103.27 A property of a particular unit-distance graph

Introduction

The *Petersen graph*, as seen in Figure 1, is well-known in graph theory. It is an undirected graph comprising 10 vertices and 15 edges. As is shown in Figure 1, the Petersen graph can be drawn in such a way as to possess five mirror lines and five-fold rotational symmetry, in which case it consists of a regular pentagon connected to, and enclosing, a regular pentagonal 'star'. This is in fact the usual way of drawing the Petersen graph. It is, incidentally, also a *cubic* graph, meaning that each of its vertices has degree 3.

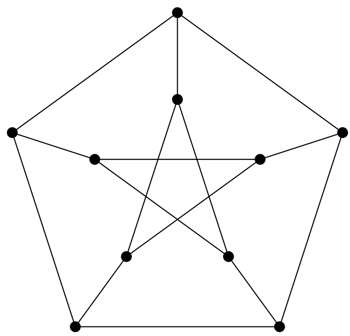

FIGURE 1: The 'usual' picture of the Petersen graph

As is now explained, we are interested here in a particular geometric property of the Petersen graph as opposed to a graph-theoretic one. First, Figure 1 is scaled so that the edges of its outer regular pentagon are each 1 unit in length. We next enlarge the pentagonal star about its centre until each of its edges is also 1 unit in length. The result of this can be seen in Figure 2. The enlarged pentagonal star is then rotated clockwise by an angle of θ about its centre until we end up with the drawing of the Petersen graph given by Figure 3 in which all 15 edges are of length 1 unit.

Any graph that can be drawn in the plane in such a way that its edges are all straight lines of length 1 unit is known as a *unit-distance* graph. It therefore follows from the previous paragraph that the Petersen graph is an instance of such a graph. It seems to have first appeared in the literature in 1965 [1], although in a slightly different context to the one being considered here. Further examples of unit-distance graphs are given by cycle graphs and hypercube graphs.

The aim of this Note is to derive a numerical expression for θ, the angle of rotation of the pentagonal star. This turns out to be interesting in that while the resulting expression is extremely simple, my method for obtaining it proved to be relatively lengthy and cumbersome. An anonymous referee kindly suggested a more straightforward and elegant solution, which is the version given here. However, because of the particularly simple form for θ, it would seem possible that there exist elegant solutions with even greater brevity. Readers may be interested in pursuing this further.

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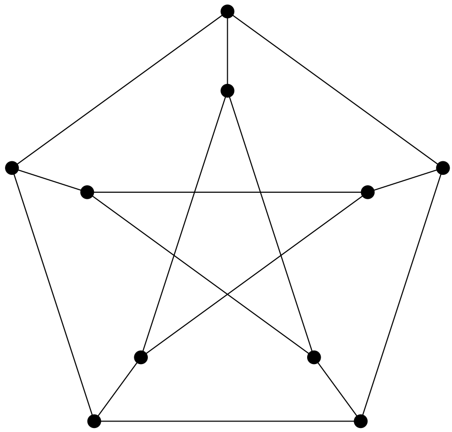

FIGURE 2: A drawing of the Petersen graph with ten of its edges of length 1 unit with the remaining five equal in length but shorter than 1 unit

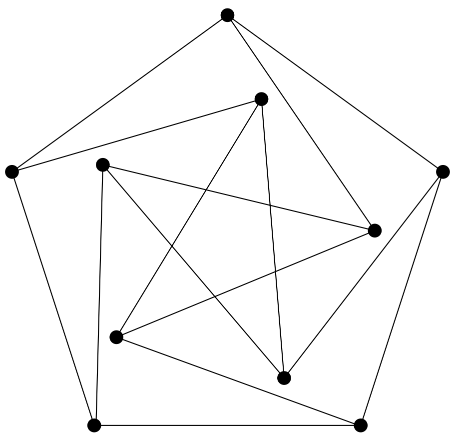

FIGURE 3: The Petersen graph drawn as a unit-distance graph

Calculations

In order to calculate θ, we make use of Figure 4, a partially-labelled version of Figure 3. Point O is used to denote the geometric centre of the graph, noting that it *does not* represent a vertex. If X is a point or a vertex, and similarly for Y, then XY denotes the length of the line joining X to Y.

First, since $AB = 1$ and ΔAOB is isosceles, it follows, by way of the sine rule and the trigonometric identities

$$\sin x = \cos\left(\frac{\pi}{2} - x\right) \qquad \text{and} \qquad \sin 2x = 2 \sin x \cos x,$$

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that

$$OB = \frac{AB \sin OAB}{\sin AOB}$$

$$= \frac{\sin \frac{3\pi}{10}}{\sin \frac{2\pi}{5}}$$

$$= \frac{\cos \frac{\pi}{5}}{2 \sin \frac{\pi}{5} \cos \frac{\pi}{5}}$$

$$= \frac{1}{2 \sin \frac{\pi}{5}}.$$

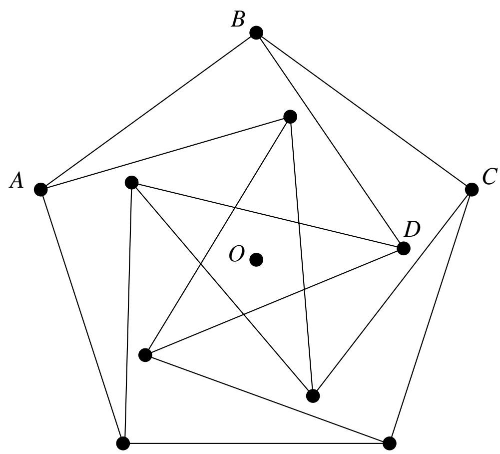

FIGURE 4: A partially-labelled Petersen graph drawn as a unit-distance graph

Next, it is fairly straightforward to show that the diagonals of any regular pentagon of edge-length 1 unit are each of length ϕ units [2] where ϕ, known as the *golden ratio*, is given by

$$\phi = \frac{1 + \sqrt{5}}{2}.$$

In Figure 4, for example, we have $AC = \phi$. Furthermore, if we join up its five vertices in turn, the pentagonal star may be regarded as a regular pentagon. Note that each of its diagonals has length 1 unit, and that the outer pentagon is thus the inner one scaled up by a factor of ϕ. This gives $OB : \phi = OD : 1$. Therefore,

$$OB^2 + OD^2 = \frac{1}{4 \sin^2 \frac{\pi}{5}} + \frac{1}{4\phi^2 \sin^2 \frac{\pi}{5}}$$

$$= \frac{1}{4 \sin^2 \frac{\pi}{5}}\left(1 + \frac{1}{\phi^2}\right)$$

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$$= \frac{1}{4\left(1 - \cos^2 \frac{\pi}{5}\right)} \cdot \frac{\phi^2 + 1}{\phi^2}$$

$$= \frac{1}{4 - \phi^2} \cdot \frac{\phi^2 + 1}{\phi^2}$$

$$= \frac{1}{(2 - \phi)(2 + \phi)} \cdot \frac{2 + \phi}{1 + \phi}$$

$$= 1,$$

where we have made use of the well-known results

$$\cos\frac{\pi}{5} = \frac{\phi}{2} \qquad \text{and} \qquad \phi^2 = \phi + 1.$$

Finally, since $DB = 1$, it follows from the converse of Pythagoras' theorem that

$$\angle BOD = \frac{\pi}{2},$$

which is the required angle of rotation θ.

Acknowledgement

The author would like to thank the aforementioned anonymous referee for suggesting an alternative solution, which was somewhat simpler and more elegant than my own version.

References

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2. M. Griffiths, Mathematics suggested by a logo: both rich and beautiful? *Teaching Mathematics and its Applications* **29** (2010) pp. 216–229.

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