THE EXPECTATION OF LIFE AND ITS RELATIONSHIP TO MORTALITY

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SUMMARY

The complete expectation of life at birth $\mathring{e}_0$ is frequently used as a measure of the level of mortality of a population. It is also used for assessing trends in mortality and trends in mortality differentials. Although the relationship between mortality and expectation of life is essentially reciprocal, the exact connexion is rather more complicated, and becomes important when, for example, trends in differentials are analysed.

In this paper, the relationship between mortality and expectation of life is explored in some detail, and formulae are developed for analysing the effects of mortality changes on expectation of life, and trends in mortality differentials on $\mathring{e}_0$ differentials.

Unlike Keyfitz (1977), who concentrates on the proportional change in $\mathring{e}_0$ corresponding to equal proportional changes in mortality at all ages, we study the relationship between absolute changes in mortality, generally different at different ages, and the corresponding absolute change in $\mathring{e}_0$.

It is demonstrated that two populations may experience diminishing mortality differentials and at the same time widening $\mathring{e}_0$ differentials. Numerical examples are given using Australian data over the periods 1921–71 and 1971–79.

Although the formulae in the paper relate solely to the expectation of life at birth, the methods and formulae are readily adapted to expectations of life at other ages and indeed, temporary expectations.

1. INTRODUCTORY MATHEMATICS

An improvement of ϕ in the force of mortality in the small age-range $(x, x+\Delta x)$ causes the expectation of life at birth in the population under consideration to increase by an amount

$$ {}_xp_0\, \mathring{e}_x\, \phi\, \Delta x, \tag{1}$$

assuming that there are no changes in mortality at other ages. This formula is well-known and leads to the following approximate formula for the gain in expectation of life at birth in a population between time 1 and time 2.

$$\mathring{e}_0^2 - \mathring{e}_0^1 \doteqdot \int_0^\infty (\mu_x^1 - \mu_x^2)\, {}_xp_0^1\, \mathring{e}_x^1\, \mathrm{d}x. \tag{2}$$

In this, and other formulae in this paper, a superscript 1 will indicate that the

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function concerned is evaluated at time 1 and a superscript 2 will indicate that the function is evaluated at time 2.

Formula (2) is reasonably accurate, provided the improvements in mortality are modest. It always underestimates the gain in expectation of life when mortality improvements are positive. The reason the formula is only approximate and underestimates the actual gain in expectation of life when positive improvements in mortality take place at all or most ages is that it ignores interaction effects between mortality improvements at the different ages.

An exact formula, explaining the gain in expectation of life in terms of mortality improvements at the individual ages can also be derived, and from this formula we can separate the main effects (formula (2)) and the interactions of various orders. We begin by defining

$$M_x = \int_0^x \mu_t \, dt = -\ln {}_xp_0. \tag{3}$$

It is then simple to show that

$$\mathring{e}_0^2 - \mathring{e}_0^1 = \int_0^\infty \{\exp(M_x^1 - M_x^2) - 1\} \, {}_xp_0^1 \, dx. \tag{4}$$

Noting that ${}_xp_0^1$ is the derivative with respect to x of $-{}_xp_0^1 \, \mathring{e}_x^1$, we may integrate (4) by parts to obtain

$$\mathring{e}_0^2 - \mathring{e}_0^1 = \int_0^\infty (\mu_x^1 - \mu_x^2) \exp(M_x^1 - M_x^2) \, {}_xp_0^1 \, \mathring{e}_x^1 \, dx. \tag{5}$$

When the exponential term in (5) is expanded in terms of the powers of $M_x^1 - M_x^2$, we see that the main effects on the gain in expectation of life of the mortality improvements, are given by equation (2), the first-order interaction terms by

$$\int_0^\infty (M_x^1 - M_x^2)(\mu_x^1 - \mu_x^2) \, {}_xp_0^1 \, \mathring{e}_x^1 \, dx, \tag{6}$$

and the j^{th}-order interactions by

$$\frac{1}{j!} \int_0^\infty (M_x^1 - M_x^2)^j (\mu_x^1 - \mu_x^2) \, {}_xp_0^1 \, \mathring{e}_x^1 \, dx. \tag{7}$$

Over the period 1921–71, Australian females enjoyed a gain in expectation of life at birth of some 11·29 years. The main effects of the mortality improvements over the 50-year period on the expectation of life at birth and the effects of

Table 1. *Gain in expectation of life of Australian females, 1921–71: main effects and interactions*

	Contribution to $\mathring{e}_0$ gain	% of gain
Main effects	9·8636	87·4
First-order interaction	1·2307	10·9
Second-order interaction	·1730	1·5
Higher-order interactions	·0227	·2
Total	11·2900	100·0

interactions between these mortality improvements are summarized in Table 1.* We see that even in this rather extreme example, almost 90% of the gain in expectation of life at birth is attributable to the main effects of mortality improvements. Only 12·6% is due to the various interactions.

2. MORTALITY IMPROVEMENTS IN INDIVIDUAL AGE-GROUPS

The change in expectation of life at birth between time 1 and time 2 can also be expressed in either of the following two forms:

$$\mathring{e}_0^2 - \mathring{e}_0^1 = \int_0^\infty (\mu_x^1 - \mu_x^2)\, {}_xp_0^2\, \mathring{e}_x^1\, dx; \tag{8}$$

$$\mathring{e}_0^2 - \mathring{e}_0^1 = \int_0^\infty (\mu_x^1 - \mu_x^2)\, {}_xp_0^1\, \mathring{e}_x^2\, dx. \tag{9}$$

Formula (8) may be obtained directly from (5), and (9) deduced from (8) by interchanging the superscripts 1 and 2.

Both these equations are exact, and represent weighted averages of the mortality improvements at the individual ages over the period between time 1 and time 2. When mortality is improving, the weights in (8) and (9) both exceed the weight in (1), and have the effect of adding the interaction terms in with the main effects of mortality change.

Interaction terms tend to be difficult to interpret, and they are certainly not easy to explain to the layman. We saw in section 1 that even in quite extreme cases, the interaction effects of mortality improvements at different ages are

* The numerical calculations may be performed as follows:

$${}_nQ_x = \int_0^n \mu_{x+t}\, dt = -\ln(l_{x+n}/l_x).$$

Then the j^{th}-order interaction term (7) is evaluated as

$$\frac{1}{j!}\{(M_{\frac12}^1 - M_{\frac12}^2)^j ({}_1Q_0^1 - {}_1Q_0^2)\, {}_{\frac12}p_0^1\, \mathring{e}_{\frac12}^1 + (M_3^1 - M_3^2)^j ({}_4Q_1^1 - {}_4Q_1^2)\, {}_3p_0^1\, \mathring{e}_3^1 +$$
$$(M_{7\frac12}^1 - M_{7\frac12}^2)^j ({}_5Q_5^1 - {}_5Q_5^2)\, {}_{7\frac12}p_0^1\, \mathring{e}_{7\frac12}^1 +$$
$$(M_{12\frac12}^1 - M_{12\frac12}^2)^j ({}_5Q_{10}^1 - {}_5Q_{10}^2)\, {}_{12\frac12}p_0^1\, \mathring{e}_{12\frac12}^1 + \ldots\}.$$

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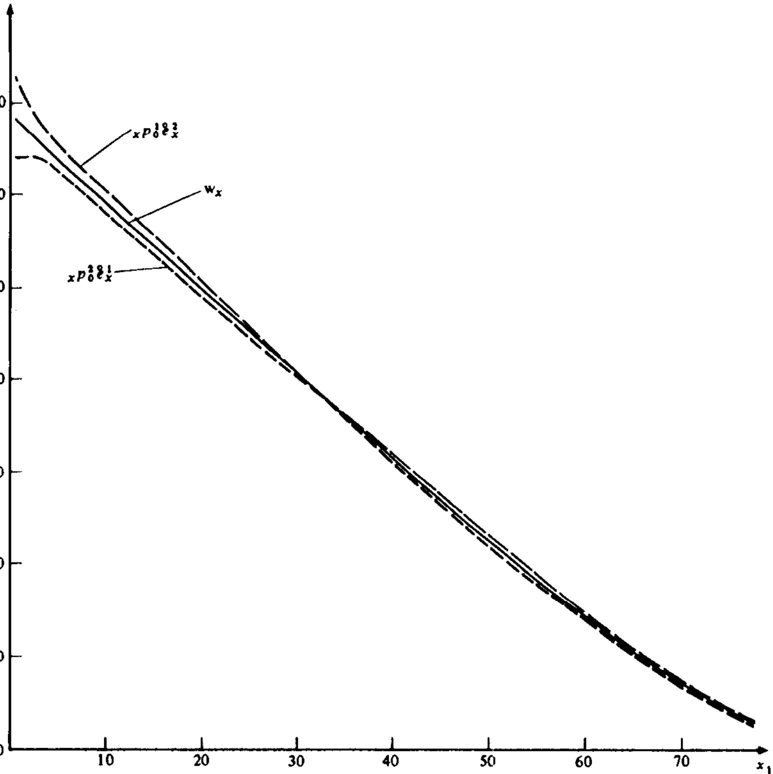

Figure 1. Alternative weights: Australian females, 1921 (time 1) to 1971 (time 2).

relatively minor. By merging the relatively minor interaction terms with the main effects, formulae (8) and (9) allow a simple, yet not unrealistic, analysis of $\mathring{e}_0$ gains in terms of mortality improvements by age.

Depending upon whether (8) or (9) is adopted as the basis of the analysis, the weight at age x is $_xp_0^2\,\mathring{e}_x^1$ at each age x or $_xp_0^1\,\mathring{e}_x^2$ at each age x. These weights are of comparable magnitude and approximate a straight line over the main part of the life-span (Fig. 1). There appears to be no theoretical reason to prefer one to the other, and in this paper we shall use as the weight their simple arithmetic mean. The relationship between the expectation of life at birth and the corresponding changes in mortality then becomes

$$\mathring{e}_0^2-\mathring{e}_0^1 = \int_0^\infty (\mu_x^1-\mu_x^2)\, w_x \,dx \tag{10}$$

with

$$w_x = \tfrac{1}{2}({}_xp_0^2\,\mathring{e}_x^1+{}_xp_0^1\,\mathring{e}_x^2). \tag{11}$$

Table 2. *Improvement in expectation of life at birth—Australian males, 1921–71*

Age Group (1)	Mortality level ${}_nQ_x \times 10^5$ 1921 (2)	1971 (3)	Mortality improvement (2)–(3) (4)	Weight (5)	Contribution to change in $\mathring{e}_0$ (4)×(5)×10⁻⁵ (6)	% Contribution to change in $\mathring{e}_0$ 100×(6)÷8·7500 (7)
0	7,399	1,968	5,431	62·94	3·4183	39
1–4	2,823	399	2,424	60·52	1·4670	17
5–9	995	230	765	56·29	·4306	5
10–14	791	215	576	51·63	·2974	3
15–19	1,120	780	340	46·98	·1597	2
20–24	1,585	936	649	42·38	·2750	3
25–29	1,842	691	1,151	37·84	·4355	5
30–34	2,088	772	1,316	33·36	·4390	5
35–39	2,661	1,106	1,555	28·95	·4502	5
40–44	3,492	1,765	1,727	24·62	·4252	5
45–49	4,871	3,019	1,852	20·41	·3780	4
50–54	6,503	5,089	1,414	16·35	·2312	3
55–59	9,424	8,424	1,000	12·52	·1252	1
60–64	14,399	13,675	724	9·01	·0652	1
65–69	21,355	21,415	−60	5·95	−·0036	−0
70–74	32,876	33,533	−657	3·48	−·0229	−0 }7
75–79	53,904	51,230	2,674	1·71	·0457	1
80–84	85,503	75,935	9,568	·66	·0631	1
85–89	130,761	112,950	17,811	·18	·0321	0
90–99	483,582	349,877	133,705	·01	·0134	0
Error due to approximations in numerical methods					·0247	0
Total					8·7500*	100

* The male expectation of life improved from 59·15 in 1921 to 67·90 in 1971.

The integral in (10) is not generally convenient for numerical purposes. Let us therefore define

$$ {}_nQ_x = \int_0^n \mu_{x+t}\, dt, \tag{12} $$

and note that for numerical evaluation purposes

$$ {}_nQ_x = -\ln (l_{x+n}/l_x). \tag{13} $$

Then

$$ \mathring{e}_0^2 - \mathring{e}_0^1 \doteqdot ({}_1Q_0^1 - {}_1Q_0^2)\, w_{\frac{1}{2}} + ({}_4Q_1^1 - {}_4Q_1^2)\, w_3 + ({}_5Q_5^1 - {}_5Q_5^2)\, w_{7\frac{1}{2}} + ({}_5Q_{10}^1 - {}_5Q_{10}^2)\, w_{12\frac{1}{2}} + \dots \tag{14} $$

and the approximation is very accurate. Indeed, formula (14) allowed the author to detect and correct minor errors‡ in the published Australian male and female expectations of life at birth 1970–72.

‡ The errors were in fact known to the Australian Bureau of Statistics. Understandably, however, errata slips appear not to have reached all holders of copies of these tables.

Formula (14) is used in Table 2 to analyse the contributions mortality improvements in the various age-groups have made to the gain in expectation of life of Australian males over the 50-year time period from 1921 to 1971. The total improvement in expectation of life at birth was 8·75 years, and we see from Table 2 that some 56% of the gain (or 4·88 years) was the result of mortality improvements under age 5, and that mortality improvements at ages 5–50 contributed almost equally to a further 47% of the $\mathring{e}_0$ gain. The older ages contributed almost nothing.

The female analysis (Table 3) presents a striking contrast. Of a total gain in expectation of life at birth of 11·29 years, 38% (or 4·29 years) was the result of mortality improvements under age 5. The remaining 62% was spread almost equally over all the remaining age-groups. Indeed, 24% of the gain (or 2·75 years) was the result of mortality improvements over age 50.

The analyses for the period 1971–79 are shown in Tables 4 and 5. Over this period, both sexes enjoyed a gain in expectation of life at birth of around 3 years. Particularly noteworthy is the large contribution to the male $\mathring{e}_0$ gain of mortality improvements over age 50 (61%). The contrast with the earlier period is striking.

Table 3. *Improvement in expectation of life at birth—Australian females, 1921–71*

Age Group (1)	Mortality level ${}_nQ_x \times 10^5$ 1921 (2)	1971 (3)	Mortality improvement (2)–(3) (4)	Weight (5)	Contribution to change in $\mathring{e}_0$ (4)×(5)×10^{-5} (6)	% Contribution to change in $\mathring{e}_0$ 100×(6)÷11·2900 (7)	
0	5,729	1,512	4,217	68·35	2·8822	26	
1–4	2,448	311	2,137	65·86	1·4073	12	
5–9	909	166	743	61·56	·4574	4	
10–14	616	136	480	56·82	·2727	3	
15–19	938	308	630	52·10	·3282	3	
20–24	1,418	308	1,110	47·40	·5262	5	
25–29	1,769	330	1,439	42·75	·6152	6	
30–34	2,037	456	1,581	38·14	·6030	5	
35–39	2,433	713	1,720	33·60	·5779	5	
40–44	2,745	1,130	1,615	29·13	·4704	4	
45–49	3,403	1,825	1,578	24·74	·3905	3	
50–54	4,599	2,804	1,795	20·48	·3676	3	
55–59	6,409	4,250	2,159	16·35	·3530	3	
60–64	9,399	6,592	2,807	12·44	·3493	3	
65–69	15,349	10,653	4,696	8·84	·4151	4	
70–74	26,041	18,433	7,608	5·68	·4321	4	24 (ages 50–99)
75–79	44,064	31,622	12,442	3·15	·3918	3	
80–84	72,813	54,140	18,673	1·41	·2633	2	
85–89	112,989	86,062	26,927	·47	·1266	1	
90–99	422,470	304,351	118,119	·04	·0472	1	
Error due to approximations in numerical methods					·0130	0	
Total					11·2900*	100	

* The female expectation of life improved from 63·31 in 1921 to 74·60 in 1971.

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Table 4. *Improvement in expectation of life at birth—Australian males, 1971–79*

Age Group (1)	Mortality level $_nQ_x \times 10^5$ 1971 (2)	1979 (3)	Mortality improvement (2)–(3) (4)	Weight (5)	Contribution to change in $\mathring{e}_0$ (4) × (5) × 10^{-5} (6)	% Contribution to change in $\mathring{e}_0$ 100 × (6) ÷ 2·8900 (7)	
0	1,968	1,265	703	68·82	·4838	17	
1–4	399	259	140	66·39	·0929	3	
5–9	230	153	77	61·98	·0477	2	
10–14	215	184	31	57·08	·0177	1	
15–19	780	629	151	52·20	·0788	3	
20–24	936	867	69	47·35	·0327	1	
25–29	691	718	−27	42·53	−·0115	−0	
30–34	772	673	99	37·76	·0374	1	
35–39	1,106	846	260	33·02	·0859	3	
40–44	1,765	1,400	365	28·33	·1034	4	
45–49	3,019	2,471	548	23·71	·1299	4	
50–54	5,089	4,144	945	19·22	·1816	6	
55–59	8,424	6,684	1,740	14·92	·2596	9	
60–64	13,675	10,662	3,013	10·94	·3296	11	
65–69	21,415	17,253	4,162	7·32	·3047	11	
70–74	33,533	27,134	6,399	4·51	·2886	10	61
75–79	51,230	40,998	10,232	2·37	·2425	8	
80–84	75,935	60,773	15,162	1·01	·1531	5	
85–89	112,950	100,418	12,532	·32	·0401	1	
90+	349,877	349,877	0	·03	·0000	0	
Error due to approximations in numerical methods					−·0085	−0	
Total					2·8900*	100	

* The male expectation of life improved from 67·90 in 1971 to 70·79 in 1979.

3. CAUSE OF DEATH

It is natural to inquire as to which causes of death have made the larger contributions to the features observed in Tables 2, 3, 4 and 5. To do this, we substitute the cause-specific forces of mortality into the right-hand side of equation (10). For practical purposes, it is adequate to use as the cause-specific $_nQ_x$ value in (14), the $_nQ_x$ value for all causes multiplied by the proportion of deaths in the age-group from the specific cause.

This approach was used in Tables 6 and 7, which analyse the $\mathring{e}_0$ gains for Australian males and females by age and cause (selected causes) over the time periods 1921–71 and 1971–79 respectively. We note in Table 6 for the period 1921–71:

1. the substantial $\mathring{e}_0$ gains for both sexes resulting from the reduction in infectious-disease mortality;

Table 5. *Improvement in expectation of life at birth—Australian females, 1971–79*

Age Group (1)	Mortality level $_nQ_x \times 10^5$ 1971 (2)	Mortality level $_nQ_x \times 10^5$ 1979 (3)	Mortality improvement (2)–(3) (4)	Weight (5)	Contribution to change in $\mathring{e}_0$ $(4)\times(5)\times10^{-5}$ (6)	% Contribution to change in $\mathring{e}_0$ $100\times(6)\div3\cdot1600$ (7)	
0	1,512	1,011	501	75·66	·3791	12	
1–4	311	220	91	73·21	·0666	2	
5–9	166	123	43	68·78	·0296	1	
10–14	136	99	37	63·87	·0236	1	
15–19	308	220	88	58·96	·0519	2	
20–24	308	290	18	54·05	·0097	0	
25–29	330	277	53	49·17	·0261	1	
30–34	456	314	142	44·30	·0629	2	
35–39	713	483	230	30·45	·0907	3	
40–44	1,130	807	323	34·62	·1118	4	
45–49	1,825	1,352	473	29·85	·1412	4	
50–54	2,804	2,134	670	25·14	·1684	5	
55–59	4,250	3,271	979	20·56	·2013	6	
60–64	6,592	5,164	1,428	16·13	·2303	7	
65–69	10,653	8,207	2,446	11·97	·2928	9	
70–74	18,433	13,459	4,974	8·18	·4069	13	68 (50–90+)
75–79	31,622	22,962	8,660	4·94	·4278	14	
80–84	54,140	40,279	13,861	2·50	·3465	11	
85–89	86,062	75,080	10,982	·97	·1065	3	
90+	304,351	304,351	0	·11	·0000	0	
Error due to approximations in numerical methods					−·0137	−0	
Total					3·1600*	100	

* The female expectation of life improved from 74·60 in 1971 to 77·76 in 1979.

2. the appreciable $\mathring{e}_0$ losses for both sexes, but especially the males, resulting from increased circulatory system mortality;
3. the overall $\mathring{e}_0$ gains for both sexes arising from accident (external cause) mortality, despite the increased accident mortality in the age-range 15–24;
4. the substantial $\mathring{e}_0$ gain for females as a result of improved puerperal mortality;
5. the negative effect for males and positive effect for females of changes in cancer mortality;

and in Table 7 for the period 1971–79:

1. the substantial $\mathring{e}_0$ gains for both sexes resulting from improved circulatory system mortality;
2. the gains from improved cancer mortality; and
3. the disturbing $\mathring{e}_0$ reduction for females caused by increased accident and external cause mortality.

While the improvement in Australian male and female circulatory disease mortality over the period 1971–79 appears to be well documented in official

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Table 6. *Contribution (years of life) of selected causes of death to the improvement in expectation of life at birth—Australian males and females, 1921–71**

Age Group	Class I infections	Class II neoplasms	Class VII circulatory	Class XI pregnancy	Class XVII accidents	Other classes	Totals
			Males				
0–4	·7396	·0093	·0214	—	·1626	3·9525	4·8854
5–14	·2075	–·0117	·0372	—	·1192	·3758	·7280
15–24	·2164	–·0098	·0569	—	–·1426	·3139	·4348
25–49	·9025	·0126	–·0984	—	·1980	1·1132	2·1279
50+	·4493	–·1961	–1·8361	—	·1265	2·0059	·5495
Totals	2·5153	–·1957	–1·8190	—	·4637	7·7613	8·7256
			Females				
0–4	·8545	·0061	·0031	0	·1169	3·3091	4·2897
5–14	·2125	–·0079	·0625	·0017	·0455	·4159	·7302
15–24	·3696	–·0031	·0681	·1493	–·0404	·3108	·8543
25–49	·8172	·1999	·1267	·4750	·0647	·9735	2·6570
50+	·3506	·2216	–·9699	0	·0388	3·1049	2·7460
Totals	2·6044	·4166	–·7095	·6260	·2255	8·1142	11·2772

* Note that the International Classification of Diseases changed several times over this period and to this extent the results shown in this table must be treated with some caution.

Table 7. *Contribution (years of life) of selected causes of death to the improvement in expectation of life at birth—Australian males and females 1971–79*

Age Group	Class II neoplasms	Class VII circulatory	Class XVIII accidents	Other classes	Totals
		Males			
0–4	·0105	·0013	·0385	·5264	·5767
5–14	·0141	–·0019	·0376	·0156	·0654
15–24	·0161	·0066	·0726	·0162	·1115
25–49	·0203	·1757	·1015	·0477	·3452
50+	–·0365	1·4059	·1018	·3286	1·7998
Totals	·0245	1·5876	·3520	·9345	2·8986
		Females			
0–4	·0119	·0022	·0175	·4141	·4457
5–14	·0150	·0014	·0021	·0346	·0531
15–24	·0178	·0041	–·0231	·0628	·0616
25–49	·0535	·1654	–·1141	·3279	·4327
50+	·0634	1·7438	–·0086	·3820	2·1806
Totals	·1616	1·9169	–·1262	1·2214	3·1737

publications and elsewhere, the extent of its contribution to the substantial improvement in expectation of life at birth over the period does not. The negative effect on the female expectation of life at birth of the changed accidental and external cause mortality (admittedly small) does not appear to have been noted by other authors. The emergence since 1947 of a pronounced 'accident hump' near age 20 for Australian females is evident, however, in Heligman and Pollard (1980).

4. DIFFERENTIALS AND THEIR TRENDS—A PARADOX

In Table 8, we exhibit the quinary mortality rates ${}_5q_x$ for two populations A and B at two points of time: time 1 and time 2, some years later. It is immediately clear from columns 6 and 7 in this table that population B has a marked advantage over population A as far as mortality is concerned, although the advantage has diminished somewhat over time. Indeed, there seems to be a *reduction* in the differential of about 5% at all ages. Population A is *gaining* on population B in the mortality stakes.

The expectations of life at birth of populations A and B are readily calculated to be 67·94 and 74·20 respectively at time 1, and 69·34 and 75·90 at time 2. In

Table 8. *Change in mortality differentials for two populations A and B*

Age	Time 1		Time 2		Differential ${}_5q_x^A - {}_5q_x^B$	
x	${}_5q_x^A$	${}_5q_x^B$	${}_5q_x^A$	${}_5q_x^B$	Time 1	Time 2
(1)	(2)	(3)	(4)	(5)	(6)	(7)
0	·02685	·02146	·02550	·02038	·00539	·00512
5	·00260	·00194	·00244	·00181	·00066	·00063
10	·00244	·00142	·00219	·00122	·00102	·00097
15	·00630	·00253	·00536	·00178	·00377	·00358
20	·00814	·00302	·00686	·00210	·00512	·00476
25	·00740	·00345	·00641	·00266	·00395	·00375
30	·00839	·00482	·00750	·00411	·00357	·00339
35	·01141	·00730	·01038	·00659	·00411	·00379
40	·01824	·01132	·01651	·00994	·00692	·00657
45	·02953	·01787	·02662	·01554	·01166	·01108
50	·04895	·02720	·04351	·02285	·02175	·02066
55	·07946	·04051	·06972	·03272	·03895	·03700
60	·12697	·06463	·11064	·05236	·06134	·05828
65	·18841	·10426	·16737	·08743	·08415	·07994
70	·27228	·17115	·24700	·15092	·10113	·09608
75	·38303	·27694	·35651	·25572	·10609	·10079
80	·52337	·42676	·49922	·40744	·09661	·09178
85	·67682	·59834	·65720	·58264	·07848	·07456
90	·80474	·74863	·79071	·73741	·05611	·05330
95	·89218	·85726	·88345	·85028	·03492	·03317
100	·95000	·92758	·94440	·92310	·02242	·02130

other words, there was an *increase* in the $\mathring{e}_0$ differential of about 5% from 6·26 years to 6·56 years over the period. The complete expectation of life at birth, as a mortality indicator, would suggest that population A is *falling behind* population B in the mortality stakes, which is contrary to our earlier conclusion. In case it be thought that this paradox can only be observed with quite pathological life tables, it should be pointed out that the mortality rates at time 1 in Table 8 are in fact the Australian male and female values 1960–62. The small improvements in these rates between time 1 and time 2 are admittedly hypothetical ones chosen to demonstrate the problem clearly and unambiguously over the whole life-span. The phenomenon is, however, frequently observed with real populations over sections of the life table. They are certainly not pathological rates chosen to demonstrate an unusual phenomenon.

A further numerical example is given in Table 9. The populations are hypothetical with forces of mortality independent of age. Note that in this case population A has gained on population B both in absolute amount and in percentage terms when the force of mortality is used for comparison purposes, but it has lost ground if $\mathring{e}_0$ is used for comparison purposes.

To investigate this paradox, let us consider two populations A and B with forces of mortality at age μ_x^{A1} and μ_x^{B1} respectively at time 1. We shall assume that population A experiences the heavier mortality, and that

$$\mu_x^{A1} \geqslant \mu_x^{B1} \tag{15}$$

for all x; furthermore there is strict inequality over at least part of the age-range.

Table 9. *Change in mortality differentials between two populations with forces of mortality independent of age*

	μ	$\mathring{e}_0$
Time 1		
Population A	·01700	58·824
Population B	·01500	66·667
Differential*	·00200	7·843
Time 2		
Population A	·01663	60·132
Population B	·01470	68·027
Differential*	·00193	7·895
Absolute change		
Population A	−·00037	+1·308
Population B	−·00030	+1·360
Differential*	−·00007	+0·052
% Change		
Population A	−2·2	+2·2
Population B	−2·0	+2·0
Differential*	−3·5	+0·7

* The differentials are $\mu^A - \mu^B$ and $\mathring{e}_0^B - \mathring{e}_0^A$ respectively.

Let us also assume that over the period between time 1 and time 2, the force of mortality at age x falls by the same amount $\lambda(x)$ in both populations. Using a superscript 2 to denote mortality functions at time 2, we have:

$$l_x^{A1} = \exp\{-\int_0^x \mu_t^{A1}\,dt\}; \tag{16}$$

$$\mathring{e}_0^{A1} = \int_0^\infty l_x^{A1}\,dx; \tag{17}$$

$$\mu_x^{A2} = \mu_x^{A1} - \lambda(x); \tag{18}$$

$$\begin{aligned} l_x^{A2} &= \exp\{-\int_0^x (\mu_t^{A1} - \lambda(t))\,dt\} \\ &= l_x^{A1}\,\Lambda(x), \end{aligned} \tag{19}$$

where

$$\Lambda(x) = \exp\{\int_0^x \lambda(t)\,dt\}; \tag{20}$$

and

$$\begin{aligned} \mathring{e}_0^{A2} &= \int_0^\infty l_x^{A2}\,dx \\ &= \int_0^\infty l_x^{A1}\,\Lambda(x)\,dx. \end{aligned} \tag{21}$$

Corresponding expressions for population B are obtained by replacing the superscript A by B.

The differential at time 1 is

$$\mathring{e}_0^{B1} - \mathring{e}_0^{A1} = \int_0^\infty (l_x^{B1} - l_x^{A1})\,dx, \tag{22}$$

and at time 2, the differential becomes

$$\mathring{e}_0^{B2} - \mathring{e}_0^{A2} = \int_0^\infty (l_x^{B1} - l_x^{A1})\,\Lambda(x)\,dx. \tag{23}$$

The function $\lambda(x)$, representing the improvement in the forces of mortality at age x is assumed to be non-negative over the whole age-range, and positive over at least part of the range. It follows that $\Lambda(x)$ is greater than or equal to one over the whole age-range, and strictly greater than one over at least part of that range, so that

$$\mathring{e}_0^{B2} - \mathring{e}_0^{A2} = \int_0^\infty (l_x^{B1} - l_x^{A1})\,\Lambda(x)\,dx$$

$$> \int_0^\infty (l_x^{B1} - l_x^{A1})\, dx$$

$$= \mathring{e}_0^{B1} - \mathring{e}_0^{A1}. \quad (24)$$

In other words *equal* absolute reductions in the forces of mortality cause a *widening* of the $\mathring{e}_0$ differential, and we can deduce as a corollary that with a *reduction* in the μ_x differential at all ages it is possible to observe an *increase* in the $\mathring{e}_0$ differential—the result demonstrated above numerically.

How important is this effect? Let us denote by γ_x the change in the force of mortality at age x between time 1 and time 2 common to both sexes. Clearly

$$\gamma_x = \min\text{ (male change, female change)} \quad (25)$$

when both sexes achieve positive improvements in mortality, and

$$\gamma_x = \max\text{ (male change, female change)} \quad (26)$$

when both sexes experience a deterioration (negative change) in mortality. In other cases γ_x is zero. The male and female changes in mortality over and above the common change γ_x will be denoted by α_x and β_x respectively.

From equation (10), we see that the change in the $\mathring{e}_0$ sex differential between time 1 and time 2 is

$$(\mathring{e}_0^{F2} - \mathring{e}_0^{M2}) - (\mathring{e}_0^{F1} - \mathring{e}_0^{M1})$$

$$= \int_0^\infty \gamma_x (w_x^F - w_x^M)\, dx + \left\{ \int_0^\infty \beta_x w_x^F\, dx - \int_0^\infty \alpha_x w_x^M\, dx \right\}, \quad (27)$$

where w_x^M and w_x^F are respectively the male and female weights in formula (11).

The first term on the right-hand side of (27) summarizes the effect on the $\mathring{e}_0$ differential of mortality changes common to both sexes.

The second term summarizes the effect of the change in the mortality differential at age x on the $\mathring{e}_0$ differential. Indeed, when both sexes experience mortality changes in the same direction at age x, and the mortality differential changes by an amount δ_x, γ_x is non-zero, and either

$$\alpha_x = 0 \text{ with } \beta_x = \delta_x$$

or

$$\beta_x = 0 \text{ with } \alpha_x = -\delta_x$$

depending on whether or not the female mortality change exceeds the male change in absolute value. Either way, the second-term contribution in (27) to the change in $\mathring{e}_0$ differential is directly proportional to the change in mortality differential δ_x.

This approach has been used in Table 10 to analyse the change in the Australian $\mathring{e}_0$ sex differential over the period 1971–79. The numerical methods for evaluating the components of (27) are those of § 2 above, and the data come from

Table 10. *Contributions to the change in the Australian $\mathring{e}_0$ sex differential, 1971–79*

Age group (1)	Mortality change Males (2)	Mortality change Females (3)	Common change (4)	Additional change Males (5)	Additional change Females (6)	Contribution to $\mathring{e}_0$ change: Common change (7)	Contribution to $\mathring{e}_0$ change: Change in differential (8)	Total contribution to $\mathring{e}_0$ change (9)
0	703	501	501	202	0	·0343	−·1390	−·1047
1–4	140	91	91	49	0	·0062	−·0325	−·0263
5–9	77	43	43	34	0	·0029	−·0211	−·0182
10–14	31	37	31	0	6	·0021	·0038	·0059
15–19	151	88	88	63	0	·0059	−·0329	−·0270
20–24	69	18	18	51	0	·0012	−·0241	−·0229
25–29	−27	53	0	−27	53	·0000	·0375	·0375
30–34	99	142	99	0	43	·0065	·0190	·0255
35–39	260	230	230	30	0	·0148	−·0099	·0049
40–44	365	323	323	42	0	·0203	−·0119	·0084
45–49	548	473	473	75	0	·0290	−·0178	·0112
50–54	945	670	670	275	0	·0397	−·0529	−·0132
55–59	1,740	979	979	761	0	·0552	−·1135	−·0583
60–64	3,013	1,428	1,428	1,585	0	·0741	−·1734	−·0993
65–69	4,162	2,446	2,446	1,716	0	·1137	−·1256	−·0119
70–74	6,399	4,974	4,974	1,425	0	·1825	−·0643	·1182
75–79	10,232	8,660	8,660	1,572	0	·2226	−·0373	·1853
80–84	15,162	13,861	13,861	1,301	0	·2065	−·0131	·1934
85–89	12,532	10,982	10,982	1,550	0	·0714	−·0050	·0664
90+	0	0	0	0	0	·0000	·0000	·0000
Subtotals						1·0889	−·8140	·2749
Error due to approximate numerical method						—	—	−·0049
Total								·2700

Notes:
(2) = col (4) of Table 4.
(3) = col (4) of Table 5.
(7) = (4) × {col (5) of Table 5 − col (5) of Table 4} × 10^{-5}.
(8) = {(6) × col (5) of Table 5} − {(5) × col (5) of Table 4} × 10^{-5}.
(9) = (7) + (8).

Tables 4 and 5. We see that reductions in mortality differentials led to a narrowing of the $\mathring{e}_0$ sex differential of some ·8200 years, but that mortality improvements common to both sexes widened the gap by an even greater amount: 1·0889 years. The net effect was an increase of ·27 years in the $\mathring{e}_0$ sex differential.

The analysis may be extended to include cause of death, but the interpretation of the results becomes even more difficult. The contribution of the mortality changes common to both sexes to the change in $\mathring{e}_0$ differential can in fact be thought of as an interaction effect. Retherford (1972) adopts this nomenclature in his cause of death analysis. The interpretation, however, remains difficult.

Because of the above common mortality reduction effect (interaction effect) on the $\mathring{e}_0$ differential, the expectation of life should only be used with great caution for summarizing changes in *mortality* differentials.

5. CONCLUDING REMARKS

The complete expectation of life at birth is often used as a convenient summary measure of the mortality of a population. It does, of course, suffer from the usual disadvantages of single-figure indices (Keyfitz & Golini, 1975). At the same time it does have a number of advantages, not least of which is its ease of interpretation. Even the experienced observer has little feeling for the difference between, say, q_x values of ·00307 and ·00921. On the other hand, both layman and expert have some feeling for the expectation of life, and differences between such expectations.

Although the relationship between mortality and expectation of life is essentially reciprocal, the exact connection is rather more complicated, and becomes important when, for example, expectation of life is used as a summary measure of mortality in the analysis of mortality trends, and trends in mortality differentials.

In this paper, we have shown that the change in expectation of life of a population may be expressed as a weighted function of mortality changes at individual ages *plus* the interaction effects of those mortality changes. The interaction effects are not easy to interpret and are difficult to explain to laymen. They are usually relatively minor, however, and for most practical purposes may be merged with the main effects (equation (10)).

Trends in mortality differentials, on the other hand, may become clouded by interaction effects, when measured in terms of trends in $\mathring{e}_0$ differentials. It is dangerous, therefore, to use the expectation of life for this purpose.

Finally, it should be pointed out that very often it is not really mortality in which we are interested, but rather the length of time individuals survive. The use of expectation of life is then clearly appropriate.

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