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On the Hausdorff dimension of general Cantor sets

BY A. F. BEARDON
Imperial College, London

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1. *Introduction and notation.* In this paper a generalization of the Cantor set is discussed. Upper and lower estimates of the Hausdorff dimension of such a set are obtained and, in particular, it is shown that the Hausdorff dimension is always positive and less than that of the underlying space. The concept of local dimension at a point is introduced and studied as a function of that point.

Sets of this nature often occur in function theory and, in particular, such a set can occur as a singular set of some properly discontinuous group (e.g. (1), page 35; (8), page 109). Using some results contained in this paper the author has obtained certain results in this connexion and hopes to publish these results later.

We shall implicitly assume that the underlying space is Euclidean space of dimension N and shall use the following notation throughout.

(*a*) The diameter of a set E is denoted by $|E|$.

(*b*) The distance between two sets A and B is denoted by

$$\rho(A,B) = \inf\{|a-b| : a\in A, b\in B\};$$

also

$$\rho(x,A) = \rho(\{x\},A).$$

(*c*) The closed N-dimensional sphere of centre a, radius r is denoted by $S(a,r)$.

2. *Hausdorff measures.* In (3), Hausdorff defined the concept of an outer measure with respect to any function $h(t)$ which is continuous and increasing for $t \geqslant 0$ and is such that $h(0) = 0$. In particular we take $h(t) = t^\alpha$ for any $\alpha > 0$.

For any set E let

$$\delta - m^\alpha(E) = \inf \sum_{n=1}^{\infty} |I_n|^\alpha,$$

where the infimum is taken over all coverings of E by arbitrary sets I_n with $|I_n| \leqslant \delta$. Define $m^\alpha(E)$, the outer measure of E with respect to $h(t) = t^\alpha$ to be

$$m^\alpha(E) = \lim_{\delta\to 0} \delta - m^\alpha(E) = \sup_{\delta>0} \delta - m^\alpha(E).$$

It is well known that if $\alpha < \beta$ and $m^\alpha(E) < \infty$ then $m^\beta(E) = 0$. Thus there exists a unique non-negative number, $d(E)$, called the Hausdorff dimension of E such that

$$m^\alpha(E) = 0 \quad \text{for} \quad \alpha > d(E)$$

and

$$m^\alpha(E) = \infty \quad \text{for} \quad d(E) > \alpha > 0.$$

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In order to examine the local structure of E we define $d_x(E)$, the local dimension of E at x, ((5)), by

$$d_x(E) = \lim_{\epsilon \to 0} d(E \cap S(x, \epsilon)).$$

In particular, if E is a closed set and $x \notin E$, then $d_x(E) = 0$.

We remark here that $m^\alpha(E)$ can be computed from open coverings of E, i.e. we may restrict the sets I_n to form an open cover of E. If E is compact we need only consider finite open coverings of E.

In (6), Taylor discusses the known relationships existing between Hausdorff measures and generalized capacities and that paper contains the following result.

Theorem 1. *Let E be a compact set. Then*

(i) *if E has positive Hausdorff dimension then it has positive logarithmic capacity and*

(ii) *if E has Hausdorff dimension greater than one then it has positive Newtonian capacity.*

3. *General Cantor sets.* These sets are generalizations of the classical Cantor set.

Definition 1 ((7); (8), page 106). *A set E is said to be a general Cantor set if and only if it can be expressed in the form*

$$E = \bigcap_{n=1}^{\infty} \bigcup_{i_1, \ldots, i_n = 1}^{K} \Delta_{i_1 \ldots i_n},$$

where $K \geqslant 2$ is an integer and where the $\Delta_{i_1 \ldots i_n}$ are connected, compact sets satisfying

(i) $\Delta_{i_1 \ldots i_n} \supset \Delta_{i_1 \ldots i_n i_{n+1}}$,

(ii) $\Delta_1, \ldots, \Delta_K$ *are mutually disjoint,*

(iii) *there exists a constant A, $1 > A > 0$, such that*

$$|\Delta_{i_1 \ldots i_n i_{n+1}}| \geqslant A\, |\Delta_{i_1 \ldots i_n}| \quad (i_{n+1} = 1, \ldots, K),$$

(iv) *there exists a constant B, $1 > B > 0$, such that for $s \neq t$,*

$$\rho(\Delta_{i_1 \ldots i_n s}, \Delta_{i_1 \ldots i_n t}) \geqslant B\, |\Delta_{i_1 \ldots i_n}|.$$

Definition 2. *A general Cantor set is called a spherical Cantor set if and only if for each choice of $i_1, \ldots, i_n$ $\Delta_{i_1 \ldots i_n}$ is an N-dimensional sphere.*

The classical Cantor set is an example of a spherical Cantor set with the constants satisfying $N = 1$, $K = 2$ and $A = B = \frac{1}{3}$. Unless otherwise stated a general Cantor set will always have a decomposition in the notation of Definition 1. Using a similar terminology to Good ((2)), we say

(*a*) a fundamental interval is a set of the form $\Delta_{i_1 \ldots i_n}$,

(*b*) a fundamental system, $\mathscr{F}$, of E is a finite disjoint collection of fundamental intervals whose union covers E,

(*c*) the order of the fundamental interval $\Delta_{i_1 \ldots i_n}$ is n and

(*d*) the upper (and lower) order of a fundamental system, $\mathscr{F}$, of E is the maximum (and minimum) value of the order of a fundamental interval in $\mathscr{F}$. Further, define

$$\delta_n = \max |\Delta_{i_1 \ldots i_n}| \quad (i_1, \ldots, i_n = 1, \ldots, K).$$

We shall need the following result.

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LEMMA 1. *For any general Cantor set*, $\delta_n \to 0$ as $n \to \infty$.

Proof. The proof is trivial for $N = 1$ but a more complicated argument is needed to prove the general case.

For each n, let $\Delta^*_{i_1\ldots i_n}$ be such that

$$|\Delta^*_{i_1\ldots i_n}| = \delta_n$$

and let $\{j_n\}$ be a sequence with the property that for each value of m there exist infinitely many values of n such that

$$\Delta_{j_1\ldots j_m} \supset \Delta^*_{i_1\ldots i_n}.$$

Such a sequence can easily be inductively defined. Suppose now that

$$\delta_n \geqslant \delta \geqslant 0.$$

Then it follows that $$|\Delta_{j_1\ldots j_m}| \geqslant \delta.$$

Now define $\{i_n\}$ such that $i_n \neq j_n$ and choose $x_n \in \Delta_{j_1\ldots j_n i_{n+1}}$. Then for all m and n, $m > n$,

$$\begin{aligned} |x_n - x_m| &\geqslant \rho(\Delta_{j_1\ldots j_n i_{n+1}}, \Delta_{j_1\ldots j_m i_{m+1}}) \\ &\geqslant \rho(\Delta_{j_1\ldots j_n i_{n+1}}, \Delta_{j_1\ldots j_n j_{n+1}}) \\ &\geqslant B\,|\Delta_{j_1\ldots j_n}| \\ &\geqslant B\delta. \end{aligned}$$

Since the set $\{x_n\}$ is a bounded infinite set we see that $\delta = 0$. The lemma follows on observing that

$$\delta_1 \geqslant \delta_2 \geqslant \ldots \geqslant \delta_n \geqslant \ldots \geqslant 0.$$

From Lemma 1 we deduce that any general Cantor set is a discrete set ((1), page 277). The known results concerning the densities of these sets are summarized in the following theorems.

THEOREM 2 ((7); (8), page 106). (i) *Let E be a spherical Cantor set and let $N \geqslant 3$. Then $m^N(E) = 0$. Further, if $AK > 1$ then E has positive Newtonian capacity.*

(ii) *Let E be a general Cantor set and let $N = 1$ or 2. Then $m^N(E) = 0$ and E has positive logarithmic capacity.*

THEOREM 3 ((4)). *Let $\Delta_0, \Delta_1, \ldots, \Delta_K$ be geometrically similar compact, connected sets such that*

(i) $\Delta_0 \supset \Delta_j$ $(j = 1, \ldots, K)$, $(K \geqslant 2)$ *and*

(ii) $\Delta_1, \ldots, \Delta_K$ *are disjoint.*

By the similarity there exist transformations F_j $(j = 1, \ldots, K)$ of Δ_0 onto Δ_j with F_j the composition of a translation, a rotation and a (negative) magnification. Define

$$\Delta_{i_1\ldots i_n} = F_{i_1}\ldots F_{i_n}(\Delta_0)$$

and $$E = \bigcap_{n=1}^{\infty} \bigcup_{i_1,\ldots,i_n=1}^{K} \Delta_{i_1\ldots i_n}.$$

Then we have $0 < m^d(E) < \infty$ and $d = d(E)$ where d satisfies

$$|\Delta_0|^d = |\Delta_1|^d + \ldots + |\Delta_K|^d.$$

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It is clear that such a set is a general Cantor set and we say that E is derived from the pattern $\Delta_0, \Delta_1, \ldots, \Delta_K$. This theorem includes the well-known result of Hausdorff ((3)), i.e. that the dimension of the classical Cantor set is $\log 2 . (\log 3)^{-1}$.

We shall see that for any general Cantor set E,

$$0 < d(E) < N$$

and further, if $AK > 1$ then $d(E) > 1$. Theorem 1 then shows that both parts of Theorem 2 have been considerably extended.

The first result is to show that $d(E)$ is completely determined by the diameters of the fundamental intervals used in the definition of E.

LEMMA 2. *Let E be a general Cantor set and suppose that $\delta - M^\alpha(E)$ and $M^\alpha(E)$ are defined as for $\delta - m^\alpha(E)$ and $m^\alpha(E)$ respectively with the added restriction that the covering $\{I_n\}$ is a fundamental system of E. Then*

$$M^\alpha(E) \geqslant m^\alpha(E) \geqslant B^\alpha M^\alpha(E)$$

and so $M^\alpha(E)$ can be used to define $d(E)$.

Proof. Without loss of generality we can restrict our consideration to finite open coverings, say $\{I_1, \ldots, I_q\}$, of E (E is clearly a compact set) such that

$$I_j \cap E \neq \phi$$

and $$|I_j| < \min\{\rho(\Delta_s, \Delta_t) : s \neq t\} \quad (j = 1, \ldots, q).$$

Thus I_j has a non-empty intersection with precisely one fundamental interval of order one. From Lemma 1 we see that any neighbourhood of any point of $E \cap I_j$ contains at least two fundamental intervals of order n for some sufficiently large n. Define $s(j)$ to be the largest positive integer with the property that I_j has a non-empty intersection with precisely one fundamental interval of each of the orders $1, \ldots, s(j)$. Then if the latter of these fundamental intervals is Δ_j^* we note that I_j has a non-empty intersection with two fundamental intervals of order $s(j)+1$ and these are both contained in the same fundamental interval Δ_j^*; thus by Definition 1 (iv),

$$|I_j| \geqslant B|\Delta_j^*|.$$

It is easily proved that $$\bigcup_{j=1}^{q} \Delta_j^* \supset E,$$

and so we can replace the covering $\{I_1, \ldots, I_q\}$ by the fundamental system $\{\Delta_1^*, \ldots, \Delta_q^*\}$ of E which is such that

$$B^\alpha \sum_{j=1}^{q} |\Delta_j^*|^\alpha \leqslant \sum_{j=1}^{q} |I_j|^\alpha.$$

Lemma 2 now follows.

It is seen from Lemma 2 that $d(E)$ is completely determined by the diameters of the $\Delta_{i_1 \ldots i_n}$ and in this sense is independent of their geometrical shape. We next prove the basic result of this section and in doing so will use similar techniques to those contained in (2). The more elaborate methods contained in (2) do not, however, appear to yield any substantial improvement to the following result.

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THEOREM 4. *Let E be a general Cantor set. Then for all $n = 1, 2, \ldots$ and all*

$$i_1, \ldots, i_n = 1, \ldots, K$$

(i) $\sum_{j=1}^{K} |\Delta_{i_1\ldots i_n j}|^\alpha \leqslant |\Delta_{i_1\ldots i_n}|^\alpha$ *implies $d(E) \leqslant \alpha$ and*

(ii) $\sum_{j=1}^{K} |\Delta_{i_1\ldots i_n j}|^\beta \geqslant |\Delta_{i_1\ldots i_n}|^\beta$ *implies $d(E) \geqslant \beta$.*

Proof. The hypotheses of (i) imply that

$$\sum_{i_1,\ldots,i_n=1}^{K} |\Delta_{i_1\ldots i_n}|^\alpha$$

is bounded above for all n. Lemmas 1 and 2 show that

$$m^\alpha(E) < \infty,$$

which proves (i). The second half of the proof is more difficult. By Lemma 2 it is sufficient to consider fundamental systems of E. Suppose that $\mathscr{F} = \{\Delta_1^*, \ldots, \Delta_p^*\}$ is one such system and let m and n be the upper and lower orders of $\mathscr{F}$. In particular there exists

$$\Delta_{i_1\ldots i_m} \in \mathscr{F}.$$

By the disjointness of $\mathscr{F}$ we see that

$$\Delta_{i_1}, \Delta_{i_1 i_2}, \ldots, \Delta_{i_1\ldots i_{m-1}} \notin \mathscr{F}$$

and so, by the covering property of $\mathscr{F}$, it follows that

$$\Delta_{i_1\ldots i_{m-1}1}, \Delta_{i_1\ldots i_{m-1}2}, \ldots, \Delta_{i_1\ldots i_{m-1}K} \in \mathscr{F}.$$

We replace these K sets of $\mathscr{F}$ by the set $\Delta_{i_1\ldots i_{m-1}}$ to form a new covering $\mathscr{F}^*$. The hypothesis of (ii) implies that

$$\sum_{\Delta\in\mathscr{F}} |\Delta|^\beta \geqslant \sum_{\Delta\in\mathscr{F}^*} |\Delta|^\beta,$$

and repeated applications of this result yield

$$\sum_{\Delta\in\mathscr{F}} |\Delta|^\beta \geqslant \sum_{i_1,\ldots,i_n=1}^{K} |\Delta_{i_1\ldots i_n}|^\beta.$$

Further applications yield
$$\sum_{\Delta\in\mathscr{F}} |\Delta|^\beta \geqslant \sum_{j=1}^{K} |\Delta_j|^\beta$$

and so
$$M^\beta(E) \geqslant \sum_{j=1}^{K} |\Delta_j|^\beta > 0.$$

Finally, by Lemma 2, $d(E) \geqslant \beta$.

Theorem 4 is now proved and we list some immediate consequences.

COROLLARY 1. *Suppose that*

$$0 < A_j \leqslant \frac{|\Delta_{i_1\ldots i_n j}|}{|\Delta_{i_1\ldots i_n}|} \leqslant C_j \leqslant 1, \quad A_j < 1 \quad (j = 1, \ldots, K).$$

Then $d(E)$ satisfies
$$\sum_{j=1}^{K} A_j^{d(E)} \leqslant 1 \leqslant \sum_{j=1}^{K} C_j^{d(E)}.$$

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Proof. If
$$\sum_{j=1}^{K} A_j^{\alpha} = 1$$
then
$$\sum_{j=1}^{K} |\Delta_{i_1\ldots i_n j}|^{\alpha} \geqslant |\Delta_{i_1\ldots i_n}|^{\alpha},$$
and so $d(E) \geqslant \alpha$, i.e.
$$\sum_{j=1}^{K} A_j^{d(E)} \leqslant 1.$$
The second inequality is proved similarly.

In particular we may take
$$A_1 =, \ldots, = A_K = A.$$
Then
$$d(E) \geqslant \frac{\log K}{-\log A} > 0$$
and so every general Cantor set has positive Hausdorff dimension.

COROLLARY 2. *Suppose that*
$$\frac{|\Delta_{i_1\ldots i_n j}|}{|\Delta_{i_1\ldots i_n}|} \to A_j < 1$$
uniformly as $n \to \infty$. *Then* $d(E)$ *satisfies*
$$A^{d(E)} +, \ldots, + A_K^{d(E)} = 1.$$

Proof. This follows easily from Corollary 1 and the obvious fact that for any n,
$$d(E) = \max\{d(i_1, \ldots, i_n): i_1, \ldots, i_n = 1, \ldots, K\},$$
where $d(i_1, \ldots, i_n) = d(E \cap \Delta_{i_1\ldots i_n})$.

In particular, if
$$A_1 =, \ldots, = A_K = A$$
then
$$d(E) = \frac{-\log K}{\log A}.$$

COROLLARY 3. *Suppose that*
$$\{\Delta_{i_1\ldots i_n}: i_1, \ldots, i_n = 1, \ldots, K\} \quad \textit{and} \quad \{I_{i_1\ldots i_n}: i_1, \ldots, i_n = 1, \ldots, K\}$$
yield two general Cantor sets E *and* F *respectively in the sense of Definition* 1 *and that*
$$|\Delta_{i_1\ldots i_n}| \geqslant |I_{i_1\ldots i_n}|.$$
Then $d(E) \geqslant d(F)$. *In particular, if*
$$|\Delta_{i_1\ldots i_n}| = |I_{i_1\ldots i_n}|$$
then $d(E) = d(F)$.

This result is a direct corollary of Lemma 2. In this sense $d(E)$ is independent of the value of B. This same independence is to some extent exhibited in the result of Theorem 4.

Finally, Theorem 3 is an immediate consequence of the more general Theorem 4.

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4. *The range of values of $d(E)$.* We are concerned here with the evaluation of the possible range of values of $d(E)$. General Cantor sets may be classified as follows: (*a*) the class of general Cantor sets with $N = 1$ (these are automatically spherical Cantor sets); (*b*) the class of spherical Cantor sets with $N \geqslant 2$; and finally (*c*) the class of remaining general Cantor sets. In this section we shall see that there are certain distinct properties of $d(E)$ associated with each class of general Cantor sets E. The following two theorems are well known and are included here for the sake of completeness.

THEOREM 5. *Let $N = 1$ and suppose that α, $1 > \alpha > 0$, and a positive integer $M \geqslant 2$ are given. Then there exists a general Cantor set E with $K = M$ and $d(E) = \alpha$.*

Proof. Choose A such that
$$MA^{\alpha} = 1,$$
then
$$MA < 1$$
and so we can find M disjoint closed subintervals $\Delta_1, \ldots, \Delta_M$ of $\Delta_0 = [0, 1]$ each having length A. Regarding this as a pattern (in the sense of Theorem 3) which generates a general Cantor set E we see that $d(E) = \alpha$.

In the notation of the above proof let Δ_0^N be the N-fold Cartesian product of Δ_0 with itself and let $\Delta_1^N, \ldots, \Delta_{M^n}^N$ be the M^n distinct sets of the form
$$\Delta_{i_1} \times \ldots \times \Delta_{i_n} \quad (1 \leqslant i_1, \ldots, i_n \leqslant K).$$
Then regarding this as a pattern generating a general Cantor set F we see from Theorem 3 that $d(F) = Nd(E)$. This can be used to prove the following result.

THEOREM 6. *Let $N \geqslant 2$ and α, $0 < \alpha < N$, be given. Then there exists a general Cantor set E such that $d(E) = \alpha$.*

THEOREM 7. *Let $N \geqslant 2$ and α, $0 < \alpha < N$, be given. Then there exists a spherical Cantor set E such that $d(E) = \alpha$.*

Proof. Let C be a closed N-dimensional cube of side one contained in $S(0, 1)$. We can subdivide this cube into n^N disjoint cubes, each of side $(n+1)^{-1}$ and each having its faces parallel to those of the original cube. Now each of the smaller cubes contains a sphere of radius $\frac{1}{2}(n+1)^{-1}$, i.e. we can find n^N disjoint spheres
$$S(a_j, \tfrac{1}{2}(n+1)^{-1}) \quad (j = 1, \ldots, n^N),$$
contained in $S(0, 1)$. Regarding this as a pattern yielding a spherical Cantor set F we see that
$$d(F) = \frac{N \log n}{\log 2(n+1)}.$$

For any given α, $0 < \alpha < N$, choose n sufficiently large so that
$$N > \frac{N \log n}{\log 2(n+1)} \geqslant \alpha.$$

Define θ, $0 < \theta \leqslant 1$, by
$$\frac{N \log n}{\log 2(n+1) - \log \theta} = \alpha,$$

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and note that the pattern formed by $S(0, 1)$ and the n^N spheres $S(a_j, \theta/(2(n+1)))$ yields a spherical Cantor set E such that

$$d(E) = \frac{N \log n}{\log 2(n+1) - \log \theta}$$
$$= \alpha.$$

The latter three theorems imply that for E in each class, $d(E)$ can take any value in the range $0 < d(E) < N$. We next show that in all cases this is precisely the range of values $d(E)$ can take, i.e. the bounds 0 and N are the best possible in the general case and are never attained. We remark here that we have already shown that for any general Cantor set E, $d(E) > 0$ (Theorem 4, Corollary 1).

THEOREM 8. *Let E be a general Cantor set with $N = 1$. Then*

$$0 < \frac{\log K}{-\log A} \leqslant d(E) \leqslant \frac{\log K}{\log K - \log(1-B)} < 1.$$

Proof. Since $N = 1$ it is immediate that

$$\sum_{j=1}^{K} |\Delta_{i_1 \ldots i_n j}| \leqslant (1-B)\,|\Delta_{i_1 \ldots i_n}|,$$

and so for any α, $0 < \alpha < 1$, we have

$$\sum_{j=1}^{K} \left\{\frac{|\Delta_{i_1 \ldots i_n j}|}{|\Delta_{i_1 \ldots i_n}|}\right\}^{\alpha} \leqslant K\left(\frac{1-B}{K}\right)^{\alpha}.$$

Applying Theorem 4 to the case

$$\alpha = \frac{\log K}{\log K - \log(1-B)}$$

the upper bound follows.

The classical Cantor set has dimension $\log 2 \,.\, (\log 3)^{-1}$ and thus these bounds are best possible.

THEOREM 9. *Let $N \geqslant 2$ and let E be a general Cantor set. Then*

$$0 < \frac{\log K}{-\log A} \leqslant d(E) \leqslant N - \frac{\log\{1-(B/4+2B)^N\}}{\log A} < N.$$

Proof. The proof is long and in essence consists of first representing E in a form in which more information relating $m^N(\Delta_{i_1 \ldots i_n})$ to $|\Delta_{i_1 \ldots i_n}|$ is available. This information is then used to deduce the required result.

Define

$$I_{i_1 \ldots i_n} = \{x : \rho(x, \Delta_{i_1 \ldots i_n}) \leqslant \tfrac{1}{4}B\,|\Delta_{i_1 \ldots i_n}|\}.$$

Then clearly
$$|I_{i_1 \ldots i_n}| = (1 + \tfrac{1}{2}B)\,|\Delta_{i_1 \ldots i_n}|$$

and hence
$$E = \bigcap_{n=1}^{\infty} \bigcup_{i_1, \ldots, i_n = 1}^{K} I_{i_1 \ldots i_n} \qquad \text{(i)}$$

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is a suitable representation for the general Cantor set E with the constant

$$B^* = \frac{B}{2+B}$$

replacing the constant B of Definition 1 (iv). If $x \in \Delta_{i_1\ldots i_n}$ then

$$\begin{aligned} I_{i_1\ldots i_n} &\supset S(x, \tfrac{1}{4}B\,|\Delta_{i_1\ldots i_n}|) \\ &= S(x, \tfrac{1}{2}B^*\,|I_{i_1\ldots i_n}|), \end{aligned} \tag{ii}$$

and so $$\gamma\,|I_{i_1\ldots i_n}|^N \geqslant m^N(I_{i_1\ldots i_n}) \geqslant \gamma(\tfrac{1}{2}B^*)^N\,|I_{i_1\ldots i_n}|^N, \tag{iii}$$

where γ is the N-dimensional measure of an N-dimensional sphere of unit radius. Since the sets
$$\{x\colon \rho(x, \Delta_{i_1\ldots i_n i_{n+1}}) < \tfrac{1}{2}B\,|\Delta_{i_1\ldots i_n}|,\ i_{n+1} = 1, \ldots, K\}$$

are open and disjoint and since each of them has a non-empty intersection with $\Delta_{i_1\ldots i_n}$ it follows from the connectedness of $\Delta_{i_1\ldots i_n}$ that they do not cover $\Delta_{i_1\ldots i_n}$. Hence there exists a point $y \in \Delta_{i_1\ldots i_n}$ such that for any $i_{n+1} = 1, \ldots, K$

$$\rho(y, \Delta_{i_1\ldots i_n i_{n+1}}) \geqslant \tfrac{1}{2}B\,|\Delta_{i_1\ldots i_n}|,$$

i.e. with $$\rho(y, I_{i_1\ldots i_{n+1}}) \geqslant \tfrac{1}{2}B^*\,|I_{i_1\ldots i_n}|.$$

By (i) and (ii) it follows that

$$m^N(I_{i_1\ldots i_n}) \geqslant \sum_{j=1}^{K} m^N(I_{i_1\ldots i_n j}) + \gamma(\tfrac{1}{2}B^*\,|I_{i_1\ldots i_n}|)^N$$

and so by (iii) $$m^N(I_{i_1\ldots i_n})\{1-(\tfrac{1}{2}B^*)^N\} \geqslant \sum_{j=1}^{K} m^N(I_{i_1\ldots i_n j}).$$

Writing $$Q = 1-(\tfrac{1}{2}B^*)^N$$

we see that $$m^N(I_{i_1\ldots i_n}) \geqslant Q^{-1}\sum_{j=1}^{K} m^N(I_{i_1\ldots i_n j})$$

and repeated applications of this inequality clearly yield

$$Q^{-n}\sum_{i_1,\ldots,i_n=1}^{K} m^N(I_{i_1\ldots i_n}) = O(1) \quad \text{as} \quad n \to \infty.$$

Applying (iii) we see that

$$Q^{-n}\sum_{i_1,\ldots,i_n=1}^{K} |I_{i_1\ldots i_n}|^N = O(1) \quad \text{as} \quad n \to \infty. \tag{iv}$$

From Definition 1 (iii) there exists a constant D such that

$$|I_{i_1\ldots i_n}| \geqslant DA^n$$

and so for any β, $0 < \beta < N$,

$$\begin{aligned} |I_{i_1\ldots i_n}|^N &\geqslant |I_{i_1\ldots i_n}|^{N-\beta}\,|I_{i_1\ldots i_n}|^\beta \\ &\geqslant |I_{i_1\ldots i_n}|^{N-\beta}\,(DA^n)^\beta. \end{aligned}$$

Thus from (iv), $$(Q^{-1}A^\beta)^n \sum_{i_1,\ldots,i_n=1}^{K} |I_{i_1\ldots i_n}|^{N-\beta} = O(1) \quad \text{as} \quad n \to \infty,$$

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and so
$$d(E) \leqslant N - \beta$$
providing
$$Q \leqslant A^{\beta},$$
and this is so if
$$\beta \leqslant \frac{\log Q}{\log A}.$$
It now follows that
$$d(E) \leqslant N - \frac{\log Q}{\log A} < N,$$
which completes the proof of Theorem 9.

To summarize: in each of the three classes of general Cantor sets O and N are the lower and upper bounds for the dimension and these bounds are the best possible in the general case. We now show that provided that $N \geqslant 2$ a stronger result is true for the class of spherical Cantor sets. Three lemmas are first proved.

LEMMA 3. *Let $N \geqslant 2$ and define*
$$F_N(M) = \sup d(E),$$
where E is a spherical Cantor set with $K = M$. Then $F_N(M)$ is unaltered if we add the further restriction that E should be derived from a pattern in the sense of Theorem 3.

Proof. Define
$$F_N^*(M) = \sup d(E),$$
where E, a spherical Cantor set with $K = M$, is derived from a pattern in the sense of Theorem 3. Clearly, then,
$$F_N^*(M) \leqslant F_N(M).$$

Let E be a spherical Cantor set with $K = M$. Since all of the $\Delta_{i_1 \ldots i_n}$ are spheres it is geometrically evident that
$$|\Delta_{i_1 \ldots i_n i_{n+1}}| \leqslant (1-B)\,|\Delta_{i_1 \ldots i_n}|,$$
and so we can define unique, positive numbers $\alpha_{i_1 \ldots i_n}$ satisfying
$$\sum_{j=1}^{K} |\Delta_{i_1 \ldots i_n j}|^{\alpha_{i_1 \ldots i_n}} = |\Delta_{i_1 \ldots i_n}|^{\alpha_{i_1 \ldots i_n}}.$$
Now if
$$\beta > \sup\{\alpha_{i_1 \ldots i_n}\}$$
then
$$\sum_{j=1}^{K} |\Delta_{i_1 \ldots i_n j}|^{\beta} \leqslant |\Delta_{i_1 \ldots i_n}|^{\beta}$$
and so, by Theorem 4,
$$d(E) \leqslant \beta,$$
and hence
$$d(E) \leqslant \sup\{\alpha_{i_1 \ldots i_n}\}.$$
For each sequence $i_1, \ldots, i_n$ we can regard the spheres
$$\Delta_{i_1 \ldots i_n}, \Delta_{i_1 \ldots i_n 1}, \ldots, \Delta_{i_1 \ldots i_n K}$$
as a pattern yielding a spherical Cantor set $E_{i_1 \ldots i_n}$ of dimension $\alpha_{i_1 \ldots i_n}$. Thus
$$\alpha_{i_1 \ldots i_n} \leqslant F_N^*(M) \quad (K = M)$$
and so
$$d(E) \leqslant F_N^*(M).$$

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From this we deduce $$F_N(M) \leqslant F_N^*(M)$$

and hence $$F_N(M) = F_N^*(M).$$

This is the statement of Lemma 3.

We next show that the dimension of the spherical Cantor set is small whenever the pattern is such that one of the subsets has a diameter close to that of the whole.

LEMMA 4. *Let E be a spherical Cantor set derived from the pattern*

$$S(0,1),\ S(a_1, r_1), \ldots, S(a_K, r_K)$$

and suppose that for some j $(1 \leqslant j \leqslant K)$,

$$r_j \geqslant \frac{4K^2}{4K^2+9}.$$

Then $d(E) \leqslant 1\frac{1}{2}$.

Proof. Without loss of generality we take

$$r_1 \geqslant \frac{4K^2}{4K^2+9}.$$

Then for $j = 2, \ldots, K$ we have $$r_j < 1 - r_1.$$

Suppose that $$r_1 = 1 - \epsilon,$$

where $$0 < \epsilon \leqslant \frac{9}{4K^2+9}.$$

Then for $j = 2, \ldots, K$, $$r_j < \epsilon.$$

Let α be defined by $$\sum_{j=1}^{K} r_j^\alpha = 1$$

and β by $$(1-\epsilon)^\beta + (K-1)\,\epsilon^\beta = 1.$$

Since $$r_1 = 1-\epsilon, \quad r_j < \epsilon \quad (j = 2, \ldots, K),$$

it follows that $$\sum_{j=1}^{K} r_j^\alpha < (1-\epsilon)^\alpha + (K-1)\,\epsilon^\alpha$$

and thus $$\alpha < \beta.$$

Writing $$t\epsilon = 1,$$

we see that $$t^\beta - (t-1)^\beta = K-1.$$

Now by the Mean Value Theorem,

$$t^{\frac{3}{2}} - (t-1)^{\frac{3}{2}} > \tfrac{3}{2}\sqrt{(t-1)}$$
$$> K-1.$$

It follows that $\alpha < \beta < \frac{3}{2}$ as required.

LEMMA 5. *Let δ $(1 > \delta > 0)$ be a constant and let the disjoint spheres $S(a_j, r_j)$ $(j = 1, \ldots, K)$ be contained in $S(0,1)$ such that for each $j = 1, \ldots, K$, $r_j \leqslant \delta$. Then there exists a function of δ and K, say $V_\delta(K)$, such that*

$$\sum_{j=1}^{K} r_j^N \leqslant V_\delta(K) < 1.$$

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We remark that $\sum_{j=1}^{K} r_j^N$ is, in this case, the ratio of the sum of the volumes of $S(a_j, r_j)$ $(j = 1, \ldots, K)$ to the whole volume of $S(0, 1)$. Further, if δ is a function of K we write $V(K)$ for $V_\delta(K)$.

Proof. This is by induction on K and is trivial for $K = 2$. We assume the result for $K = 2, \ldots, m$ and choose $\mu_m > 0$ such that

$$V_\delta(m) + \mu_m < 1.$$

Consider now $m+1$ spheres satisfying the hypotheses of the lemma. Without loss of generality we may assume $r_1 \geqslant r_2 \geqslant, \ldots \geqslant, r_m \geqslant r_{m+1}$. Now if

$$r_{m+1} \leqslant \mu_m$$

then

$$\begin{aligned} r_1^N + \ldots + r_m^N + r_{m+1}^N &\leqslant r_1^N + \ldots + r_m^N + \mu_m \\ &\leqslant V_\delta(m) + \mu_m \\ &< 1. \end{aligned} \tag{i}$$

Alternatively, we have

$$0 < \mu_m \leqslant r_j \leqslant \delta < 1 \quad (j = 1, \ldots, m+1). \tag{ii}$$

For such collections of spheres we may regard each collection as representing a point in some Euclidean space (taking the radii and the coordinates of the centres of the spheres as the coordinates in the Euclidean space) and, if we allow the spheres to touch, the set of points in the Euclidean space (corresponding to the collections of spheres satisfying (ii) and the hypotheses of the lemma) is clearly compact. The function

$$\sum_{j=1}^{m+1} r_j^N$$

defined on this set is continuous and hence attains its least upper bound which cannot be one since this would imply the N-dimensional measure of the $m+1$ spheres equals that of the sphere $S(0, 1)$. This together with (i) proves the lemma.

Theorem 10. *Let $N \geqslant 2$. Then if $F_N(M)$ is as defined in Lemma 3, $F_N(M) < N$ and $F_N(M) \to N$ as $M \to \infty$.*

Thus if E is any spherical Cantor set with K as in Definition 1, then

$$d(E) \leqslant F_N(K) < N.$$

Proof. Clearly $F_N(M)$ is a non-decreasing function of M and by Theorems 7 and 9

$$F_N(M) \to N \quad \text{as} \quad M \to \infty.$$

It thus suffices to prove $F_N(M) < N$.

By Lemma 3 we may restrict our attention to spherical Cantor sets derived from a pattern. There is no loss of generality in assuming this pattern to be of the form

$$S(0, 1), \quad S(a_1, r_1), \quad \ldots, \quad S(a_K, r_K).$$

Further, Lemma 4 implies that we may assume

$$r_j \leqslant \frac{4K^2}{4K^2+9} \quad (j = 1, \ldots, K)$$

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and in this case Lemma 5 implies there exists a function $V(K) < 1$ such that

$$\sum_{j=1}^{K} r_j^N \leqslant V(K) < 1.$$

In this case we need the following result.

LEMMA 6. *Suppose that* $N \geqslant 2$ *and let* $\epsilon > 0$ *be such that*

$$\sum_{j=1}^{K} r_j^{N-\epsilon} = 1 \quad (0 < r_j < 1, \quad j = 1, \ldots, K).$$

Then
$$\sum_{j=1}^{K} r_j^N \geqslant 1 - K\epsilon.$$

Proof. By the Mean Value Theorem

$$1 - r_j^\epsilon = \epsilon x^{\epsilon - 1}$$

for some $x \in (r_j, 1)$ and so
$$1 - r_j^\epsilon \leqslant \epsilon \max\{1, r_j^{\epsilon-1}\}.$$

Thus
$$\begin{aligned} 1 - \sum_{j=1}^{K} r_j^N &= \sum_{j=1}^{K} r_j^{N-\epsilon}(1 - r_j^\epsilon) \\ &\leqslant \epsilon \sum_{j=1}^{K} \max\{r_j^{N-1}, r_j^{N-\epsilon}\} \\ &< K\epsilon. \end{aligned}$$

Theorem 3 and Lemma 6 together imply $d(E) = N - \epsilon$ where $V(K) \geqslant 1 - K\epsilon$. Thus

$$d(E) \leqslant N - \left\{\frac{1 - V(K)}{K}\right\}.$$

Since
$$\tfrac{3}{2} \leqslant N - \left\{\frac{1 - V(K)}{K}\right\}$$

it follows that
$$F_N(K) \leqslant N - \left\{\frac{1 - V(K)}{K}\right\}$$

and Theorem 10 is proved.

This latter result shows that for N greater than or equal to 2, spherical Cantor sets can have dimension approximately equal to N only if K is correspondingly large. Theorem 6, however, shows that there exist general Cantor sets with dimension arbitrarily close to N and with $K = 2^N$, i.e. K need not be correspondingly large.

5. *The function* $d_x(E)$. We end this paper with a short discussion of $d_x(E)$ regarded as a function of x. We note that $d_x(E)$ is defined in (5) and the following results generalize some of the work contained in that paper. We first prove a lemma.

LEMMA 7. *Let* E *be a general Cantor set and let* $x \in E$. *Then for all sufficiently small positive* ϵ, *there exist positive integers* m *and* n *such that*

$$\{x\} \subset E \cap \Delta_{i_1 \ldots i_n} \subset E \cap S(x, \epsilon) \subset E \cap \Delta_{i_1 \ldots i_m},$$

where
$$\{x\} = \bigcap_{p=1}^{\infty} \Delta_{i_1 \ldots i_p}.$$

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In particular,
$$d_x(E) = \lim_{p\to\infty} d(E \cap \Delta_{i_1\ldots i_p}).$$

Proof. The existence of n follows from Lemma 1. Now if $s_1 \neq t_1$,
$$\begin{aligned}\rho(\Delta_{i_1\ldots i_p s_1\ldots s_q}, \Delta_{i_1\ldots i_p t_1\ldots t_q}) &\geqslant \rho(\Delta_{i_1\ldots i_p s_1}, \Delta_{i_1\ldots i_p t_1})\\ &\geqslant B\,|\Delta_{i_1\ldots i_p}|\\ &\geqslant B^* A^{p+q}\end{aligned}$$
for some positive constant B^*. Thus if ϵ is sufficiently small there exists a positive integer m such that
$$B^* A^m > \epsilon.$$
In this case, if $y \in E \cap S(x,\epsilon)$ then $\quad y \in E \cap \Delta_{i_1\ldots i_m}$
and so
$$E \cap S(x,\epsilon) \subset E \cup \Delta_{i_1\ldots i_m}.$$
Lemma 7 is now proved.

Let E be a spherical Cantor set, F a spherical Cantor set with $N = 1$, $K = 2$ and suppose that E is derived from the system
$$\{I_{i_1\ldots i_n} : n = 1, 2, \ldots;\ i_1, \ldots, i_n = 1, 2\}$$
and F from the system
$$\{\Delta_{i_1\ldots i_n} : n = 1, 2, \ldots;\ i_1, \ldots, i_n = 1, 2\}.$$
Then there exists a natural homeomorphism from E to F defined by $x^* \to x$ where $x^* \in E$, $x \in F$ and where
$$\{x\} = \bigcap_{n=1}^{\infty} \Delta_{i_1\ldots i_n}, \quad \{x^*\} = \bigcap_{n=1}^{\infty} I_{i_1\ldots i_n}.$$

Definition 3. *In the above notation let $f(x^*)$ be a function defined on E such that $0 < f(x^*) < 1$ for $x^* \in E$. Then we say that F is equivalent to (E,f) if and only if for all $x \in F$,*
$$d_x(F) = f(x^*).$$

It is easily shown from Lemma 7 and the definition of $d_x(E)$ that if, for a fixed general Cantor set F, we regard $d_x(F)$ as a function of x defined on the whole space then
$$\limsup_{y\to x} d_y(F) = d_x(F).$$
In particular, $d_x(F)$ is an upper semi-continuous function of x.

Theorem 12. *Let E be the classical Cantor set and let $f(x^*)$ be defined and continuous on E and such that $0 < f(x^*) < 1$. Then there exists a general Cantor set F equivalent to (E,f).*

Theorem 13. *There exists a general Cantor set F such that $d_x(F)$ is discontinuous at a point of F when regarded as a function with domain F.*

Before proving these theorems we remark that it seems probable that for any function $f(x^*)$ satisfying

(*a*) $\limsup_{y^*\to x^*} f(y^*) = f(x^*)$,

(*b*) $f(x^*) = 0$ for all $x^* \notin E$, and

(*c*) $0 < f(x^*) < 1$ for all $x^* \in E$

there exists a general Cantor set F with F equivalent to (E,f). This, however, seems difficult to prove. We end with the proofs of Theorems 12 and 13.

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Proof (of Theorem 12). Since E is compact there exist constants m and M such that for $x^* \in E$,

$$0 < m \leqslant f(x^*) \leqslant M < 1.$$

Define
$$M(i_1, \ldots, i_n) = \sup\{f(x^*): x^* \in E \cap I_{i_1 \ldots i_n}\},$$
$$m(i_1, \ldots, i_n) = \inf\{f(x^*): x^* \in E \cap I_{i_1 \ldots i_n}\}$$
and
$$a(i_1 \ldots i_n) = \exp\left\{\frac{-2\log 2}{M(i_1, \ldots, i_n) + m(i_1, \ldots, i_n)}\right\}.$$

The continuity of $f(x^*)$ implies that if

$$\{x^*\} = \bigcap_{n=1}^{\infty} I_{i_1 \ldots i_n}$$

then $M(i_1, \ldots, i_n)$ decreases to $f(x^*)$ and $m(i_1, \ldots, i_n)$ increases to $f(x^*)$ as $n \to \infty$. Further we note that

$$a(i_1 \ldots i_n) \geqslant 2^{-1/m} > 0$$

and
$$a(i_1 \ldots i_n 1) + a(i_1 \ldots i_n 2) \leqslant 2^{1-(1/M)} < 1.$$

Thus we can construct a general Cantor set F with $N = 1$, $K = 2$ and such that

$$|\Delta_{i_1 \ldots i_n i_{n+1}}| = a(i_1 \ldots i_n i_{n+1}) \, |\Delta_{i_1 \ldots i_n}|.$$

Consider now a fixed sequence $\{j_n\}$ with

$$\{x\} = \bigcap_{n=1}^{\infty} \Delta_{j_1 \ldots j_n}.$$

By the monotonicity of $M(i_1, \ldots, i_n)$ and of $m(i_1, \ldots, i_n)$ we see that for all $i_1, \ldots, i_{m+1}$,

$$2^{-1/[m(j_1, \ldots, j_n)]} \leqslant \frac{|\Delta_{j_1 \ldots j_n i_1 \ldots i_{m+1}}|}{|\Delta_{j_1 \ldots j_n i_1 \ldots i_m}|} \leqslant 2^{-1/[M(j_1, \ldots, j_n)]},$$

and so for the general Cantor set $F \cap \Delta_{j_1 \ldots j_n}$,

$$m(j_1, \ldots, j_n) \leqslant d(F \cap \Delta_{j_1 \ldots j_n}) \leqslant M(j_1, \ldots, j_n).$$

From Lemma 7 we deduce that $d_x(F) = f(x^*)$ as required.

Corollary ((5)). *There exists a general Cantor set with the property that if $x, y \in F$ and $x \neq y$, then $d_x(F) \neq d_y(F)$.*

Proof (of Theorem 13). We shall construct such a set. Let $\sigma_n = i_1, \ldots, i_n$ denote the sequence given *by* $i_1 = i_2 = i_3 =, \ldots, = i_n = 1$. It is easily seen that we can construct a general Cantor set, E, from $\{\Delta_{i_1 \ldots i_n}: n = 1, 2, \ldots; i_1, \ldots, i_n = 1, 2\}$ which is such that

$$\frac{|\Delta_{\sigma_n 2 j_1 \ldots j_{n+1}}|}{|\Delta_{\sigma_n 2 j_1 \ldots j_n}|} = \begin{cases} \frac{1}{5} & \textit{if} \;\; j_1 = 1, \\ \frac{1}{3} & \text{if} \;\; j_1 = 2, \end{cases}$$

and
$$|\Delta_{\sigma_{n+1}}| = \tfrac{1}{3}\,|\Delta_{\sigma_n}|.$$

This defines the ratios $|\Delta_{i_1 \ldots i_n i_{n+1}}|/|\Delta_{i_1 \ldots i_n}|$ for all choices of $i_1, \ldots, i_{n+1}$. Note that for any n,

$$d(E \cap \Delta_{\sigma_n 22}) = \frac{\log 2}{\log 3}$$

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and
$$d(E \cap \Delta_{\sigma_n 21}) = \frac{\log 2}{\log 5},$$

and so if
$$\{x\} = \bigcap_{n=1}^{\infty} \Delta_{j_1 \ldots j_n},$$

where
$$j_1 j_2 \ldots = \sigma_m 2 s_1 s_2 \ldots,$$

then
$$d_x(E) = \begin{cases} \dfrac{\log 2}{\log 5} & \text{if} \quad s_1 = 1, \\ \dfrac{\log 2}{\log 3} & \text{if} \quad s_1 = 2. \end{cases}$$

Clearly then, $d_x(E)$ when regarded as a function defined on E is discontinuous at the point x_0 where
$$\{x_0\} = \bigcap_{n=1}^{\infty} \Delta_{\sigma_n}.$$

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