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Very Ampleness of Line Bundles and Canonical Embedding of Coverings of Manifolds

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Abstract. Let L be an ample line bundle on a Kähler manifolds of nonpositive sectional curvature with K as the canonical line bundle. We give an estimate of m such that $K + mL$ is very ample in terms of the injectivity radius. This implies that m can be chosen arbitrarily small once we go deep enough into a tower of covering of the manifold. The same argument gives an effective Kodaira Embedding Theorem for compact Kähler manifolds in terms of sectional curvature and the injectivity radius. In case of locally Hermitian symmetric space of noncompact type or if the sectional curvature is strictly negative, we prove that K itself is very ample on a large covering of the manifold.

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Let L be an ample line bundle on an algebraic manifold M. Let K be the canonical line bundle of the manifold. It follows by definition that there is a constant m such that $K + mL$ is ample. It is natural to ask for the smallest value of such m. In case that the line bundle is the canonical line bundle, the question is about the smallest k such that kK is very ample. Let M be a nonpositively curved algebraic manifold with a profinite fundamental group so that there is a tower of coverings over M corresponding to normal subgroups of finite index. In this paper, we show that for sufficiently large covering manifold in the tower, m can be taken to be 1. In the particular case of a sufficiently large covering of the Hermitian symmetric manifold of nonpositive curvature, we show that actually $k = 1$, that is, the canonical line bundle K itself is ample for a covering manifold of a sufficiently large covering index. The same conclusion holds for similar examples of Kähler manifolds with negative Riemannian sectional curvature. An effective version of the Kodaira embedding theorem which gives an estimate of k or m in terms of curvature bounds and the injectivity radius of a general manifold is also obtained.

Here are the main results of this article.

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THEOREM 1. *Let M be a complex manifold of complex dimension n. Suppose the sectional curvature R of M satisfies $-a^2 \leqslant R \leqslant 0$. Let the curvature R_L of the holomorphic line bundle L satisfy $c < R_L$. Let the injectivity radius of the manifold M be bounded from below by τ. Then, $K + mL$ is very ample for $m \geqslant 2n\delta_{\tau,a,c}$, where*

$$\delta_{\tau,a,c} = \frac{1}{c}\left[\frac{16 + 16\log 2}{\tau^2} + \frac{4\log 2}{\tau} a \coth\left(\frac{a\tau}{2}\right)\right]. \tag{1}$$

Moreover, $K + mL$ generates the Nth *order jet of M at any point of M for $m \geqslant ((N/2) + n)\delta_{\tau,a,c}$.*

Remarks.(1) We remark that $\delta_{\tau,a,c} \to 0$ as $\tau \to \infty$.

(2) (Effective Kodaira Embedding Theorem) If we relax the curvature condition to $-a^2 \leqslant R \leqslant b^2$, where $b > 0$, the conclusion is that $K + mL$ is very ample for $m \geqslant 2n\delta_{\tau,a,b,c}$. Letting $\tau_o = \min(\tau, \pi/2b)$, $\delta_{\tau,a,b,c}$ is estimated by

$$\delta_{\tau,a,b,c} = \frac{1}{c}\left[\frac{16 + 16\log 2}{\tau_o^2} + \frac{4\log 2}{\tau_o} a \coth\left(\frac{a\tau_o}{2}\right) - \frac{b\tau_o \cot(b\tau_o) - 1}{\tau_o^2}\right], \tag{2}$$

Moreover, $K + mL$ generates the nth order jet of M at every point of M for $m \geqslant ((N/2) + n)\delta_{\tau,a,b,c}$. The number τ_o is used instead of τ so that $\cot(b\tau_o)$ will not become too negative. This is an effective version of Kodaira embedding Theorem.

As some applications of Theorem 1, we assume that the fundamental group $\pi_1(M)$ of M is profinite in the sense that there exists a sequence of normal subgroups Γ_i satisfying $\Gamma_{i+1} < \Gamma_i$, $\Gamma_0 = \pi_1(M)$ and $\cap_{i=0}^{\infty}\Gamma_i = \emptyset$. Let $\tilde{M}$ be the universal covering of M. Then $M_i = \tilde{M}/\Gamma_i$ is a covering space of M with the covering map denoted by $p_i: M_i \to M$. As $\cap_{i=0}^{\infty}\Gamma_i = \emptyset$, we conclude that the injectivity radius of M_i tends to ∞ as $i \to \infty$ due to the discreteness of $\pi_1(M)$. We call $\{M_i\}$ a tower of coverings for M with the injectivity radius increasing to ∞. The following is an immediate corollary of Theorem 1:

THEOREM 2. *Let M be a nonpositively curved algebraic manifold with profinite fundamental group. There exists i_o such that for all $i \geqslant i_o$, $K_{M_i} + p_i^*L$ is very ample. In fact, there is i_j such that $K_{M_i} + p_i^*L$ generate the* jth *jet space of M for $i \geqslant i_j$. Furthermore, the same is true for $K_{M_i} + \varepsilon p_i^*L$ for any small rational $\varepsilon > 0$ such that εp_i^*L is a line bundle on M_i.*

Since the fundamental group of Hermitian symmetric manifolds of noncompact type are discrete subgroups of general linear groups, they have to be profinite. Hence, the above conclusion is readily applicable in this case. However, this is superseded by the following theorem:

THEOREM 3. *Let* $\{M_j\}$ *be a tower of covering of Hermitian symmetric manifolds of noncompact type. There exists a constant* $i_0 \geqslant 0$ *such that* K_{M_j} *is very ample for* $j \geqslant i_0$. *Moreover, given any* $l > 0$, *there exists* $i_l \geqslant 0$ *such that* K_{M_j} *generates the* kth *jet* $J^k(M_j)$ *of* M_j *for* $j \geqslant i_l$.

Similar statements for Kähler manifolds with sectional curvature pinched between two negative numbers are also true.

THEOREM 4. *Let* $\{M_j\}$ *be a tower of covering over a Kähler manifold* M *with sectional curvature* R *satisfying* $-a^2 < R < -b^2 < 0$. *There exists a constant* $i_0 \geqslant 0$ *such that* K_{M_j} *is very ample for* $j \geqslant i_0$. *Moreover, given any* $l > 0$, *there exists* $i_l \geqslant 0$ *such that* K_{M_j} *generates the* kth *jet* $J^k(M_j)$ *of* M_j *for* $j \geqslant i_l$.

Following from the definition of the Seshadri constant for a line bundle, which will be explained in Section 1, we get the following conclusion:

COROLLARY 1. *For the examples in Theorems 3 and 4, the Seshadri constant for the canonical line bundle is at least* 1.

The organization of the article is as follows. In Section 1, we first use L^2 estimates to show that the value of m, so that $K + mL$ is ample, can be effectively estimated by the injectivity of the manifold and the curvature form of L. In this way, we also estimate the Seshadri constant of the line bundle. Then we apply the results to a tower of coverings of profinite nonpositively curved manifolds to get the result that $K + L$ is very ample after one goes deep enough into the covering space. In particular, this includes the class of Hermitian symmetric manifolds of noncompact type. In Section 2, we relate the L^2 geometry of the universal covering to conclude that K is actually very ample for the covering of a sufficiently large covering index for the manifolds stated in Theorems 3 and 4.

1. Some Criteria for Very Ampleness of Line Bundles on General Manifolds

The main tool is the following L^2-estimates due to Hörmander [Ho].

LEMMA 1. *Let* M *be a compact Kähler manifold with a Kähler metric* ω *and let* K_M *be the canonical line bundle. Let* φ *be a function on* M. *Let* (L, h) *be a Hermitian line bundle on* M. *Assume that*

$$c_1(L, h) + \sqrt{-1}\partial\bar{\partial}\varphi - c_1(K_M) > c\omega.$$

Let g *be a* $\bar{\partial}$*-closed* L*-valued* $(0, 1)$*-form on* M *with* $\int_M \|g\|_h^2 e^{-\varphi} < 0$. *Then the equation*

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$\bar{\partial} f = g$ has a solution satisfying the L^2- estimate

$$\int_M \|f\|_h e^{-\varphi} < \int_M \frac{\|g\|_h^2 e^{-\varphi}}{c}.$$

We also need the following Hessian comparison theorem as stated in [G-W], p. 19:

LEMMA 2. *Let (M_1, o_1) and (M_2, o_2) be Riemannian manifolds with poles at o_1, o_2 and of equal dimension. Suppose that the radial curvature of a point on a normal geodesic γ_1 on M_1 starting from o_1 is at least the radial curvature of the point on a corresponding normal geodesic γ_2 on M_2. Then for every increasing function f, the following Hessian comparison is valid.*

$$D^2 f(\rho_1)(\gamma_1(t)) \leqslant D^2 f(\rho_2)(\gamma_2(t)).$$

Proof of Theorem 1. We need to consider the lower bound of the eigenvalues of the complex Hessian $Lf(X, Y) = D^2 f(X, Y) + D^2 f(JX, JY)$, where J is the complex structure involved. Since the injectivity radius of M is at least τ, we can place a geodesic ball $B(x, \tau)$ of radius τ centered at each point x of M within which there is no cut locus or conjugate locus. For a fixed $\varepsilon > 0$, let $\chi(t)$ be a C^∞ bumping function defined on the interval $[0, \infty)$, satisfying

$$\chi(t) = 1, t \leqslant \frac{\tau}{2}, \qquad \chi(t) = 0, t \geqslant \tau,$$

$$-\frac{2+\varepsilon}{\tau} \leqslant \chi'(t) \leqslant 0, \qquad |\chi''(t)| \leqslant \frac{4(2+\varepsilon)}{\tau^2}.$$

Then $\chi(t)$ is a decreasing function, with support in $[0, \tau]$. The function χ can be constructed as follows. Construct a step function $s(t)$,

$$s(t) = -\frac{8}{\tau^2} \quad \text{for} \quad t \in \left(\frac{\tau}{2}, \frac{3\tau}{4}\right), \qquad s(t) = \frac{8}{\tau^2} \quad \text{for} \quad t \in \left(\frac{3\tau}{4}, \tau\right)$$

and $s(t) = 0$ outside the range. Let $s_1(t)$ be the integral of $s(t)$ with initial condition $s_1(0) = 0$, and $s_2(t)$ be the integral of $s_1(t)$ with $s_2(0) = 1$. Smoothing $s(t)$, the resulting $s_2(t)$ gives a candidate for χ. Let $\rho_x(y)$ be the distance of y from x with respect to the Kähler metric. Define a function ψ_x on M by $\psi_x = (\log(4\rho_x^2/\tau^2))\chi \circ \rho_x$. ψ_x is supported only on $B(x, \tau)$. For simplicity of notations, we will suppress x in the formula below. Note that

$$D^2 f(\rho)(X, Y) = f''(\rho) d\rho(X) \otimes d\rho(Y) + f'(\rho) D^2 \rho(X, Y)$$

for two vectors X, Y on M. Let M_μ be the space form of constant Riemannian sec-

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tional curvature μ. We let X' be the vector on M_μ corresponding to vectors X.

$$
\begin{aligned}
D^2\psi(X,X) &= D^2\left[\log\left(\frac{4\rho_x^2}{\tau^2}\right)\chi\circ\rho_x\right](X,X)\\
&= \chi\circ\rho_x D^2\log\rho^2(X,X) + 2D\log\rho^2 D(\chi\circ\rho_x)(X,X)+\\
&\quad + \log\left(\frac{4\rho_x^2}{\tau^2}\right)D((\chi\circ\rho_x)(X,X))
\end{aligned}
$$

Applying Lemma 2 by comparing it with the flat space M_0 and denoting the restriction of g to the geodesic sphere perpendicular to the radial direction by h, we get

$$
\begin{aligned}
L\log\rho^2(X,X) &= D^2\log\rho^2(X,X) + D^2\log\rho^2(JX,JX)\\
&\geqslant D^2\log\rho^2_{M_{-b}}(X',X') + D^2\log\rho^2_{M_0}((JX)',(JX)')\\
&\geqslant -\frac{2}{\rho^2}\mathrm{d}\rho\otimes d\rho(X,X) - \frac{2}{\rho^2}\mathrm{d}\rho\otimes d\rho(JX,JX)+ \qquad (3)\\
&\quad + \frac{2}{\rho^2}h(X,X) + \frac{2}{\rho^2}h(JX,JX)\\
&\geqslant 0.
\end{aligned}
$$

Comparing with M_{-a} and using the fact that $\tau \geqslant \rho_x \geqslant \tau/2$ in the region where $\chi' \neq 0$, we have

$$
\begin{aligned}
\log\left(\frac{4\rho_x^2}{\tau^2}\right)D^2(\chi\circ\rho)(X,X) &\geqslant \log\left(\frac{4\rho_x^2}{\tau^2}\right)D^2(\chi\circ\rho_{M_{-a}})(X',X')\\
&= \log\left(\frac{4\rho_x^2}{\tau^2}\right)\chi''(\rho_{M_{-a}})\mathrm{d}\rho_{M_{-a}}\otimes\mathrm{d}\rho_{M_{-a}}(X',X')+\\
&\quad + \log\left(\frac{4\rho_x^2}{\tau^2}\right)\chi'(\rho_{M_{-a}})D^2\rho_{M_{-a}}(X',X')
\end{aligned}
$$

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$$\geqslant \log\left(\frac{4\rho_x^2}{\tau^2}\right)\chi''(\rho_{M_{-a}})\mathrm{d}\rho_{M_{-a}}\otimes \mathrm{d}\rho_{M_{-a}}(X',X')-$$

$$-\log\left(\frac{4\rho_x^2}{\tau^2}\right)\frac{2+\varepsilon}{\tau}a\coth(a\rho)g(X,X)$$

$$\geqslant -\log 4\frac{4(2+\varepsilon)}{\tau^2}\mathrm{d}\rho\otimes\mathrm{d}\rho(X,X)-$$

$$-\log 4\frac{2+\varepsilon}{\tau}a\coth\left(\frac{a\tau}{2}\right)g(X,X)$$

and

$$D\log\rho\otimes D(\chi\circ\rho)(X,X)=\frac{D\rho}{\rho}\otimes D(\chi\circ\rho)(X,X)$$

$$=\frac{1}{\rho}\chi' D\rho\otimes D\rho(X,X)|_{\rho\geqslant\tau 2}$$

$$\geqslant -\frac{2(2+\varepsilon)}{\tau^2}\mathrm{d}\rho\otimes\mathrm{d}\rho(X,X).$$

Combining the above inequalities, we get

$$L\psi_{M,x}(X,X)$$

$$\geqslant -\log 4\frac{4(2+\varepsilon)}{\tau^2}[\mathrm{d}\rho\otimes\mathrm{d}\rho(X,X)+\mathrm{d}\rho\otimes\mathrm{d}\rho(JX,JX)]-$$

$$-\log 42+\varepsilon\tau a\coth\left(\frac{a\tau}{2}\right)[g(X,X)+g(JX,JX)]-\frac{8(2+\varepsilon)}{\tau^2}\mathrm{d}\rho\otimes\mathrm{d}\rho(X,X)$$

$$\geqslant -\left[(8+4\log 4)\frac{2+\varepsilon}{\tau^2}+\log 4\frac{2+\varepsilon}{\tau}a\coth\left(\frac{a\tau}{2}\right)\right](g(X,X)+g(JX,JX)).$$

We can now apply the L^2-estimates to construct sections which separate points and generate the first jet of the tangent bundle. Let x, y be arbitrary points on M. The functions ψ_x and ψ_y, as constructed above, are supported in $B(x,\tau)$ and $B(y,\tau)$, respectively. Note that for $\rho(x,w)$ sufficiently small, $\rho(x,w)^2$ is $|x-w|^2(1+\mathcal{O}(|x-w|)$, where $\mathcal{O}(|x-w|)$ is a bounded term tending to 0 as w approaches x. Hence, so does ψ_x. Let $\varphi = n(\psi_x+\psi_y)$. As h is the Hermitian metric for L and $h_1=\det g^{-1}$ is the metric for K_M, $h^\varepsilon h_1 \mathrm{e}^{-\varphi}$ is a metric for K_M+mL. It

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follows from our choice of φ that

$$mc_1(L,h)+\sqrt{-1}\partial\bar{\partial}\varphi+c_1(K_M)-c_1(K_M)>\varepsilon_1\omega$$

with some positive ε_1 provided that the following inequality is satisfied:

$$\varepsilon_1=mc-2n\left[(8+4\log 4)\frac{2+\varepsilon}{\tau^2}+\log 4\frac{2+\varepsilon}{\tau}a\coth\left(\frac{a\tau}{2}\right)\right]>0. \quad (4)$$

Let $\lambda=\min(\tau,\frac{1}{2}\rho(x,y))$. The line bundle K_M+mL is trivial on the support of $B(x,\lambda)$ which is the ball of radius λ centered at x. Let s be the canonical section of the bundle $(K_M+mL)|_{B(x,\lambda)}$. Consider now

$$\zeta(w)=\chi\left(\frac{\rho(x,w)}{1/2\rho(x,y)}\right)s(w)$$

as a C^∞ section of K_M+mL, which is 1 in a small neighbourhood of x and 0 in a small neighbourhood of y. $\bar{\partial}\zeta$ is an integrable $\bar{\partial}$-closed $K_M+\varepsilon L$-valued 1-form, as $\bar{\partial}\zeta$ is zero around x and y. Hence, from L^2-estimates as stated in Lemma 1, there is a solution of $\bar{\partial}f=\bar{\partial}\zeta$ satisfying

$$\int_M\|f\|_h^2e^{-\varphi}<\int_M\frac{\|\bar{\partial}\zeta\|_h^2e^{-\varphi}}{\varepsilon_1}<\infty.$$

From the pole order of φ at x and y, we conclude that $f(x)=f(y)=0$. Hence, $\zeta-f$ is a holomorphic section of K_M+mL which is 1 at x and 0 at y.

To prove that sections of K_M+mL generate 1-jet at any $x\in M$, let χ_1 be a bumping function as χ supported in a normal coordinate chart of x so that $z=0$ corresponds to x. Let $\zeta_i(z)=z_i\chi_1(z)s(z)$ and extend by 0 so that ζ_i is a well-defined C^∞ section of K_M+mL on M. Let $\varphi=(n+\frac{1}{2})\psi_x$. Then the same argument as above shows that we can solve $\bar{\partial}f=\bar{\partial}\zeta_i$ with f vanishing to order 2 at x corresponding to our choice of $(n+\frac{1}{2})\psi_x$ in φ once the inequality (4) is satisfied. Hence, ζ_i-f is a holomorphic section of K_M+mL satisfying $\partial/\partial z_i(\zeta_i-f)(x)=1$. As z_i can be an arbitrary holomorphic coordinate function at x, this shows that the sections of K_M+mL generates the 1-jet and, hence, together with earlier discussions the very ampleness of the line bundle if Equation (4) is satisfied. Note that we can always find an ε satisfying Equation (4) once equation (1) is true.

For the generation in the Nth order jet, it suffices for us to consider $\zeta_{i_1\cdots i_n}(z)=z_{i_1}\cdots z_{i_n}\chi_1(z)s(z)$ instead of $\zeta_i(z)$ in the earlier argument. This concludes the proof of Theorem 1. □

Proof of Remark to Theorem 1. We use $\tau_o=\min(\tau,\pi/2b)$ instead of τ in the proof of Theorem 1. The only modification to this case is the estimates for $L^2\log\rho^2(X,X)$ in Equation (3). Instead of comparing with M_0, we need to compare with M_b. Instead of Equation (3), we get the new estimate

$$\begin{aligned} &L^2 \log \rho^2(X, X) \\ &\quad \geqslant -\frac{2}{\rho^2}[\mathrm{d}\rho \otimes \mathrm{d}\rho(X, X) + \mathrm{d}\rho(JX, JX)] + \\ &\qquad + \frac{2}{\rho} b \cot(b\rho)[h(X, X) + h(JX, JX)]. \end{aligned} \tag{5}$$

Note that if X is a radial tangent vector, JX is tangential to the geodesic sphere. We now observe that $((\rho b \cot(b\rho) - 1)/\rho^2)' \leqslant 0$ and, hence, the minimum of $(\rho b \cot(b\rho) - 1)/\rho^2$ is achieved at τ_o. This concludes the proof of Remark 1. □

As a detour, we consider the Seshadri constant of a line bundle L which is defined as follows (cf [De]): For each $x \in M$, let $\pi: \tilde{X} \to X$ be the blow-up of X at x and E be the exceptional divisor. Let

$$\begin{aligned} s(L, x) &= \sup\{\varepsilon \geqslant 0 | \pi^* L - \varepsilon E \text{ is } \mathit{nef}\} \\ &= \inf_{C \in x} \frac{L \cdot C}{v(C, x)} \end{aligned}$$

where $v(C, x)$ is the multiplicity of C at x and the infimum is taken over all curves passing through x. The relation between the Seshadri constant and very ampleness is related by the following Lemma, which follows immediately from the definition of very ampleness (cf. [De], p.68).

LEMMA 3. *Suppose mL is very ample. Then $s(L) \geqslant \frac{1}{m}$.*

As a corollary, we get

COROLLARY 2. *Assume that M is an algebraic manifold with sectional curvature satisfying $-a \leqslant K \leqslant 0$ and the curvature of L is at least c with respect to the Kähler metric. Then the Seshadri constant of an ample line bundle L is bounded from below by $c/(2n\delta_{\tau,a,c} + na^2)$, where $\delta_{\tau,a,c}$ is the function considered in Theorem 1.*

This follows immediately from Theorem 1, Lemma 3 and the estimates

$$\left(\frac{na}{c} + m\right) c_1(L) \geqslant c_1(K) + m c_1(L).$$

2. Very Ampleness of Canonical Line Bundles in Some Hermitian Symmetric Manifolds and Negatively Curved Manifolds

In the following, we first assume that M is a Hermitian symmetric manifold of noncompact type and give a proof of Theorem 3. Later on we will modify the proof

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to handle the case of negatively curved Kähler manifolds. $\{M_j\}$ is a tower of covering over $M_0 = M$.

Before we go to the proof of Theorem 3, we need some preliminaries. On a compact manifold M, the space of L^2 sections of K_M is finite-dimensional. Let $s_i, i = 1, \ldots, n$ be an orthonormal basis with respect to the Hermitian inner product $(s_i, s_j) = \int_M s_i \wedge \overline{s_j}$. The Bergmann kernel is defined to be $H_M(x, y) = \sum_{i=1}^n s_i(x) \wedge \overline{s_j(y)}$ on $M \times M$ and is independent of the basis chosen. For the universal covering $\tilde{M}$, the space of L^2-holomorphic section of $K_{\tilde{M}}$ form a Hilbert space with respect to a similar inner product $(t_i, t_j) = \int_{\tilde{M}} t_i \wedge t_j$. Take an orthonormal basis $t_i, i \in N$ and form the Bergman kernel $H_{\tilde{M}}(x, y) = \sum_{i \in N} t_i(x) \wedge \overline{t_j(y)}$. It is well known that the L^2-cohomology of a Hermitian symmetric space of noncompact type is trivial except for those corresponding to holomorphic n-forms which are infinite-dimensional. Hence, Theorem 1.1.1 of [Y] can be phrased as the following lemma:

LEMMA 4. *The dimension of the space of holomorphic n-forms on M_j is asymptotically proportional to the volume of M_j with the proportional constant given by the von Neumann dimension of L^2-holomorphic n-forms on $\tilde{M}$.*

Hence, both $H_{\tilde{M}}$ and H_{M_j} are nontrivial. Let us now identify a point $x \in M$ with a point $\tilde{x} \in \tilde{M}$ in the fundamental domain of M in $\tilde{M}$. Let $p_{j,0}\colon M_j \to M_0$ be the covering map. $p_{j,0}^{-1}x$ consists of a finite number of points in M_j. Let x_i be one of the points in $p_{j,0}^{-1}x$. $H_{M_j}(x_j, x_j)$ is independent of the point chosen as representative since the Bergman kernel is invariant under deck transformation which is an isometry. The following result is essentially due to Donnelly [Do]:

LEMMA 5. *$H_{M_j}(x_j, y_j)$ converges pointwise to $H_{\tilde{M}}(x, y)$ in a C^∞ way.*

Donnelly stated in [Do] that $H_{M_j}(x_j, x_j)$ converges pointwise to $H_{\tilde{M}}(x, x)$ uniformly. As the kernel functions are holomorphic with respect to the first variable and antiholomorphic with respect to the second variable, it follows easily from power series expansion the uniform convergence of $H_{M_j}(x_j, y_j)$ to $H_{\tilde{M}}(x, y)$. Then we conclude the convergence in a C^∞ way from Schauder estimates.

Proof of Theorem 3.

Base point freeness

Let us first prove that the global sections $\Gamma(M_j, K_{M_j})$ generate s_i^j for sufficiently large M_j. Let $s_i^j, i = 1, \ldots, N_j \leqslant \infty$ be an orthonormal basis of $\Gamma(M_j, K_{M_j})$, here $0 \leqslant j \leqslant \infty$ with $M_\infty = \tilde{M}$, and B_j be the base locus of $\Gamma(M_j, K_{M_j})$. As $p_{j+1,j}\colon M_{j+1} \to M_j$ is a

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holomorphic covering map, $p_{j+1,j}{}^*(s_i^j)$ is a holomorphic section of $K_{M_{j+1}}$ for each section s_i^j of K_{M_j}. Let D be a fundamental domain of $M = M_0$ in the universal covering $\tilde{M}$ and $p_j: \tilde{M} \to M_j$ as before. It follows that $p_{j+1}^{-1}(B_{j+1}) \cap D \subset p_j^{-1}(B_j) \cap D$. Hence, $p_j^{-1}(B_j) \cap D = \cap_{k=0}^{j} p_j^{-1}(B_k) \cap D$ is a decreasing set. We claim that $\cap_{j=0}^{\infty} p_j^{-1}(B_j) \cap D = \emptyset$ so that from the relative compactness of D, $p_j^{-1}(B_{j+1}) \cap D$ is empty for all sufficiently large j. To prove the claim, note that the Bergmann kernel function specified at $x = y$ can be expressed as

$$H_{M_j}(x, x) = \sup_{f \in \Gamma(M_j), \|f\|=1} |f(x)|^2$$

$$H_{\tilde{M}}(x, x) = \sup_{f \in \Gamma^{(2)}(\tilde{M}), \|f\|=1} |f(x)|^2.$$

This follows from the fact that the Bergmann kernel is independent of the choice of base and, hence, we may choose s_1 with maximal value at the point x. Suppose $x \in \cap_{j=0}^{\infty} p_j^{-1}(B_j) \cap D \neq \emptyset$ so that $H_{M_j}(x) = 0$ for each j. From the above lemma, it follow that $H_{\tilde{M}}(x) = 0$ as well. However, since $\tilde{M}$ is homogeneous, the base locus of $K_{\tilde{M}}$ is empty and, hence, such a x does not exist. This concludes the proof of the claim and, hence, the statement that the global sections generate the bundle.

Separation of points

LEMMA 6. *Assume that the L^2-canonical sections of the universal covering separates points. Also assume that for any $c > 0$, there exists a number $\kappa > 0$ such that for every pair of points $x, y \in \tilde{M}$ of distance $d(x, y) \geqslant c$, there is always a holomorphic section $s \in \Gamma^{(2)}(\tilde{M}, K)$ satisfying $\|s\|_{L^2} = 1$, $s(x) = 0$, $\|s(y)\| \geqslant \kappa$. Then K_{M_j} separates M_j for all sufficiently large j.*

Proof. Take a nested sequence of domains D_j on $\tilde{M}$ so that each D_j is a fundamental domain of M_j. From the above discussion on base-point freeness, we may assume that sections of $\Gamma(M_j, K)$ is base point free for all $j \geqslant 0$. Let $t_1, \ldots t_N$ be a basis of $\Gamma(M_0, K)$.

Consider first the case that $x, y \in M_j$ both lying in some fundamental domain of M_o when pulled back to $\tilde{M}$. We may assume that $x, y \in D_0$ after a biholomorphism if necessary. Since $H_{M_j}(w, z) = \sum_i s_i^j(w)\overline{s_i^j(z)}$ converges to $H_{M_\infty}(w, z)$ uniformly on any relatively compact set containing w, z according to Lemma 1, and for $0 \leqslant j \leqslant \infty$,

$$\sum_i (s_i^j(x) - s_i^j(y))\overline{(s_i^j(x) - s_i^j(y))}$$

$$= H_{M_j}(x, x) - H_{M_j}(y, x) - H_{M_j}(x, y) + H_{M_j}(y, y),$$

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we conclude the convergence of

$$\sum_i (s_i^j(x) - s_i^j(y))\overline{(s_i^j(x) - s_i^j(y))} \to \sum_i (s_i^\infty(x) - s_i^\infty(y))\overline{(s_i^\infty(x) - s_i^\infty(y))}.$$

For $0 \leqslant j \leqslant \infty$, define

$$C_j = \{(x, w) \in p_j(D_0) \cap M_j \times p_j(D_0) |$$

$$s^j(x) = s^j(w) \quad \text{for all } s^j \in \Gamma^{(2)}(M_j, K_{M_j})\}.$$

We easily see that C_j is nested in the sense that $C_{j+1} \subset C_j$. If C_j is nonempty for each $0 \leqslant j < \infty$, the above convergence of the kernel functions implies that sections in $\Gamma^{(2)}(\tilde{M}, K_{\tilde{M}})$, $\tilde{M} = M_\infty$, are not base-point free, contradictory to our assumption.

Consider now the case that $d(x, y) \geqslant \tau(M_0)$, the injectivity radius of M_0, for points $x, y \in M_j$. Let $t_1, \ldots t_N$ be a basis of $\Gamma(M_0, K)$. We claim that, after including a finite number of sections from linear combination of the above sections if necessary, we can assume that for every pair of points $z, w \in M_0$, there is an l such that $t_l(z) \neq 0$, $t_l(w) \neq 0$, l depending on z, w. For the claim, let

$$E = \{(z, w) \in M_0 \times M_0 | t_l(z)t_l(w) = 0, \quad \text{for all } 1 \leqslant l \leqslant N\} \qquad .$$

For generic point $(z, w) \in E$, we can always find t_{l_1}, t_{l_2} such that $t_{l_1}(z) \neq 0$, $t_{l_2}(w) \neq 0$. By taking suitable linear combinations of t_{l_1} and t_{l_2}, we get a new t_{N+1} which neither vanishes on z nor w. Adding t_{N+1} cuts down the dimension of E by 1. The claim follows by applying the above argument repeatly using the fact that M_0 is algebraic. We use the same notation t_l, $1 \leqslant l \leqslant N$, to denote its pull-backs to M_j for each $j > 0$. For continuity, we conclude that for any two points $z, w \in M_j$, there exists δ_1 and δ_2 such that $\delta_2 \geqslant \|t_l(z)\|, \|t_l(w)\| \geqslant \delta_1 > 0$. For the sections s_i^j of $\Gamma(M_j, K)$, let $f_i^{j,l} = s_i^j / t_l$, which are meromorphic functions on M_j for each l. Let

$$\tilde{H}^l_{M_j}(z, w) = \sum_i f_i^{j,l}(z)\overline{f_i^{j,l}(w)} = \frac{H_{M_j}(z, w)}{t_l(z)t_l(w)}.$$

From the uniform convergence of the Bergmann kernel on compacta, given any $\varepsilon > 0$, there exists j_o such that for $j \geqslant j_o$,

$$| H_{\tilde{M}}(z, z) - H_{M_j}(z, z)| \leqslant \varepsilon$$

and,

$$|H_{\tilde{M}}(z, w) - H_{M_j}(z, w)| \leqslant \varepsilon$$

where z, w are two arbitrary points on the manifold M_j. Equivalently, $\varepsilon = \varepsilon(j)$ can be made sufficiently small so that the above inequalities hold when j is sufficiently large.

This implies that for any two points $z, w \in M_j$, there is an l such that

$$|\tilde{H}^l_{\tilde{M}}(z, w) - \tilde{H}^l_{M_j}(z, w)| \leqslant \frac{\varepsilon}{\delta_1^2}.$$

Assume now that $d(x, y) \geqslant \tau = c$. We conclude that

$$\begin{aligned}\sum_i |f_i^{j,l}(x) - f_i^{j,l}(y)|^2 &= \tilde{H}^l_{M_j}(x, x) - \tilde{H}^l_{M_j}(y, x) - \tilde{H}^l_{M_j}(x, y) + \tilde{H}^l_{M_j}(y, y)\\ &\geqslant \tilde{H}^l_{\tilde{M}}(x, x) - \tilde{H}^l_{\tilde{M}}(y, x) - \tilde{H}^l_{\tilde{M}}(x, y) +\\ &\quad + \tilde{H}^l_{\tilde{M}}(y, y) - \frac{4\varepsilon}{\delta_1^2}\\ &= \sum_i |f_i^{\infty,l}(x) - f_i^{\infty,l}(y)|^2 - \frac{4\varepsilon}{\delta_1^2}\\ &\geqslant \frac{1}{\delta_2^2}\sum_i |s_i^{\infty}(x) - s_i^{\infty}(y)|^2 - \frac{4\varepsilon}{\delta_1^2}\\ &\geqslant \frac{\kappa}{\delta_2^2} - \frac{4\varepsilon}{\delta_1^2}.\end{aligned}$$

Here we use our assumption in the last step. Note that $\varepsilon = \varepsilon(j)$ tends to 0 uniformly as j tends to ∞. Hence, for all j sufficiently large, the above expression is positive and, hence, the sections of M_j separate x, y. Together with the previous argument for x, y both lying in some fundamental domain of M_0, we conclude that the sections of $\Gamma(M_j, K)$ separate points for j large enough.

We now apply the above Lemma to the case of a bounded symmetric domain. By homogeneity, we may assume that $x = 0$, the origin in the standard realization. For the point $y \in \tilde{M}$ we simply choose

$$s = \frac{y\mathrm{d}z^1 \wedge \cdots \wedge \mathrm{d}z^n}{\int_{\tilde{M}} |y|^2}.$$

The denominator is finite as it is a bounded domain. Obviously s has norm 1, with its value at y bounded from below by some $\kappa > 0$ once its distance from $x = 0$ is sufficiently large. Hence the conditions of the Lemma are satisfied. This concludes the proof of the separation of points by the sections.

Generation of Jets

We need to prove that, given a positive integer k, there is a sufficiently large j_o such that sections of K_{M_j} generate the k-jet of M_j for all $j \geqslant j_o$. Similarly, we define

$$D_j = \{x \in M_j \mid \Gamma(M_j, K_{M_j}) \text{ does not generate } J_k(x)\},$$

where $J_k(x)$ denotes the kth jet of x. At a point $x_j \in D_j$, it follows by definition that

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there is a multiderivative

$$\partial_{i_1\cdots i_k} = \frac{\partial^{i_1+\cdots+i_k}}{\partial z^{i_1}\cdots\partial z^{i_k}}$$

such that $\partial_{i_1\cdots i_k} s_j(x_j) = 0$ for every section $s_j \in \Gamma(M_j, K_{M_j})$. This is reflected by the statement that $\partial_{i_1\cdots i_k} H_{M_j}(x_j, y) = 0$ for each $y \in M_j$. Again, the set D_j forms a nested set when pulled-back to the universal covering in the sense that $D_{j+1} \subset D_j$. Hence, by similar argument as before and using the previous lemma shows that if our statement is not true, there is a point $x \in \tilde{M}$ and a differential operator $\partial_{i_1\cdots i_k}$ such that $\partial_{i_1\cdots i_k} H_{\tilde{M}}(x, y) = 0$ for every y, say, in a neighbourhood of x in $\tilde{M}$. Hence

$$\sum_{i=1}^{\infty} \partial_{i_1\cdots i_k} t_i(x) \wedge \overline{t_i(y)} = 0.$$

Letting $y = x$ implies that $\partial_{i_1\cdots i_k} t(x) = 0$ for each section $t \in \Gamma^{(2)}(\tilde{M}, K_{\tilde{M}})$. However, homogeneity again implies that this should hold for every $x \in \tilde{M}$, contradicting the fact that for generic point on $\tilde{M}$, the sections in $\Gamma^{(2)}(\tilde{M}, K_{\tilde{M}})$ generate kth order jets.

This concludes the proof of Theorem 3. □

Proof of Theorem 4. M is a Kähler manifold with sectional curvature R satisfying $-a^2 \leqslant R \leqslant -b^2 < 0$. We only need to verify that the universal covering $\tilde{M}$ satisfies the same L^2-cohomological properties as the Hermitian symmetric spaces. It follows from the result of Gromov and Stern [G], that there is no L^2-harmonic forms on the universal covering $\tilde{M}$ of M except for L^2 holomorphic n-forms which form an infinite-dimensional vector space. This latter fact is also stated in [GW]. In fact, the argument there can be modified to construct L^2-holomorphic sections of the canonical line bundle which generate a given high jet of $\tilde{M}$ in the following way: Let $x \in \tilde{M}$. We can construct a smooth increasing function $\phi_x(w)$ on $\tilde{M}$ satisfying

$$0 \leqslant \phi_x \leqslant 1 \text{ and } \partial\bar{\partial}\phi \geqslant b^2\left(\cosh\frac{b\rho}{2}\right)^{-4}\omega,$$

where ω is the Kähler form ([GW], Theorem H). Let h_1 be the canonical metric on $K_{\tilde{M}}$ and φ the same as used in the proof of Theorem 1. Consider a metric $h_1 e^{-\varphi-k\phi}$ of $K_{\tilde{M}}$ and apply the L^2-estimates of Hörmander as in the proof of Theorem 1. We easily conclude that the sections generate an arbitrarily high jet of $\tilde{M}$ at the point x for k sufficiently large.

To prove that the sections separate points on M_j, it suffices to check the conditions of Lemma 6. Assume that $d(x, y) \geqslant c$ on $\tilde{M}$. As in the proof of Theorem 1, we used L^2-estimates to construct a section of K vanishing at x but nonvanishing at y. Following the notations of the proof of Theorem 1, we need to solve $\bar{\partial}f = \bar{\partial}\zeta$ for

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a test function

$$\zeta = \chi\left(\frac{\rho(x, w)}{1/2\rho(x, y)}\right)s(w)$$

and $s_1 = \zeta - f$ is then a holomorphic section vanishing at 0 but having the value 1 at y. From the L^2-estimates, we get

$$\int_M \|f\|_h^2 e^{-\varphi - k\phi} < \int_M \frac{\|\bar{\partial}\zeta\|_h^2 e^{-\varphi - k\phi}}{\varepsilon_1} < \infty.$$

and, hence,

$$\|s_1\|^2 \leqslant \|\zeta\|^2 + \|f\|^2 \leqslant \|\zeta\|^2 + C_1 \int_M \frac{\|\bar{\partial}\zeta\|_h^2 e^{-\varphi - k\phi}}{\varepsilon_1}$$

with some absolute constant C_1. As $d(x, y) \geqslant c$ and $D\zeta = 0$ on a small ball of the radius depending only on c around x and y, we immediately have the estimates of $D\zeta$ and, hence, $\|s_1\|$ in the above is finite with a constant upper bound C determined only by c. It suffices for us to divide s_1 by C to get a section satisfying the conditions of Lemma 6. The arguments of Theorem 3 can then be carried over to conclude the proof of Theorem 4. □

Corollary 1 follows immediately from Theorems 3 and 4 and Lemma 3.

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This work is motivated in part by the results of Hwang and To [HW] where they gave effective estimates for the Seshadri constant of the canonical line bundle on complex ball quotients and thereby prove statements similar to Theorem 2 and Corollary 1 for complex ball quotients.

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