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On the instability of a class of generalized strong curvature singularities[1]

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1. *Introduction*

Cosmic censorship hypothesis [**1**] is a major unsolved problem in classical general relativity. According to this hypothesis singularities occurring in generic space-times should not be naked. This means that there should not be singularities to the future of a regular initial surface that are visible to observers at infinity A mathematically precise statement of the hypothesis is that a space-time should be future asymptotically predictable from a partial Cauchy surface ([**2**] p. 310).

There exist a number of examples of naked singularities in some exact solutions of the Einstein equations. However it has been stressed by Penrose[**3**] that the exact solutions with special symmetries have a rather limited value for verification of the cosmic censorship hypothesis and what is required is an understanding of the generic case. The cosmic censorship hypothesis concerns space-time singularities occurring in physically realistic situations. One of us [**4**], [**5**] has recently proposed a definition of such singularities. This definition has been motivated by the definition of strong curvature singularities by Tipler and its modifications by Królak, but it is significantly different. The new singularities that we call generalized strong curvature singularities can be classified into four types. By the theorem quoted below only one type of the generalized strong curvature singularities can be naked. In this paper we shall argue that the naked class of generalized strong curvature singularities will not occur in generic space-times.

In Section 2 we shall recall the definition of generalized strong curvature singularities, their classification and the cosmic censorship theorem. In Section 3 we shall construct the space G of functions which characterizes various types of singularities, and we shall prove that the subspace of G corresponding to the naked singularities is a nowhere dense subset.

2. *Generalized strong curvature singularities*

First let us recall the definition of the generalized strong curvature singularities.

Definition 1. Let λ be a future-incomplete null (timelike) geodesic; λ is said to terminate in the future at a generalized strong curvature singularity if for each

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sequence of null (timelike) inextendible geodesics λ_n converging to λ at least one of the following conditions holds:

(1) every neighbourhood of each future-endless segment of λ contains some point $q_n \in \lambda_n$ such that the expansion $\hat{\Theta}$ of a future-directed null (timelike) geodesic congruence emanating from q_n and containing λ_n becomes negative on λ_n;

(2) there exists a subsequence λ'_n of λ_n such that $I^-(\lambda'_n) = I^-(\lambda)$ for each n.

The following definition gives us a complete classification of the singularities introduced above.

Definition 2. Let λ be a null (timelike) geodesic terminating in the future at the generalized strong curvature singularity, and let (λ_n) be a sequence of inextensible null (timelike) geodesics converging to λ. Denote by $\tilde{q}(\lambda_n)$ some point on a geodesic $\lambda_n \in (\lambda_n)$, such that the expansion $\hat{\Theta}$ of a future-directed null (timelike) geodesic congruence emanating from $\tilde{q}(\lambda_n)$ and containing λ_n becomes negative on λ_n.

Then λ is said to be of type:

A, if each sequence (λ_n) contains some subsequence (λ_k) converging to λ, such that each $\lambda_k \in I^-(\lambda)$;

B, if λ is not of type A and, for each sequence (λ_n), all points $\tilde{q}(\lambda_n)$ belong to the set $I^-(\lambda)$;

C, if λ is not of type A and each sequence (λ_n) admits some point $\tilde{q}(\lambda_n)$ not in $I^-(\lambda)$;

D, if λ is not of type A, B or C.

The following censorship theorem has been proved.

THEOREM 1 [**4**], [**5**]. *Let S be a partial Cauchy surface in a weakly asymptotically simple and empty space-time (M, g) such that:*

(i) *S has an asymptotically simple past,*

(ii) *each generator of $\mathscr{I}^+$ admits a past inextensible segment contained in* $\operatorname{int} D^+(S, \tilde{M})$, *where $\mathscr{M}$ is space-time with the null boundary $\mathscr{I}$.*

Suppose moreover that the following conditions hold:

(1) *$R_{ab} k^a k^b \geqslant 0$ for every null vector k^a,*

(2) *the generic condition, i.e. every null geodesic contains a point at which $k_{[a} R_{b]cd[e} k_{f]} k^c k^d \neq 0$, where k^a is tangent to the geodesic,*

(3) *if there exists a future-incomplete null geodesic $\gamma \subset \operatorname{int} D^+(S)$, then every null geodesic λ, such that $I^-(\lambda) = I^-(\gamma)$, terminates in the generalized strong curvature singularity.*

If (M, g) is not asymptotically future predictable from S, then there must exist a null geodesic $\lambda \subset \operatorname{int} D^+(S)$, which terminates in the generalized strong curvature singularity and is of type D.

By this theorem only singularities of the type D can be naked.

3. *The G-space*

Let $\gamma\colon [0, t_s) \ni t \mapsto \gamma(t) \in \operatorname{int} D(S)$, where S is a partial Cauchy surface, be a null geodesic terminating in a generalized strong curvature singularity in the future. Denote by L_γ the family of all sequences (γ_i) of inextensible null geodesics converging to γ such that all members of (γ_i) intersect the boundary $\dot{J}^-(\gamma)$ but none of them is

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contained in $J^-(\gamma)$. It follows from Definition 2 above that if γ is not of type A, then L_γ is not empty.

As $\mathrm{int}\,D(S)$ is globally hyperbolic, there must exist a foliation $S(\tau)$ of this set such that each member of the foliation is a Cauchy surface for $\mathrm{int}\,D(S)$ and such that, for some τ_0, $S(\tau_0) = S$. Let $S(\tau_1)$ be a slice through $\gamma(0)$. The affine parameter t on γ introduces in a natural way a reparametrization of $S(\tau)$ for all $\tau \geqslant \tau_1$, namely by $S(t)$ we shall denote the slice $S(\tau)$ which intersects $\gamma(t)$. Let $N := J^+[S(\tau_1)] \cap \mathrm{int}\,D^+(S)$ and define d by $d: N \times N \ni (x, y) \mapsto t_x - t_y$, where t_x and t_y are the values of the affine parameter t on γ such that $x \in S(t_x)$ and $y \in S(t_y)$.

Fix a sequence (γ'_i) from the family L_γ. For each value of the affine parameter t on γ we shall distinguish all geodesics $\gamma_k(t) \subset (\gamma'_i)$ such that on each $\gamma_k(t)$ there exists a point $q \in J^+[S(t)]$ such that the expansion $\hat{\Theta}$ of a future-directed null geodesic congruence emanating from q becomes negative on γ_k (the existence of such points follows from Definition 1).

We write $R(t)$ for the set $\dot{J}^-(\gamma) \cap \gamma_k(t)$. Let $g'(t)$ be defined as follows:

$$g'(t) = \inf\{d(x, \gamma(t)) : x \in R(t)\}.$$

The following function characterizes the strength of singularity:

$$g: [0, t_s] \ni t \mapsto \inf\{g'(t) : t \in [0, t_s)\}.$$

By this procedure we can associate a function g to every member of the family L_γ; we denote by G_γ the set of all these functions. We divide G_γ into two parts: $G^1_\gamma := \{g \in G_\gamma : g(t) \geqslant 0, \forall t \in [0, t_s]\}$ and $G^2_\gamma = G_\gamma - G^1_\gamma$.

By the construction above we can establish the following connections between the possible types of geodesic γ and certain properties of the space G_γ.

PROPOSITION 1. *If γ is of type B, then $G^1_\gamma = G_\gamma$; if γ is of type C, then $G^2_\gamma = G_\gamma$ and if γ is of type D, then $G^1_\gamma \neq \varnothing$ and $G^2_\gamma \neq \varnothing$.*

Remark. The relationships established in Proposition 1 do not depend on the choice of the foliation $S(\tau)$ of the set $\mathrm{int}\,D(S)$ used in the construction of the function g.

It follows from Theorem 1 that predictability of space-times can be violated only if there exist null geodesics of type D. Therefore it would be interesting to know whether the existence of such geodesics is a stable property. Our approach to this issue is to study the stability of the type D by investigating the stability of those properties of the space G_γ which characterize type D. The first step in this approach is to postulate some appropriate topology on G_γ.

Given any topology $\mathscr{T}$ on G_γ, for any non-empty set $U \in \mathscr{T}$ we define the function $\epsilon_U: [0, t_s] \ni t \mapsto \sup\{|g_i(t) - g_j(t)| : g_i, g_j \in U\}$ by means of which we determine the following two numbers.

$$\delta^1_U = \inf\{\epsilon_U(t) : t \in [0, t_s]\},$$

$$\delta^2_U = \sup\{\epsilon_U(t) : t \in [0, t_s]\}.$$

We shall assume that G_γ admits a topology with the following properties:

(1) $\forall U \in \mathscr{T} - \{\varnothing\}$, $\delta^1_U > 0$;

(2) $\forall c > 0$, $\forall g \in G_\gamma$, $\exists U(g) \in \mathscr{T}$, $g \in U(g)$ and $\delta^2_U \leqslant c$.

Property (1) ensures that perturbations introduced on G_γ are sufficiently generic, i.e. each function $g(t)$ is changed at each point of its domain. Property (2) ensures the possibility of investigations of stability on G_γ under generic, but arbitrarily small, perturbations.

PROPOSITION 2. *Let γ be a geodesic of type C or D and let $\mathscr{T}$ be a topology on G_γ satisfying* (1) *and* (2). *Then the set G^2_γ is open and dense in the topological space* $(G_\gamma, \mathscr{T})$.

Proof. First we shall show that G^2_γ is a dense set in the space G_γ, i.e. that every non-empty neighbourhood $U \in \mathscr{T}$ of each function $g \in G_\gamma$ intersects G^2_γ. Suppose that some function g_1 belongs to G^1_γ. It follows from the construction of the functions g that $\inf g(t) \leqslant 0$ for every g, and therefore by condition (1) any neighbourhood U of g_1 must contain some function g_2 such that $\inf g_2 < 0$. By definition of G^2_γ, this means that U intersects G^2_γ.

To prove that G^2_γ is an open set in G_γ we have to show that each function $g \in G^2_\gamma$ admits a neighbourhood $V \in \mathscr{T}$ contained in G^2_γ. Given any $g \in G^2_\gamma$ let $c_g = \inf g(t)$. Clearly $c_g < 0$ by definition of G^2_γ. Since $\mathscr{T}$ satisfies (2), there must exist some neighbourhood $V \in \mathscr{T}$ of the g such that $\delta^2_V < c_g$. From the definitions of δ^1_U and δ^2_U this implies $V \subset G^2_\gamma$. ▌

In view of Proposition 2, belonging to the set G^2_γ is a generic property of the functions g. Therefore generically type D is no different from type C. Consequently, under the assumptions of Theorem 1 and Proposition 2, the future asymptotic predictability holds in generic space-times.

4. *Conclusions*

The space G characterizes some connections between the causal structure and the strength of curvature in a neighbourhood of the singularity. For methodological reasons, only those properties of any physical model are significant that will be stable under perturbations of the model. In the argument above we have shown that type D of singularities is unstable under generic perturbations introduced on the space G (Proposition 2). This result gives a new argument which supports the view that naked singularities are exceptional.

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