P_{n-1}}\left|\sum_{k=0}^{n-1}\left(\frac{p_k}{P_k} - \frac{p_n}{P_n}\right)P_k\cos(n-k)t\right| \\
&\leqslant t\sum_{n=1}^\infty \frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}(R_k - R_n)P_k\cos(n-k)t\right| \\
&\quad + t\sum_{n=1}^\infty \frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}p_k(n-k)\cos(n-k)t\right| \\
&= t\sum_{n=1}^\infty \frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}P_k\cos(n-k)t\sum_{\nu=k}^{n-1}\Delta R_\nu\right| \\
&\quad + t\sum_{n=1}^\infty \frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}p_k(n-k)\cos(n-k)t\right| \\
&= \Sigma_1 + \Sigma_2, \text{ say.}\end{aligned}$$

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By changing the order of summation and applying Lemma 1, we get

$$\begin{aligned}\Sigma_1 &= t\sum_{n=1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}P_k\cos(n-k)t\sum_{\nu=k}^{n-1}\Delta R_\nu\right| \\ &= t\sum_{n=1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left|\sum_{\nu=0}^{n-1}\Delta R_\nu\sum_{k=0}^{\nu}P_k\cos(n-k)t\right| \\ &\leqslant K\sum_{n=1}^{\infty}\frac{1}{(n+1)P_{n-1}}\sum_{\nu=0}^{n-1}|\Delta R_\nu|\,P_\nu \\ &= K\sum_{\nu=0}^{\infty}|\Delta R_\nu|\,P_\nu\sum_{n=\nu+1}^{\infty}\frac{1}{(n+1)P_{n-1}} \\ &\leqslant K\sum_{\nu=0}^{\infty}|\Delta R_\nu|\,P_\nu\sum_{n=\nu+1}^{\infty}\frac{1}{(n+1)P_n}, \quad \text{by Lemma 6,} \\ &\leqslant K\sum_{\nu=0}^{\infty}|\Delta R_\nu|,\end{aligned}$$

since by Lemma 5, hypotheses (2·1·1) and (2·1·2) imply that $c_nP_n = O(1)$. Thus,

$$\Sigma_1 \leqslant K, \tag{3·1·5}$$

by virtue of the hypothesis (2·1·1).

Now we observe that

$$\begin{aligned}\Sigma_2 &\leqslant t\sum_{n=1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}p_k(n-k)\,e^{i(k-n)t}\right| \\ &= t\sum_{n=1}^{\tau}\frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}p_k(n-k)\,e^{i(k-n)t}\right| + t\sum_{n=\tau+1}^{\infty}\frac{|e^{-int}|}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}p_k(n-k)\,e^{ikt}\right| \\ &= \Sigma_{21}+\Sigma_{22}, \text{ say.}\end{aligned}$$

But
$$\Sigma_{21} \leqslant Kt\sum_{n=1}^{\tau}\frac{n}{(n+1)P_{n-1}}\sum_{k=0}^{n-1}p_k \leqslant Kt\sum_{n=1}^{\tau}1 \leqslant K.$$

Now
$$\begin{aligned}\Sigma_{22} &\leqslant t\sum_{n=\tau+1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}p_k(n-k)\,e^{ikt}\right| \\ &\leqslant Kt^{-1}\sum_{n=\tau+1}^{\infty}\frac{1}{n(n+1)}\frac{P_n}{P_{n-1}}, \quad \text{by Lemma 4,} \\ &\leqslant K\tau\sum_{n=\tau+1}^{\infty}\frac{1}{n(n+1)} \leqslant K,\end{aligned}$$

since by Lemma 6, $P_n/P_{n-1} = O(1)$.

Thus we have

$$\Sigma_2 \leqslant K. \tag{3·1·6}$$

Following the technique of proof for $\Sigma \leqslant K$, we write

$$\begin{aligned}\Sigma' &= \sum_n\frac{1}{P_nP_{n-1}}\left|\sum_{k=0}^{n-1}(P_np_k-p_nP_k)\frac{\sin(n-k)t}{n-k}\right| \\ &\leqslant \sum_{n=1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}P_k\frac{\sin(n-k)t}{n-k}\sum_{\nu=k}^{n-1}\Delta R_\nu\right| \\ &\quad + \sum_{n=1}^{\infty}\frac{1}{(n+1)P_{n-1}}\left|\sum_{k=0}^{n-1}p_k\sin(n-k)t\right| \\ &= \Sigma_1'+\Sigma_2', \text{ say.}\end{aligned}$$

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By changing the order of summation and applying Lemma 2, we get

$$\Sigma_1' = \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{\nu=0}^{n-1} \Delta R_\nu \sum_{k=0}^{\nu} P_k \frac{\sin(n-k)t}{n-k} \right|$$
$$\leqslant K \sum_{n=1}^{\infty} \frac{1}{(n+1)P_{n-1}} \sum_{\nu=0}^{n-1} |\Delta R_\nu| P_\nu \leqslant K,$$

as in the proof of $\Sigma_1 \leqslant K$.

Next, we write

$$\Sigma_2' \leqslant \sum_{n=1}^{\tau} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=0}^{n-1} p_k \sin(n-k)t \right|$$
$$+ \sum_{n=\tau+1}^{\infty} \frac{1}{(n+1)P_{n-1}} \left| \sum_{k=0}^{n-1} p_k \sin(n-k)t \right|$$
$$= \Sigma_{21}' + \Sigma_{22}', \quad \text{say.}$$

Now, $|\sin(n-k)t| \leqslant nt$, for $0 \leqslant k < n$, therefore

$$\Sigma_{21}' \leqslant t \sum_{n=1}^{\tau} \frac{1}{P_{n-1}} \sum_{k=0}^{n-1} p_k = t \sum_{n=1}^{\tau} 1 \leqslant K.$$

We write by Abel's transformation

$$\left| \sum_{k=0}^{n-1} p_k \sin(n-k)t \right| \leqslant \sum_{k=0}^{n-2} |\Delta p_k| \left| \sum_{\nu=0}^{k} \sin(n-\nu)t \right| + p_{n-1} \left| \sum_{\nu=0}^{n-1} \sin(n-\nu)t \right|$$
$$\leqslant K\tau \sum_{k=0}^{n-2} |\Delta p_k| + K\tau p_{n-1}$$
$$\leqslant K\tau \frac{P_n}{n} + K\tau R_{n-1} \frac{P_{n-1}}{n}$$
$$\leqslant K\tau \frac{P_n}{n},$$

by the hypotheses (2·1·3) and (2·1·1).

Thus

$$\Sigma_{22}' \leqslant K\tau \sum_{n=\tau+1}^{\infty} \frac{1}{(n+1)P_{n-1}} \frac{P_n}{n}$$
$$\leqslant K\tau \sum_{n=\tau+1}^{\infty} \frac{1}{n(n+1)} \leqslant K,$$

since by Lemma 6, $P_n/P_{n-1} = O(1)$.

We have, therefore

$$\Sigma' \leqslant K \quad \text{(see also (10) and (11))}. \tag{3·1·7}$$

Combining (3·1·5), (3·1·6) and (3·1·7), we get (3·1·3) and (3·1·4), and this completes the proof of Theorem 1.

3·2. *Proof of Theorem* 2. As in the proof of Theorem 1, in order to prove Theorem 2, we have to show that uniformly in $0 < t \leqslant \pi$, (3·1·3) and (3·1·4) hold under the hypotheses of the theorem.

Now (3·1·3) follows directly from Lemma 8.

Next we consider (3·1·4). Following the technique of proof used for showing that $\Sigma \leqslant K$, in section 3·1, we observe that uniformly in $0 < t \leqslant \pi$, $\Sigma_1 \leqslant K$, by virtue of the hypotheses: $\{R_n\} \in BV$ and

$$\sum_{k=n+1}^{\infty} \frac{1}{P_k} \leqslant K \frac{n}{P_n},$$

the latter of which implies $P_n c_n = O(1)$, since $\{p_n\}$ is a positive sequence.

In order to prove that $\Sigma_2 \leqslant K$, we write by Abel's transformation

$$\begin{aligned}\sum_{k=0}^{n} p_k(n-k)\, e^{ikt} &= \sum_{k=0}^{n-1} \Delta\{p_k(n-k)\} \sum_{\nu=0}^{k} e^{i\nu t} \\ &= (1-e^{it})^{-1}\left[\sum_{k=0}^{n-1} \Delta\{p_k(n-k)\} - \sum_{k=0}^{n-1} \Delta\{p_k(n-k)\}\, e^{i(k+1)t}\right] \\ &= (1-e^{it})^{-1}\left[np_0 - \sum_{k=0}^{n-1} (n-k)\,(\Delta p_k)\, e^{i(k+1)t} - \sum_{k=0}^{n-1} p_{k+1}\, e^{i(k+1)t}\right] \\ &= (1-e^{it})^{-1}\left[np_0 - \sum_{k=0}^{n-1} \sum_{\nu=0}^{k} (\Delta p_\nu)\, e^{i(\nu+1)t} - \sum_{k=0}^{n-1} p_{k+1}\, e^{i(k+1)t}\right],\end{aligned}$$

Thus

$$\begin{aligned}&\left|\sum_{k=0}^{n-1} p_k(n-k)\cos(n-k)t\right| \\ &\leqslant \left|\sum_{k=0}^{n-1} p_k(n-k)\, e^{i(k-n)t}\right| \\ &\leqslant \frac{|e^{-int}|}{|1-e^{it}|}\left[np_0 + \sum_{k=0}^{n-1}\left|\sum_{\nu=0}^{k} (p_{\nu+1}-p_\nu)\, e^{i(\nu+1)t}\right| + \left|\sum_{k=0}^{n-1} p_{k+1} e^{i(k+1)t}\right|\right] \\ &\leqslant K\tau\left[np_0 + \sum_{k=0}^{n-1} K \sum_{\nu=0}^{\tau} (p_{\nu+1}-p_\nu) + p_n \max_{1\leqslant\nu\leqslant n}\left|\sum_{k=1}^{\nu} e^{ikt}\right|\right],\end{aligned}$$

(by Lemma 7 and Abel's Lemma, since $\{p_{\nu+1}-p_\nu\}$ is non-negative, non-increasing and $\{p_n\}$ is non-decreasing)

$$\begin{aligned}&\leqslant K\tau[np_0 + Knp_{\tau+1} + Kp_n\tau] \\ &\leqslant Kn\tau p_{\tau+1} + K\tau^2 p_n,\end{aligned}$$

since $\{p_n\}$ is non-decreasing.

Thus we have finally

$$\begin{aligned}\Sigma_2 &= t\sum_{n=1}^{\tau} \frac{1}{(n+1)\,P_{n-1}}\left|\sum_{k=0}^{n-1} p_k\,(n-k)\cos(n-k)t\right| \\ &\quad + t\sum_{n=\tau+1}^{\infty} \frac{1}{(n+1)\,P_{n-1}}\left|\sum_{k=0}^{n-1} p_k\,(n-k)\cos(n-k)t\right| \\ &\leqslant t\sum_{n=1}^{\tau} \frac{1}{P_{n-1}} \sum_{k=0}^{n-1} p_k + K\tau \sum_{n=\tau+1}^{\infty} \frac{p_n}{(n+1)\,P_{n-1}} + Kp_{\tau+1} \sum_{n=\tau+1}^{\infty} \frac{1}{P_{n-1}} \\ &\leqslant t\sum_{n=1}^{\tau} 1 + K\tau \sum_{n=\tau+1}^{\infty} \frac{R_n}{(n+1)^2}\frac{P_n}{P_{n-1}} + Kp_{\tau+1}\frac{\tau+1}{P_{\tau+1}} \\ &\leqslant K + K\tau \sum_{n=\tau+1}^{\infty} \frac{1}{(n+1)^2} + KR_{\tau+1} \\ &\leqslant K,\end{aligned}$$

by the hypotheses $\{R_n\} \in B$ and (2·1·5) and Lemma 6.

This completes the proof of Theorem 2.

The analogues of Theorem 1 and Theorem 2, for the conjugate series and the derived series of the Fourier series have been obtained very recently by the present author.

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