Problem Corner

Solutions are invited to the following problems. They should be addressed to **Nick Lord** at **Tonbridge School, Tonbridge, Kent TN9 1JP** (e-mail: **njl@tonbridge-school.org**) and should arrive not later than 10 March 2019.

Proposals for problems are equally welcome. They should also be sent to Nick Lord at the above address and should be accompanied by solutions and any relevant background information.

102.I (Isaac Sofair)

In the 'Four T Puzzle ©', four T-shaped pieces fit into the large square shown in Figure 1; they also fit into the smaller square shown in Figure 2 (where each T-shape touches two adjacent sides of the square). Find (in surd form) the ratio of the side length of the smaller square to that of the larger square.

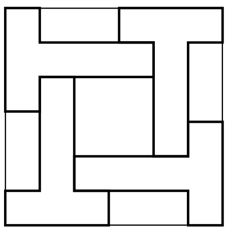


FIGURE 1

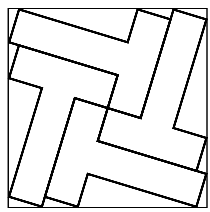


FIGURE 2

102.J (Zoltán Retkes)

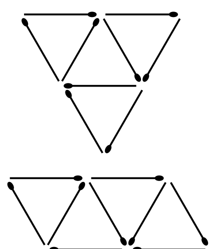
Let P_n be a set of $n \geqslant 4$ points in space with the property that every choice of four points from P_n are non-coplanar and form a tetrahedron with volume not greater than 0.037. Show that P_n lies within a tetrahedron of unit volume.

102.K (Finbarr Holland)

Suppose t is a complex number. Prove that the solutions of the equation $\frac{t^3}{z^2} + \frac{(1 - t)^3}{(1 - z)^2} = 1$ are unimodular if, and only if, $|1 - t| \leqslant 1 = |t|$.

102.L (Stan Dolan)

The diagram illustrates two ways that 4 congruent equilateral triangles can be made with a planar arrangement of 9 matches.



Given that T congruent equilateral triangles can be made with a planar arrangement of M matches, prove that

$$M \geqslant \frac{3T}{2} + \sqrt{\frac{3T}{2}}.$$

https://doi.org/10.1017/mag.2018.140 Published online by Cambridge University Press

Solutions and comments on **102.A**, **102.B**, **102.C**, **102.D** (March 2018).

102.A (Stan Dolan)

For some values of m it is possible to find numbers which:

- have m digits;
- are divisible by m;
- have no subsequence divisible by m.

Prove that the sum of the digits of such a number is divisible by m.

[A subsequence of a number is formed by deleting some, but not all, of its digits, with leading zeros not being allowed. Examples of numbers satisfying the above properties are 252, 8000006 and 2011111111111111111.]

The two solutions received from Jacob Siehler and the proposer, Stan Dolan, (below) both used the following elegant argument.

Let u be an example of a number satisfying the conditions of the problem and, for $1 \leqslant i \leqslant m$, let u_i be the number formed by the leftmost i digits of u. If $u_i \equiv u_j \pmod{m}$ for some $i < j$, then the subsequence of u formed by the digits from the $(i + 1)$th to the jth is congruent to $u \pmod{m}$ and thus divisible by m, contradicting the hypothesis on subsequences. By the pigeon-hole principle, $\{u_i : 1 \leqslant i \leqslant m\}$ is thus the complete set of residues mod m.

If it were the case that $m = fM$ with $f = 2$ or 5, then the same argument shows that u_i, $1 \leqslant i \leqslant m - 1$, are all distinct and non-zero, modulo M. Thus $m - 1 \leqslant M - 1 \leqslant \frac{1}{2}m - 1$ which is a contradiction. Since neither 2 nor 5 are factors of m, m is coprime to 10. Thus $\{10u_i : 1 \leqslant i \leqslant m\}$ and $\{u_i : 1 \leqslant i \leqslant m\}$ are both complete sets of residues mod m. Therefore the sum of the digits of u is

$$u_1 + \sum_{i=2}^{m} \left(u_i - 10u_{i-1}\right) = \sum_{i=1}^{m} u_i - \sum_{i=1}^{m-1} 10u_i$$

$$\equiv \sum_{i=1}^{m} u_i - \sum_{i=1}^{m} 10u_i \qquad \pmod{m},$$

since $u_m = u \equiv 0 \pmod{m}$

$$\equiv 0 \qquad \pmod{m}.$$

Jacob Siehler completed his proof by noting that m is odd so that

$$\sum_{i=1}^{m} u_i - 10 \sum_{i=1}^{m-1} u_i \equiv -9(1 + 2 + \dots + m - 1) \equiv 0 \pmod{m}.$$

Correct solutions were received from: J. Siehler and the proposer Stan Dolan.

https://doi.org/10.1017/mag.2018.140 Published online by Cambridge University Press

102.B (Prithwijit De)

Evaluate the following integrals:

$$(a) \qquad \int_0^{\pi/2} \frac{dx}{(\sin^3 x + \cos^3 x)^2};$$

$$(b) \qquad \int_0^{\pi/2} \frac{x}{\sin^3 x + \cos^3 x}\, dx;$$

$$(c) \qquad \int_0^{\pi/2} \cos x \ln\left(\sin^3 x + \cos^3 x\right) dx.$$

Answers: (a) $\dfrac{2}{3} + \dfrac{8\sqrt{3}}{27}\pi$, (b) $\dfrac{\pi\sqrt{2}}{6} \ln\left(\sqrt{2} + 1\right) + \dfrac{\pi^2}{12}$,
(c) $\sqrt{2} \ln\left(\sqrt{2} + 1\right) - 3 + \dfrac{\pi}{2}$.

This was a very popular problem which attracted a wide range of approaches depending on the trigonometrical manipulations and substitutions used. The solution which follows cherry-picked from those submitted.

(a) Denote the integrals in each part by I_a, I_b, I_c. Then, substituting $t = \tan x$ rewrites I_a as

$$I_a = \int_0^{\infty} \frac{(t^2 + 1)^2}{(t^3 + 1)^2}\, dt.$$

But

$$\frac{(t^2+1)^2}{(t^3+1)^2} = \frac{(t^2+1)^2 - t^2 + t^2}{(t^3+1)^2} = \frac{(t^2-t+1)(t^2+t+1)}{(t+1)^2(t^2-t+1)^2} + \frac{t^2}{(t^3+1)^2}$$
$$= \frac{1}{3}\left[\frac{1}{(t+1)^2} + \frac{2}{t^2-t+1}\right] + \frac{t^2}{(t^3+1)^2}$$

so that

$$I_a = \left[-\frac{1}{3(t + 1)} + \frac{4\sqrt{3}}{9} \tan^{-1}\frac{2t - 1}{\sqrt{3}} - \frac{1}{3\left(t^3 + 1\right)}\right]_0^{\infty}$$
$$= \frac{2}{3} + \frac{8\sqrt{3}}{27}\pi.$$

Alternatively, Michel Bataille substituted $t = u^{1/3}$ to obtain

$$I_a = \frac{1}{3}\int_0^{\infty} \frac{\left(u^{2/3} + 1\right)^2 u^{-2/3}}{(u + 1)^2}\, du$$

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$$= \frac{1}{3}\int_0^\infty \frac{u^{-2/3}}{(u + 1)^2} + \frac{2}{(u + 1)^2} + \frac{u^{2/3}}{(u + 1)^2}\,du$$

$$= \frac{1}{3}\left[B\left(\frac{1}{3}, \frac{5}{3}\right) + 2 + B\left(\frac{5}{3}, \frac{1}{3}\right)\right]$$

where

$$B\left(\frac{1}{3}, \frac{5}{3}\right) = B\left(\frac{5}{3}, \frac{1}{3}\right) = \frac{\Gamma\left(\frac{5}{3}\right)\Gamma\left(\frac{1}{3}\right)}{\Gamma(2)} = \tfrac{2}{3}\Gamma\left(\tfrac{2}{3}\right)\Gamma\left(\tfrac{1}{3}\right) = \frac{2}{3}\frac{\pi}{\sin\frac{\pi}{3}}$$

using standard gamma function formulae.

(With some restrictions on k, m, n, this method works for integrals of the form $\int_0^\infty \frac{x^k}{(1 + x^n)^m}$.)

(b) The substitution $x \to \frac{\pi}{2} - x$ shows that

$$I_b = \int_0^{\pi/2} \frac{\frac{1}{2}\pi - x}{\sin^3 x + \cos^3 x}\,dx$$

which rearranges to give

$$\frac{4}{\pi}I_b = \int_0^{\pi/2} \frac{1}{\sin^3 x + \cos^3 x}\,dx$$

$$= \int_{-\pi/4}^{\pi/4} \frac{1}{\sin^3\left(x + \frac{\pi}{4}\right) + \cos^3\left(x + \frac{\pi}{4}\right)}\,dx.$$

But

$$\sin^3\left(x + \frac{\pi}{4}\right) + \cos^3\left(x + \frac{\pi}{4}\right) = \frac{1}{2\sqrt{2}}\left[(\sin x + \cos x)^3 + (\cos x - \sin x)^3\right]$$

$$= \frac{1}{\sqrt{2}}\left(\cos^3 x + 3\cos x \sin^2 x\right)$$

$$= \frac{1}{\sqrt{2}}\cos x\left(1 + 2\sin^2 x\right)$$

so that

$$\frac{4}{\pi}I_b = \sqrt{2}\int_{-\pi/4}^{\pi/4} \frac{1}{\cos x\,(2\sin^2 x + 1)}\,dx$$

$$= \frac{\sqrt{2}}{3}\int_{-\pi/4}^{\pi/4} \frac{1}{\cos x} + \frac{2\cos x}{2\sin^2 x + 1}\,dx$$

$$= \frac{\sqrt{2}}{3}\left[\ln(\sec x + \tan x) + \sqrt{2}\tan^{-1}\left(\sqrt{2}\sin x\right)\right]_{-\pi/4}^{\pi/4}$$

$$= \frac{\sqrt{2}}{3}\left[2\ln\left(\sqrt{2} + 1\right) + \frac{\pi\sqrt{2}}{2}\right] \qquad (*)$$

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and

$$I_b = \frac{\pi\sqrt{2}}{6} \ln\left(\sqrt{2} + 1\right) + \frac{\pi^2}{12}.$$

(c) Integrating by parts,

$$I_c = \left[\sin x \ln\left(\sin^3 x + \cos^3 x\right)\right]_0^{\pi/2} - 3\int_0^{\pi/2} \frac{\sin x\left(\sin^2 x \cos x - \cos^2 x \sin x\right)}{\sin^3 x + \cos^3 x} dx$$

$$= -3\int_0^{\pi/2} \frac{\sin^3 x \cos x - \cos^2 x\left(1 - \cos^2 x\right)}{\sin^3 x + \cos^3 x}\, dx$$

$$= -3\int_0^{\pi/2} \cos x - \frac{\cos^2 x}{\sin^3 x + \cos^3 x}\, dx$$

$$= -3 + 3\int_0^{\pi/2} \frac{\cos^2 x}{\sin^3 x + \cos^3 x}\, dx.$$

The substitution $x \to \frac{\pi}{2} - x$ shows that

$$\int_0^{\pi/2} \frac{\cos^2 x}{\sin^3 x + \cos^3 x}\, dx = \int_0^{\pi/2} \frac{\sin^2 x}{\sin^3 x + \cos^3 x}\, dx = \tfrac{1}{2}\int_0^{\pi/2} \frac{1}{\sin^3 x + \cos^3 x}\, dx,$$

on adding.

Thus

$$I_c = -3 + \frac{3}{2}\int_0^{\pi/2} \frac{1}{\sin^3 x + \cos^3 x}\, dx$$

$$= -3 + \frac{3}{2}\,\frac{\sqrt{2}}{3}\left[2 \ln\left(\sqrt{2} + 1\right) + \frac{\pi\sqrt{2}}{2}\right], \text{ using } (*) \text{ from part (b)}$$

$$= \sqrt{2} \ln\left(\sqrt{2} + 1\right) - 3 + \frac{\pi}{2}.$$

Correct solutions were received from: M. Bataille, N. Curwen, S. Dolan, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson, P. Kitchenside, J. D. Mahony, J. A. Mundie, B. N. Roth, S. Sayadzade (part (a)), V. Schindler, I. D. Sfikas, G. B. Trustrum and the proposer Prithwijit De.

102.C (Peter Shiu)

Let $0 < \alpha < 1$ be an irrational number. Show that there are infinitely many Pythagorean triples (a, b, c) with $a^2 + b^2 = c^2$ such that

$$0 < \frac{a}{b} - \alpha < \frac{7}{c}.$$

This interesting result, which quantifies the fact that every right-angled triangle is as close in shape as you like to an integer-sided right-angled triangle [1, 2], clearly intrigued solvers. Jacob Siehler's solution which follows proves the stronger inequality with $\frac{4}{c}$ in place of $\frac{7}{c}$ on the right-hand side.

Let $f(x) = \dfrac{2x}{1 - x^2}$ with domain $[0, \sqrt{2} - 1]$. On this domain, f is strictly increasing with range $[0, 1]$; moreover the maximum value of $f'(x)$ is $f'\left(\sqrt{2} - 1\right) = 2 + \sqrt{2}$ $(*)$.

Let $0 < \beta < \sqrt{2} - 1$ with $f(\beta) = \alpha$. Since α is irrational and f is a rational function, β is irrational as well.

Consider the continued fraction convergents to β: these alternate either side of β and every convergent $\frac{m}{n}$ satisfies $\left|\frac{m}{n} - \beta\right| < \frac{1}{n^2}$. There are thus infinitely many convergents with

- $0 < \frac{m}{n} - \beta < \frac{1}{n^2}$
- $\beta < \frac{m}{n} < \sqrt{2} - 1$.

Let $a = 2mn$, $b = n^2 - m^2$, $c = m^2 + n^2$ be the Pythagorean triple generated by m, n so that $f\left(\dfrac{m}{n}\right) = \dfrac{2mn}{n^2 - m^2} = \dfrac{a}{b}$. By the mean value theorem with the bound $(*)$

$$0 < f\left(\frac{m}{n}\right) - f(\beta) \leqslant \left(2 + \sqrt{2}\right)\left(\frac{m}{n} - \beta\right),$$

hence

$$0 < \frac{a}{b} - \alpha < \frac{2 + \sqrt{2}}{n^2}.$$

But

$$c = n^2 + m^2 < \left[\left(\sqrt{2} - 1\right)n\right]^2 + n^2 = \left(4 - 2\sqrt{2}\right)n^2$$

or

$$\frac{1}{n^2} < \frac{4 - 2\sqrt{2}}{c}.$$

It follows that

$$0 < \frac{a}{b} - \alpha < \frac{\left(2 + \sqrt{2}\right)\left(4 - 2\sqrt{2}\right)}{c} = \frac{4}{c}.$$

This solution uses $\beta = \dfrac{\sqrt{1 + \alpha^2} - 1}{\alpha}$ (from $\dfrac{2\beta}{1 - \beta^2} = \alpha$).

Other solvers worked with $\beta = \dfrac{1 + \sqrt{1 + \alpha^2}}{\alpha}$ and $\beta = \alpha + \sqrt{1 + \alpha^2}$ corresponding to $\dfrac{2\beta}{\beta^2 - 1} = \alpha$ and $\dfrac{2\beta}{\beta^2 - 1} = \dfrac{1}{\alpha}$; these solutions often then gave the bound $0 < \dfrac{a}{b} - \alpha < \dfrac{4 + \sqrt{8}}{c} < \dfrac{7}{c}$.

References

1. P. Shiu, The shapes and sizes of Pythagorean triangles, *Math. Gaz.* **67** (March 1983) pp. 33-38.
2. R. E. Pfiefer, The density of Pythagorean rationals, *Math. Gaz.* **70** (December 1986) pp. 292–294.

Correct solutions were received from: S. Dolan, GCHQ Problem Solving Group, G. Howlett, I. D. Sfikas, J. Siehler, G. B. Trustrum, L. Wimmer and the proposer Peter Shiu.

102.D (Michael Fox)

This problem is about spheres with collinear centres and a common tangent line. The line ℓ passes through given points $(0, 0, 1)$ and $(1, m, 1)$ and it is the locus $(t, mt, 1)$. The centre of sphere S_0 is the origin. Its radius is 1, and it touches ℓ at the point where $t = 0$. For all natural numbers n, the centre of sphere S_n is $(c_n, 0, 0)$, its radius is r_n and it touches ℓ at $(t_n, mt_n, 1)$. Each S_n touches S_{n-1} externally, with $c_n > c_{n-1}$.

In any order, show that:

(a) if $2m^2$ is an integer, then so are all the r_n, t_n and c_n;

(b) the r_n, t_n, c_n are integer polynomials in m^2;

(c) if $m = \sinh u$, then $r_n = \cosh 2nu$.

Finally, in (c), express t_n and c_n in terms of hyperbolic functions.

Answer: (c) $t_n = \dfrac{2 \sinh 2nu}{\sinh 2u}$, $c_n = \dfrac{\sinh 2nu}{\tanh u}$.

Solvers of this attractive problem were evenly divided as to whether they tackled the parts in order or reverse order. The solution below is a composite one along the latter lines.

The sphere S_n with equation $(x - c_n)^2 + y^2 + z^2 = r_n^2$ touches ℓ at $(t_n, mt_n, 1)$ where $(t_n - c_n)^2 + m^2t_n^2 + 1 = r_n^2$ has equal roots for t_n.

From the discriminant

$$m^2c_n^2 = (m^2 + 1)(r_n^2 - 1) \qquad (1)$$

and from the equal roots

$$c_n = (m^2 + 1)t_n. \qquad (2)$$

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Also, since S_{n-1} touches S_n externally,

$$c_n - c_{n-1} = r_n + r_{n+1}. \tag{3}$$

From (1) and (3) we have

$$\sqrt{m^2 + 1}\left(\sqrt{r_n^2 - 1} - \sqrt{r_{n-1}^2 - 1}\right) = m\left(r_n + r_{n-1}\right).$$

Setting $m = \sinh u$ and $r_n = \cosh \theta_n$ then gives

$$\cosh u\left(\sinh \theta_n - \sinh \theta_{n-1}\right) = \sinh u\left(\cosh \theta_n + \cosh \theta_{n-1}\right)$$

which simplifies to

$$\sinh\left(\theta_n - u\right) = \sinh\left(\theta_{n-1} + u\right)$$

so that $\theta_n - u = \theta_{n-1} + u$ or $\theta_n - \theta_{n-1} = 2u$ with $\theta_0 = 0$ since $c_0 = 0$ and $r_0 = 1$. Hence $\theta_n = 2nu$ and $r_n = \cosh 2nu$.

From (1) and (2),

$$c_n = \frac{\cosh u \sinh 2nu}{\sinh u} = \frac{\sinh 2nu}{\tanh u} \text{ and } t_n = \frac{\sinh 2nu}{\sinh u \cosh u} = \frac{2\sinh 2nu}{\sinh 2u};$$

this completes (c).

Now observe that r_n, t_n, c_n all arise from a difference equation the roots of whose auxiliary quadratic are $e^{\pm 2u}$. Thus r_n, t_n, c_n all satisfy the same recurrence relation $x_{n+1} - \left(e^{2u} + e^{-2u}\right)x_n + x_{n-1} = 0$ or $x_{n+1} - 2\left(2m^2 + 1\right)x_n + x_{n-1} = 0$ with respective initial conditions $\left(r_0, t_0, c_0\right) = (1, 0, 0)$ and $\left(r_1, t_1, c_1\right) = \left(2m^2 + 1, 2, 2\left(m^2 + 1\right)\right)$. Parts (a) and (b) then follow immediately from the recurrence relation and induction.

The proposer, Michael Fox, noted that, using the standard formulae expressing $\cosh 2nu$ and $\dfrac{\sinh 2nu}{\sinh u \cosh u}$ as polynomials in $\sinh^2 u$, we can give explicit formulae for the polynomials in part (b):

$$r_n = 1 + \frac{n}{2}\left[\binom{n}{1}M + \binom{n+1}{3}\frac{M^2}{2} + \binom{n+2}{5}\frac{M^3}{3} + \dots\right],$$

$$t_n = 2\left[\binom{n}{1} + \binom{n+1}{3}M + \binom{n+2}{5}M^2 + \dots\right]$$

where $M = 4m^2$.

Those solvers tackling the parts of the problem in order used (1), (2), (3) to derive the common recurrence relation above for r_n, t_n, c_n and then solved it by the standard method.

Correct solutions were received from: N. Curwen, S. Dolan, GCHQ Problem Solving Group, G. Howlett, P. F. Johnson and the proposer Michael Fox.

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