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OPTIMAL ALLOCATION OF RELEVATIONS IN COHERENT SYSTEMS

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Abstract

In this paper we study the allocation problem of relevations in coherent systems. The optimal allocation strategies are obtained by implementing stochastic comparisons of different policies according to the usual stochastic order and the hazard rate order. As special cases of relevations, the load-sharing and minimal repair policies are further investigated. Sufficient (and necessary) conditions are established for various stochastic orderings. Numerical examples are also presented as illustrations.

Keywords: Relevation; coherent system; load-sharing model; minimal repair; stochastic orders

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Secondary 60E15; 60K10

1. Introduction

As one of the most effective strategies, redundancy allocation is widely used in the area of industrial engineering to improve the reliability of coherent systems. The basic concepts used in reliability theory were introduced in the classical monograph by Barlow and Proschan [2]. Redundancy allocations can be performed in different ways in practical scenarios such as hot standby redundancy, cold standby redundancy, the load-sharing model, minimal repair, (im)perfect repair, and so on.

On one hand, hot standby means putting a redundant component in parallel with the original component of the system so that the two components work simultaneously. A large number of researchers have carried out relevant studies of the allocation of hot standbys; see for example [15], [16], [25], [33], [36], and [37]. However, as discussed in [28], the surviving component under the hot standby policy will be subjected to greater working pressure when the other component fails. As a result, the failure probability of the latter component may become greater, which is called the *load-sharing model* in the literature. Relevant studies in this respect include [13], [31], and [34].

On the other hand, cold standby means that after the initial component fails, a new redundant unit is immediately adopted to replace the broken one, for which the intermediate replacement time is negligible. The resulting lifetime can be treated as the convolution of the lifetimes of

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the original component and the cold redundancy. For comprehensive studies of the allocation of cold standbys, interested readers may refer to [11], [22], [32], and [35]. However, it is usually difficult and/or expensive to replace a broken component with a new one. Instead, it may be more appropriate and economical to repair it. In order to take the failed component back to its previous working state, we can conduct a 'relevation redundancy' policy to replace the broken component with a 'new' component that has the same age as the failed one. The operation of relevation was originally introduced by Krakowski [20], and it was further shown by Baxter [4] that the relevation transform generates the non-homogeneous Poisson process. Later on, Shanthikumar and Baxter [30] studied some closure properties of the relevation transform, and Kapodistria and Psarrakos [18] constructed a sequence of a random variables with the help of the relevation transform. Recently, Psarrakos and Di Crescenzo [24] and Di Crescenzo, Kayal, and Toomaj [14] introduced a residual inaccuracy measure and a past inaccuracy measure based on the relevation transform, respectively. For other recent work on relevation in reliability systems, interested readers may refer to [6], [23], and [26].

If the broken component is replaced with a new spare having the same lifetime distribution as the failed one, it is called a *perfect repair*; see [12], [21], and [27]. However, in some practical situations (e.g. warranty coverage, integrated-circuit panels in a television set, and water pumps in a car), it might be more practical and more resource-saving to *minimally repair* the failed units so as to bring them back to the working state immediately before failure. For a more detailed study of the effect of minimal repair in reliability systems, one may refer to [1], [3], [23], and [38]. It will be stated later that relevation redundancy includes the minimal repair policy as a special case.

Consider a component with lifetime T and a redundancy having lifetime S, where T and S have absolutely continuous survival functions $\overline{F}$ and $\overline{G}$, respectively. Then $T+S$ can be understood as the aggregate lifetime of the original component and the cold standby redundancy, and its survival function be expressed as

$$\mathbb{P}(T+S>t)=\overline{F}(t)+\int_0^t \overline{G}(t-x)f(x)\,\mathrm{d}x,$$

where $f(\cdot)$ is the probability density function of the random variable T. If the component with lifetime T fails at time $x>0$, and is immediately replaced by another new component with lifetime S with the same age as the failed component, the reliability of the relevation transform of T and S (denoted as $T\#S$) is given by

$$\mathbb{P}(T\#S>t)=\mathbb{P}(T+S_T>t)=\overline{F}(t)+\overline{G}(t)\int_0^t \frac{f(x)}{\overline{G}(x)}\,\mathrm{d}x,$$

where $S_T:=[S-T \mid S>T]$ is the residual lifetime of S at the random time T. For more discussions of relevation redundancy, interested readers can refer to [4], [26], and [30].

It is worth noting that relevation contains the well-known load-sharing and minimal repair policies as two special cases.

(i) *Load-sharing model.* The operation of a parallel system sustains a time-dependent load, which is shared by the surviving components. As time goes by, the components fail to work one by one, and the remaining survival components will suffer an increasing shared load. In this situation, the two-component parallel system is called the 'load-sharing model' [5, 28]. As assumed in [34] and [39], the total load is usually unevenly shared by the components. Inspired by their work, as displayed in Figure 1, we consider

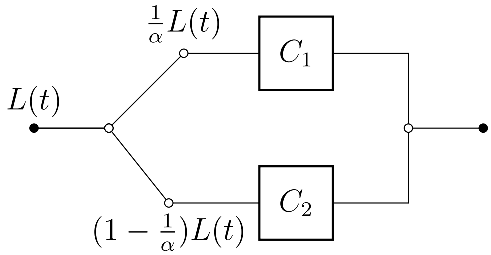

FIGURE 1: Load-sharing model.

a general load-sharing model with two components C_1 and C_2 sharing the total load $L(\cdot)$ with respective proportions $1/\alpha$ and $1-1/\alpha$, where $\alpha > 1$. We adopt the linear breakdown rule in [9], and assume that the total load $L(\cdot)$ can be regarded as a fictitious system's failure rate (see [28]). Then the lifetime of component C_1 that shares the load $L(\cdot)/\alpha$ is denoted by $T_{L/\alpha}$ with survival function

$$\overline{F}_{T_{L/\alpha}}(t) = \exp\left\{-\int_0^t \frac{L(x)}{\alpha}\,\mathrm{d}x\right\}, \quad \alpha > 1,$$

where $L(x)/\alpha$ is interpreted as the instantaneous failure rate of component C_1 at time x under the total load $L(x)$. Furthermore, when one component (say C_1) fails, the lifetime of the other surviving component (say C_2) functions continuously and bears the total load $L(\cdot)$ with survival function

$$\overline{F}_{T_L}(t) = \exp\left\{-\int_0^t L(x)\,\mathrm{d}x\right\}.$$

Therefore the survival function of the load-sharing model can be regarded as the relevation of two components C_1 and C_2 with respective failure rates $L(\cdot)/\alpha$ and $L(\cdot)$ ([7]). In this regard, the survival function of the load-sharing model can be expressed as

$$\begin{aligned}\mathbb{P}(T_{L/\alpha}\#T_L > t) &= \overline{F}_{T_{L/\alpha}}(t) - \overline{F}_L(t)\int_0^t \frac{1}{\overline{F}_L(x)}\,\mathrm{d}\overline{F}_{T_{L/\alpha}}(x) \\ &= \frac{\alpha}{\alpha-1}\overline{F}_L^{1/\alpha}(t) - \frac{1}{\alpha-1}\overline{F}_L(t), \quad \alpha > 1.\end{aligned}$$

(ii) *Minimal repair*. Consider a minimal repair policy as shown in Figure 2, where the component with lifetime T starts working first. When the component fails, the switch K will be turned on immediately (the switching time is negligible) so that it is replaced by another component with lifetime S having the same reliability function and same age as T. That is, the component is 'minimally repaired' and brought to the working state immediately before failure. Then the resulting reliability under a minimal repair reads as

$$\mathbb{P}(T + S_T > t) = \overline{F}(t) + \overline{F}(t)\int_0^t \frac{f(x)}{\overline{F}(x)}\,\mathrm{d}x = \overline{F}(t) - \overline{F}(t)\ln\overline{F}(t),$$

which is also a special case of relevation.

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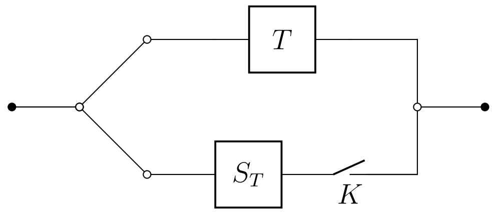

FIGURE 2: Minimal repair.

Recently, Belzunce, Martínez-Riquelme, and Ruiz [7] considered the problem of where to allocate one relevation in a coherent system in order to enhance system reliability in the sense of the usual stochastic ordering. Sufficient conditions are established on the lifetimes of the original components and the relevation by means of the usual stochastic ordering and the hazard rate ordering. In particular, both the load-sharing model and minimal repair are studied in detail. A natural interesting question arises: If we allocate more than one relevation in a coherent system, what is the best allocation policy? The main objective of this paper is to pinpoint the optimal allocation of a fixed number of relevations in coherent systems by means of the hazard rate order and the usual stochastic order, which partially answers the open problems proposed in [7]. Moreover, the special cases of the load-sharing model and the minimal repair policy are investigated explicitly.

The remainder of the paper is organized as follows. In the remaining part of this section, we provide some relevant definitions and concepts used below. In Section 2 we study the best allocation policy of two relevations in series systems and establish sufficient (and necessary) conditions for the hazard rate ordering among different allocation policies. Two special cases including the load-sharing and minimal repair policies are also studied when we consider the allocation of one relevation. Section 3 presents the results on optimal allocation of relevations in coherent systems. Section 4 concludes the paper.

Throughout, we assume that all random variables are non-negative and absolutely continuous. We shall use '$\overset{\text{sgn}}{=}$' to express that both sides of the equality have the same sign, and '$\mathrm{d}\phi(t)/\mathrm{d}t$' to denote the differentiation of $\phi(t)$ with respective to t. The terms 'increasing' and 'decreasing' mean 'non-decreasing' and 'non-increasing', respectively. We denote $\mathbb{R}=(-\infty, \infty)$ and $\mathbb{R}_+=[0, \infty)$.

Stochastic orders are a very useful tool to compare random variables arising from reliability theory, operations research, actuarial science, economics, finance, and so on. Let X and Y be two random variables with distribution functions $F(t)$ and $G(t)$, survival functions $\overline{F}(t)=1-F(t)$ and $\overline{G}(t)=1-G(t)$, probability density functions $f(t)$ and $g(t)$, hazard rate functions $h_X(t)=f(t)/\overline{F}(t)$ and $h_Y(t)=g(t)/\overline{G}(t)$, and reversed hazard rate functions $\tilde{r}_X(t)=f(t)/F(t)$ and $\tilde{r}_Y(t)=g(t)/G(t)$, respectively.

Definition 1. *X is said to be smaller than Y in the*

(i) usual stochastic order (denoted by $X\leq_{st} Y$) if $\overline{F}(t)\leq\overline{G}(t)$ for all $t\in\mathbb{R}$,

(ii) hazard rate order (denoted by $X\leq_{hr} Y$) if $h_X(t)\geq h_Y(t)$ for all $t\in\mathbb{R}$, or equivalently if $\overline{G}(t)/\overline{F}(t)$ is increasing in $t\in\mathbb{R}$,

(iii) reversed hazard rate order (denoted by $X\leq_{rh} Y$) if $\tilde{r}_X(t)\leq\tilde{r}_Y(t)$ for all $t\in\mathbb{R}$, or equivalently if $G(t)/F(t)$ is increasing in $t\in\mathbb{R}$.

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It is well known that the (reversed) hazard rate order implies the usual stochastic order, but the reversed statement is not true in general. For more comprehensive discussions of various stochastic orders and their applications, one may refer to the monographs by Shaked and Shanthikumar [29] and Belzunce, Riquelme, and Mulero [8].

The notion *totally positive of order 2* (TP_2) plays a prominent role in building various inequalities arising from many research areas. A pair of measurable non-negative real-valued functions (g_1, g_2) are said to satisfy the TP_2 property if $g_1(x_1)g_2(x_2) \geq g_1(x_2)g_2(x_1)$ for all $x_1 \leq x_2$.

The following lemma is helpful in building the main results.

Lemma 1. ([17].) *Let (g_1, g_2) be a pair of non-negative functions satisfying the TP_2 property. Let $\overline{F}_\theta$ be the survival function of the random variable X_θ, for $\theta = 1, 2$. Suppose that $\overline{F}_\theta(t)$ is TP_2 in (θ, t), and $\int_0^\infty g_i(t)\, \mathrm{d}F_\theta(t)$ exists and is finite, for $i = 1, 2$ and $\theta = 1, 2$. Further, suppose that $g_1(t)$ is increasing in t. Then $h_i(\theta) = \int_0^\infty g_i(t)\, \mathrm{d}F_\theta(t)$ is TP_2 in (i, θ), that is,*

$$\int_0^\infty g_1(t)\, \mathrm{d}F_1(t) \int_0^\infty g_2(t)\, \mathrm{d}F_2(t) \geq \int_0^\infty g_1(t)\, \mathrm{d}F_2(t) \int_0^\infty g_2(t)\, \mathrm{d}F_1(t).$$

For detailed discussions of properties of TP_2 and its applications, we refer interested readers to [17] and [19].

2. Allocation of relevations in series systems

In this section we study optimal allocation of relevations for series systems in terms of the usual stochastic order and the hazard rate order. Henceforth we assume that all components and relevations are independent of each other.

2.1. Allocation of two relevations in series systems

First we provide a result for the usual stochastic ordering among different allocations of two relevations in an n-component series system.

Theorem 1. *Let $T_1, T_2, \dots, T_n$ be the lifetimes of n components with survival functions $\overline{F}_1, \overline{F}_2, \dots, \overline{F}_n$, respectively, in a series system. Let S_1 and S_2 be the lifetimes of two relevations with survival functions $\overline{G}_1$ and $\overline{G}_2$, respectively. Denote*

$$V_1 = \min\{T_1\#S_1, T_2\#S_2, T_3, \dots, T_n\}$$

and

$$V_2 = \min\{T_1\#S_2, T_2\#S_1, T_3, \dots, T_n\}.$$

If $T_1 \leq_{hr} T_2$ and $S_1 \geq_{hr} S_2$, then $V_1 \geq_{st} V_2$.

Proof. The survival functions of V_1 and V_2 can be expressed as

$$\overline{H}_{V_1}(t) = \prod_{l=3}^{n} \overline{F}_l(t) \left\{ \left(\overline{F}_1(t) + \overline{G}_1(t) \int_0^t \frac{f_1(u)}{\overline{G}_1(u)}\, \mathrm{d}u \right) \left(\overline{F}_2(t) + \overline{G}_2(t) \int_0^t \frac{f_2(u)}{\overline{G}_2(u)}\, \mathrm{d}u \right) \right\}$$

and

$$\overline{H}_{V_2}(t) = \prod_{l=3}^{n} \overline{F}_l(t) \left\{ \left(\overline{F}_1(t) + \overline{G}_2(t) \int_0^t \frac{f_1(u)}{\overline{G}_2(u)}\, \mathrm{d}u \right) \left(\overline{F}_2(t) + \overline{G}_1(t) \int_0^t \frac{f_2(u)}{\overline{G}_1(u)}\, \mathrm{d}u \right) \right\},$$

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respectively. Observe that

$$
\begin{aligned}
&\overline{H}_{V_1}(t) - \overline{H}_{V_2}(t) \\
&\overset{\text{sgn}}{=} \left(\overline{F}_1(t) + \overline{G}_1(t)\int_0^t \frac{f_1(u)}{\overline{G}_1(u)}\,\mathrm{d}u\right)\left(\overline{F}_2(t) + \overline{G}_2(t)\int_0^t \frac{f_2(u)}{\overline{G}_2(u)}\,\mathrm{d}u\right) \\
&\quad - \left(\overline{F}_1(t) + \overline{G}_2(t)\int_0^t \frac{f_1(u)}{\overline{G}_2(u)}\,\mathrm{d}u\right)\left(\overline{F}_2(t) + \overline{G}_1(t)\int_0^t \frac{f_2(u)}{\overline{G}_1(u)}\,\mathrm{d}u\right) \\
&= \overline{F}_1(t)\overline{F}_2(t) + \overline{F}_1(t)\overline{G}_2(t)\int_0^t \frac{f_2(u)}{\overline{G}_2(u)}\,\mathrm{d}u + \overline{F}_2(t)\overline{G}_1(t)\int_0^t \frac{f_1(u)}{\overline{G}_1(u)}\,\mathrm{d}u \\
&\quad + \overline{G}_1(t)\overline{G}_2(t)\int_0^t \frac{f_1(u)}{\overline{G}_1(u)}\,\mathrm{d}u\int_0^t \frac{f_2(u)}{\overline{G}_2(u)}\,\mathrm{d}u \\
&\quad - \Bigg(\overline{F}_1(t)\overline{F}_2(t) + \overline{F}_1(t)\overline{G}_1(t)\int_0^t \frac{f_2(u)}{\overline{G}_1(u)}\,\mathrm{d}u + \overline{F}_2(t)\overline{G}_2(t)\int_0^t \frac{f_1(u)}{\overline{G}_2(u)}\,\mathrm{d}u \\
&\qquad + \overline{G}_1(t)\overline{G}_2(t)\int_0^t \frac{f_1(u)}{\overline{G}_2(u)}\,\mathrm{d}u\int_0^t \frac{f_2(u)}{\overline{G}_1(u)}\,\mathrm{d}u\Bigg) \\
&= \int_0^t \overline{F}_1(t)f_2(u)\left[\frac{\overline{G}_2(t)}{\overline{G}_2(u)} - \frac{\overline{G}_1(t)}{\overline{G}_1(u)}\right]\mathrm{d}u + \int_0^t \overline{F}_2(t)f_1(u)\left[\frac{\overline{G}_1(t)}{\overline{G}_1(u)} - \frac{\overline{G}_2(t)}{\overline{G}_2(u)}\right]\mathrm{d}u \\
&\quad + \overline{G}_1(t)\overline{G}_2(t)\left[\int_0^t \frac{f_1(u)}{\overline{G}_1(u)}\,\mathrm{d}u\int_0^t \frac{f_2(u)}{\overline{G}_2(u)}\,\mathrm{d}u - \int_0^t \frac{f_1(u)}{\overline{G}_2(u)}\,\mathrm{d}u\int_0^t \frac{f_2(u)}{\overline{G}_1(u)}\,\mathrm{d}u\right] \\
&= \int_0^t \left[\overline{F}_2(t)f_1(u) - \overline{F}_1(t)f_2(u)\right]\left[\frac{\overline{G}_1(t)}{\overline{G}_1(u)} - \frac{\overline{G}_2(t)}{\overline{G}_2(u)}\right]\mathrm{d}u \\
&\quad + \overline{G}_1(t)\overline{G}_2(t)\left[\int_0^t \frac{f_1(u)}{\overline{G}_1(u)}\,\mathrm{d}u\int_0^t \frac{f_2(u)}{\overline{G}_2(u)}\,\mathrm{d}u - \int_0^t \frac{f_1(u)}{\overline{G}_2(u)}\,\mathrm{d}u\int_0^t \frac{f_2(u)}{\overline{G}_1(u)}\,\mathrm{d}u\right] \\
&=: \phi_1(t) + \phi_2(t),
\end{aligned}
$$

where

$$\phi_1(t) = \int_0^t \left[\overline{F}_2(t)f_1(u) - \overline{F}_1(t)f_2(u)\right]\left[\frac{\overline{G}_1(t)}{\overline{G}_1(u)} - \frac{\overline{G}_2(t)}{\overline{G}_2(u)}\right]\mathrm{d}u$$

and

$$\phi_2(t) = \overline{G}_1(t)\overline{G}_2(t)\left[\int_0^t \frac{f_1(u)}{\overline{G}_1(u)}\,\mathrm{d}u\int_0^t \frac{f_2(u)}{\overline{G}_2(u)}\,\mathrm{d}u - \int_0^t \frac{f_1(u)}{\overline{G}_2(u)}\,\mathrm{d}u\int_0^t \frac{f_2(u)}{\overline{G}_1(u)}\,\mathrm{d}u\right].$$

Note that $T_1 \leq_{hr} T_2$ implies $\overline{F}_2(t)f_1(u) - \overline{F}_1(t)f_2(u) \geq 0$ for all $0 \leq u \leq t$. On the other hand, $S_1 \geq_{hr} S_2$ is equivalent to

$$\frac{\overline{G}_1(t)}{\overline{G}_2(t)} \geq \frac{\overline{G}_1(u)}{\overline{G}_2(u)} \quad \text{for all } 0 \leq u \leq t,$$

that is,

$$\frac{\overline{G}_1(t)}{\overline{G}_1(u)} - \frac{\overline{G}_2(t)}{\overline{G}_2(u)}$$

is also non-negative, for all $0 \leq u \leq t$. Hence $\phi_1(t)$ is non-negative for all $t \in \mathbb{R}_+$.

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Observe that

$$\phi_2(t) \overset{\text{sgn}}{=} \int_0^t \frac{f_1(u)}{\overline{G}_1(u)}\,\mathrm{d}u \int_0^t \frac{f_2(u)}{\overline{G}_2(u)}\,\mathrm{d}u - \int_0^t \frac{f_1(u)}{\overline{G}_2(u)}\,\mathrm{d}u \int_0^t \frac{f_2(u)}{\overline{G}_1(u)}\,\mathrm{d}u.$$

According to $S_1 \geq_{hr} S_2$, we have

$$\frac{1}{\overline{G}_1(x)\overline{G}_2(y)} - \frac{1}{\overline{G}_1(y)\overline{G}_2(x)} = \frac{\overline{G}_1(y)\overline{G}_2(x) - \overline{G}_1(x)\overline{G}_2(y)}{\overline{G}_1(x)\overline{G}_2(y)\overline{G}_1(y)\overline{G}_2(x)} \geq 0 \quad \text{for } x \leq y,$$

from which it follows that $(1/\overline{G}_1, 1/\overline{G}_2)$ is a pair of non-negative functions satisfying the TP_2 property. Further, $T_1 \leq_{hr} T_2$ implies that the survival function of $\overline{F}_\theta(t)$ is TP_2 in (θ, t). Since $1/\overline{G}_1(t)$ is increasing for $t \in \mathbb{R}_+$, according to Lemma 1 we know that $\phi_2(t)$ is non-negative for all $t \in \mathbb{R}_+$. To sum up, for all $t \in \mathbb{R}_+$,

$$\overline{H}_{V_1}(t) - \overline{H}_{V_2}(t) = \phi_1(t) + \phi_2(t) \geq 0,$$

and thus the proof is completed. □

The following numerical example illustrates Theorem 1.

Example 1. Consider allocating two relevations in a series system with two independent components. Assume that T_1 and T_2 have respective survival functions

$$\overline{F}_1(t) = \exp\left\{-\int_0^t (3x^2 + 3x + 5)\,\mathrm{d}x\right\}, \quad \overline{F}_2(t) = \exp\left\{-\int_0^t (3x^2 + 2x + 2)\,\mathrm{d}x\right\},$$

and S_1 and S_2 have responding survival functions

$$\overline{G}_1(t) = \exp\left\{-\int_0^t (5x^2 + 2x + 1)\,\mathrm{d}x\right\}, \quad \overline{G}_2(t) = \exp\left\{-\int_0^t (5x^2 + 4x + 3)\,\mathrm{d}x\right\}.$$

It is obvious that both $\overline{F}_2(t)/\overline{F}_1(t) = \mathrm{e}^{t^2/2+3t}$ and $\overline{G}_1(t)/\overline{G}_2(t) = \mathrm{e}^{t^2+2t}$ are increasing in $t \in \mathbb{R}_+$. From Figure 3(a) we can see that the difference of survival functions $\overline{H}_{V_1}(t) - \overline{H}_{V_2}(t)$ is always non-negative for all $t = -\ln u$ and $u \in [0, 1]$. Thus the relevation with lifetime S_1 should be moved to the location of the component with lifetime T_1, which is in accordance with the usual stochastic ordering result of Theorem 1. However, the hazard rate functions $h_{V_1}(t)$ and $h_{V_2}(t)$ intersect, as shown in Figure 3(b), which indicates that the hazard rate ordering does not hold between V_1 and V_2.

Remark 1. For a series system with components having ordered lifetimes $T_1 \leq_{hr} T_2 \leq_{hr} \cdots \leq_{hr} T_n$, and two relevations with lifetimes such that $S_1 \geq_{hr} S_2$, Theorem 1 states that the relevation with smaller failure rate should be put on the worst component (having largest failure rate) in order to improve system reliability in the sense of the usual stochastic ordering. This finding agrees with the classical allocation results of hot standbys, cold standbys, and minimal repairs; see for example [10], [35], and [38]. It might be of great interest to relax the assumptions in Theorem 1 when one considers the load-sharing model or the minimal repair policy. We leave them as open problems.

Let $\boldsymbol{\pi} = (\pi_1, \pi_2, \ldots, \pi_n)$ be a permutation of $\{1, 2, \ldots, n\}$, $\boldsymbol{\pi}^+ = \{1, 2, \ldots, n\}$, and $\boldsymbol{\pi}^- = \{n, n-1, \ldots, 1\}$. Consider a series system with n components having respective lifetimes

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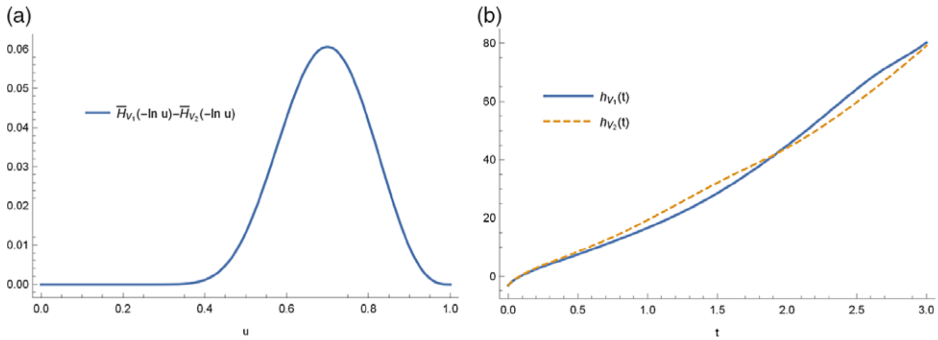

FIGURE 3: (a) Plot of the difference function $\overline{H}_{V_1}(-\ln u) - \overline{H}_{V_2}(-\ln u)$ for $u \in [0, 1]$. (b) Plot of the hazard rate functions $h_{V_1}(t)$ and $h_{V_2}(t)$ for $t \in [0, 3]$.

$T_1, T_2, \ldots, T_n$, which are allocated n relevations with lifetimes $S_1, S_2, \ldots, S_n$. Assume that each component is only allocated by one relevation, hence there exist $n!$ different relevation policies. Let $\boldsymbol{T}\#\boldsymbol{S}_{\boldsymbol{\pi}} = (T_1\#S_{\pi_1}, T_1\#S_{\pi_2}, \ldots, T_1\#S_{\pi_n})$ be the lifetime of the resulting components under the allocation policy that the relevation with lifetime S_{π_i} is allocated to the component with lifetime T_i, $i = 1, 2, \ldots, n$. The following corollary immediately follows from Theorem 1.

Corollary 1. *If* $T_1 \leq_{hr} T_2 \leq_{hr} \cdots \leq_{hr} T_n$ *and* $S_1 \geq_{hr} S_2 \geq_{hr} \cdots \geq_{hr} S_n$*, then*

$$\boldsymbol{T}\#\boldsymbol{S}_{\boldsymbol{\pi}^-} \leq_{st} \boldsymbol{T}\#\boldsymbol{S}_{\boldsymbol{\pi}} \leq_{st} \boldsymbol{T}\#\boldsymbol{S}_{\boldsymbol{\pi}^+},$$

that is, $\boldsymbol{T}\#\boldsymbol{S}_{\boldsymbol{\pi}^+}$ *is the best allocation policy, and* $\boldsymbol{T}\#\boldsymbol{S}_{\boldsymbol{\pi}^-}$ *is the worst allocation policy within the class of all admissible allocations, where*

$$\boldsymbol{T}\#\boldsymbol{S}_{\boldsymbol{\pi}^+} = \min\{T_1\#S_1, T_2\#S_2, \ldots, T_{n-1}\#S_{n-1}, T_n\#S_n\}$$

and

$$\boldsymbol{T}\#\boldsymbol{S}_{\boldsymbol{\pi}^-} = \min\{T_1\#S_n, T_2\#S_{n-1}, \ldots, T_{n-1}\#S_2, T_n\#S_1\}.$$

Proof. Note that the lexicographic permutation order starts from the identity permutation $\boldsymbol{\pi}^+ = (1, 2, \ldots, n)$. By successively swapping only two numbers we obtain all possible permutations, that is, the vector $\boldsymbol{\pi}$ can be obtained by successively exchanging two coordinates of $\boldsymbol{\pi}^+$ for finite times. The last permutation in lexicographic order will be the permutation with all numbers in reversed order, i.e. $\boldsymbol{\pi}^- = (n, n-1, \ldots, 2, 1)$. Then the result can be obtained immediately from Theorem 1. □

2.2. Allocation of one relevation in series systems: load-sharing model

Next we investigate the sufficient and necessary conditions for the usual stochastic ordering and the hazard rate ordering under the load-sharing model.

For the sake of convenience, let us define

$$\gamma_{T_i}(t) = \frac{\alpha - \overline{F}_i^{1-1/\alpha}(t)}{h_i(t)} \quad \text{for } i = 1, 2,$$

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where $\overline{F}_i(t)$ and $h_i(t)$ are the survival function and the hazard rate function, respectively, of random variable T_i.

Theorem 2. *Let $T_1, T_2, \ldots, T_n$ be the lifetimes of n components with survival functions $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_n$, respectively, in a series system. The load-sharing model of the component with lifetime T_i is denoted by $T_{i,L/\alpha}\#T_i$, for $i=1, 2$ and $\alpha > 1$. Denote $U_1 = \min\{T_{1,L/\alpha}\#T_1, T_2, \ldots, T_n\}$ and $U_2 = \min\{T_1, T_{2,L/\alpha}\#T_2, \ldots, T_n\}$. Then*

(i) *$U_1 \geq_{st} U_2$ if and only if $T_1 \leq_{st} T_2$,*

(ii) *$U_1 \geq_{hr} U_2$ if and only if $\gamma_{T_1}(t) \leq \gamma_{T_2}(t)$ for all $t \in \mathbb{R}_+$.*

Proof. (i) Let $\overline{H}_{U_1}(t)$ and $\overline{H}_{U_2}(t)$ denote the survival functions of the load-sharing policies U_1 and U_2, respectively. Then

$$\overline{H}_{U_1}(t) = \prod_{l=3}^{n} \overline{F}_l(t) \left\{ \left(\frac{\alpha}{\alpha-1} \overline{F}_1^{1/\alpha}(t) - \frac{1}{\alpha-1} \overline{F}_1(t) \right) \overline{F}_2(t) \right\}$$

and

$$\overline{H}_{U_2}(t) = \prod_{l=3}^{n} \overline{F}_l(t) \left\{ \left(\frac{\alpha}{\alpha-1} \overline{F}_2^{1/\alpha}(t) - \frac{1}{\alpha-1} \overline{F}_2(t) \right) \overline{F}_1(t) \right\}.$$

It is obvious that, for $\alpha > 1$,

$$\begin{aligned}
\phi_3(t) &:= \overline{H}_{U_1}(t) - \overline{H}_{U_2}(t) \\
&\overset{\text{sgn}}{=} \overline{F}_2(t) \left(\frac{\alpha}{\alpha-1} \overline{F}_1^{1/\alpha}(t) - \frac{1}{\alpha-1} \overline{F}_1(t) \right) - \overline{F}_1(t) \left(\frac{\alpha}{\alpha-1} \overline{F}_2^{1/\alpha}(t) - \frac{1}{\alpha-1} \overline{F}_2(t) \right) \\
&= \frac{\alpha}{\alpha-1} \overline{F}_1(t) \overline{F}_2(t) (\overline{F}_1^{1/\alpha-1}(t) - \overline{F}_2^{1/\alpha-1}(t))
\end{aligned}$$

is non-negative if and only if $\overline{F}_1(t) \leq \overline{F}_2(t)$ for all $t \in \mathbb{R}_+$. Hence the proof is completed.

(ii) The desired result is equivalent to proving the increasing monotonicity of the function

$$\begin{aligned}
\phi_4(t) := \frac{\overline{H}_{U_1}(t)}{\overline{H}_{U_2}(t)} &= \frac{\left(\frac{\alpha}{\alpha-1} \overline{F}_1^{1/\alpha}(t) - \frac{1}{\alpha-1} \overline{F}_1(t)\right) \overline{F}_2(t)}{\left(\frac{\alpha}{\alpha-1} \overline{F}_2^{1/\alpha}(t) - \frac{1}{\alpha-1} \overline{F}_2(t)\right) \overline{F}_1(t)} \\
&= \frac{\left(\frac{\alpha}{\alpha-1} \overline{F}_1^{1/\alpha-1}(t) - \frac{1}{\alpha-1}\right) \overline{F}_1(t) \overline{F}_2(t)}{\left(\frac{\alpha}{\alpha-1} \overline{F}_2^{1/\alpha-1}(t) - \frac{1}{\alpha-1}\right) \overline{F}_2(t) \overline{F}_1(t)} = \frac{\alpha \overline{F}_1^{1/\alpha-1}(t) - 1}{\alpha \overline{F}_2^{1/\alpha-1}(t) - 1}.
\end{aligned}$$

Note that

$$\begin{aligned}
\frac{\mathrm{d}\phi_4(t)}{\mathrm{d}t} &\overset{\text{sgn}}{=} (\alpha-1) h_1(t) \overline{F}_1^{1/\alpha-1}(t) (\alpha \overline{F}_2^{1/\alpha-1}(t) - 1) \\
&\quad - (\alpha-1) h_2(t) \overline{F}_2^{1/\alpha-1}(t) (\alpha \overline{F}_1^{1/\alpha-1}(t) - 1) \\
&\overset{\text{sgn}}{=} h_1(t) \overline{F}_1^{1/\alpha-1}(t) (\alpha \overline{F}_2^{1/\alpha-1}(t) - 1) - h_2(t) \overline{F}_2^{1/\alpha-1}(t) (\alpha \overline{F}_1^{1/\alpha-1}(t) - 1),
\end{aligned}$$

where $h_i(t) = f_i(t)/\overline{F}_i(t)$ is the hazard rate function of T_i, for $i=1, 2$. The non-negativity of $\mathrm{d}\phi_4(t)/\mathrm{d}t$ is equivalent to

$$h_1(t) \overline{F}_1^{1/\alpha-1}(t) \left(\alpha \overline{F}_2^{1/\alpha-1}(t) - 1 \right) \geq h_2(t) \overline{F}_2^{1/\alpha-1}(t) \left(\alpha \overline{F}_1^{1/\alpha-1}(t) - 1 \right),$$

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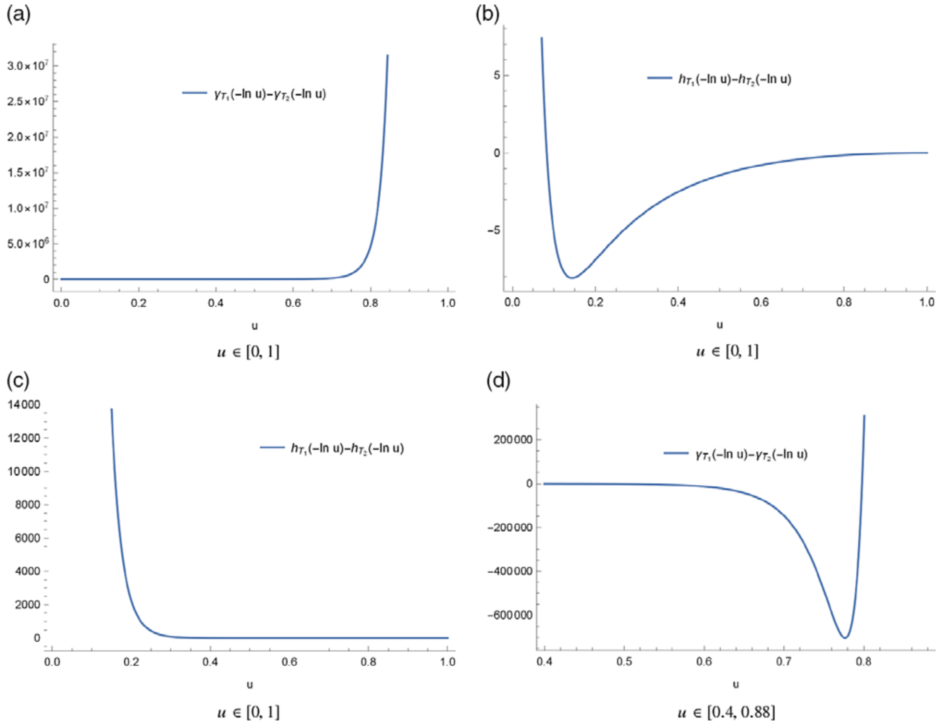

FIGURE 4: Plots of the difference functions $\gamma_{T_1}(-\ln u) - \gamma_{T_2}(-\ln u)$ and $h_{T_1}(-\ln u) - h_{T_2}(-\ln u)$ for $u \in [0, 1]$.

that is,

$$\frac{\alpha - \overline{F}_2^{\,1-1/\alpha}(t)}{h_2(t)} \geq \frac{\alpha - \overline{F}_1^{\,1-1/\alpha}(t)}{h_1(t)}.$$

It is clear that $\mathrm{d}\phi_4(t)/\mathrm{d}t \geq 0$ is equivalent to $U_1 \geq_{hr} U_2$, which holds if and only if $\gamma_{T_1}(t) \leq \gamma_{T_2}(t)$, for all $t \in \mathbb{R}_+$. □

Remark 2. Theorem 2(i) extends the result of Lemma 3 of [7] to the case of the general load-sharing model. Further, Theorem 2(ii) established sufficient and necessary conditions for the hazard rate ordering.

It is natural to ask whether the assumption in Theorem 2(ii) implies or can be implied by the hazard rate order. The following numerical example provides a negative answer.

Example 2. Assume that T_1 and T_2 have survival functions $\overline{F}_1(t) = \mathrm{e}^{-(t/\beta_1)^{\lambda_1}}$ and $\overline{F}_2(t) = \mathrm{e}^{-(t/\beta_2)^{\lambda_2}}$, respectively, for $t \in \mathbb{R}_+$.

(i) Setting $\lambda_1 = 8$, $\beta_1 = 2$, $\lambda_2 = 3$, and $\beta_2 = 1$, from Figures 4(a) and 4(b) we see that the difference function $\gamma_{T_1}(t) - \gamma_{T_2}(t)$ is always non-negative; however, the difference function $h_{T_1}(t) - h_{T_2}(t)$ crosses at the line $y = 0$, for $t = -\ln u$ and $u \in [0, 1]$. Hence $\gamma_{T_1}(t) \geq \gamma_{T_2}(t)$ implies neither $T_1 \geq_{hr} T_2$ nor $T_1 \leq_{hr} T_2$.

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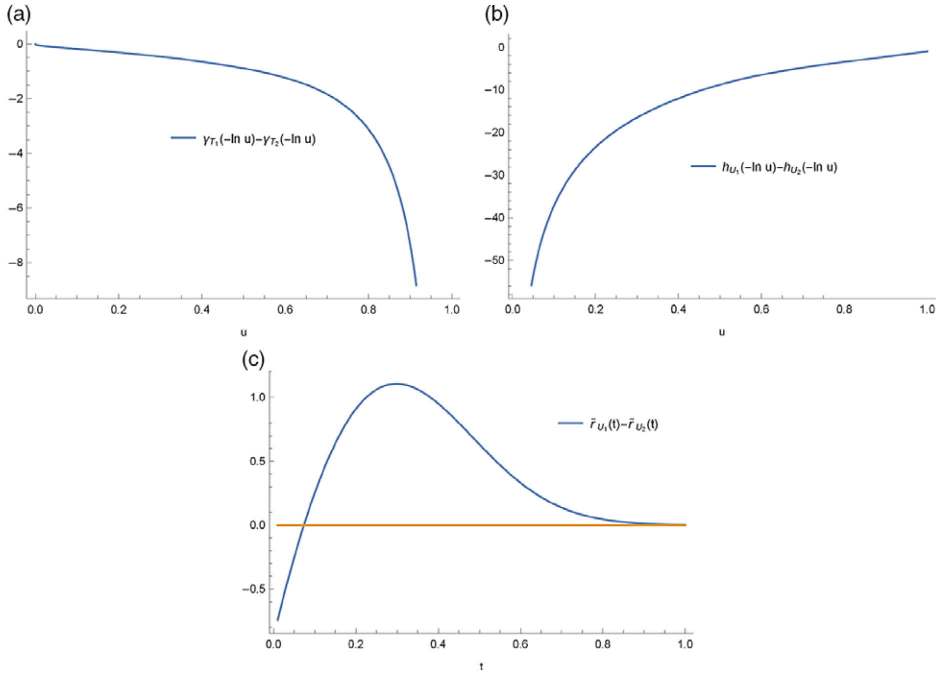

FIGURE 5: (a) Plot of the difference function $\gamma_{T_1}(-\ln u)-\gamma_{T_2}(-\ln u)$ for all $u\in[0,1]$. (b) Plot of the difference function $h_{U_1}(-\ln u)-h_{U_2}(-\ln u)$ for all $u\in[0,1]$. (c) Plot of the difference function $\tilde r_{U_1}(t)-\tilde r_{U_2}(t)$ for $t\in[0,1]$.

(ii) Taking $\lambda_1=12$, $\beta_1=1$, $\lambda_2=8$, and $\beta_2=2$, Figures 4(c) and 4(d) show that the difference function $h_{T_1}(t)-h_{T_2}(t)$ is always non-negative for all $t=-\ln u$ and $u\in[0,1]$, while the difference function $\gamma_{T_1}(t)-\gamma_{T_2}(t)$ crosses at the line $y=0$, for $t=-\ln u$ and $u\in[0.4, 0.88]$. Thus $T_1\le_{hr}T_2$ implies neither $\gamma_{T_1}(t)\ge\gamma_{T_2}(t)$ nor $\gamma_{T_1}(t)\le\gamma_{T_2}(t)$.

Therefore it follows from (i) and (ii) that the condition $\gamma_{T_1}(t)\le\gamma_{T_2}(t)$ for all $t\in\mathbb{R}_+$ does not imply or is not implied by $T_1\le_{hr}T_2$.

The next example not only illustrates the result of Theorem 2 but also indicates that the reversed hazard rate ordering does not hold in general under the same assumption.

Example 3. Consider the load-sharing model in a series system with two independent components. Assume that the survival functions of T_1 and T_2 are

$$\overline{F}_1(t)=\exp\left\{-\int_0^t(5x^2+17x+1)\,\mathrm{d}x\right\}\quad\text{and}\quad\overline{F}_2(t)=\exp\left\{-\int_0^t(2x^2+5x)\,\mathrm{d}x\right\},$$

respectively, for $t\in\mathbb{R}_+$.

(i) The difference functions $\gamma_{T_1}(t)-\gamma_{T_2}(t)$ and $h_{U_1}(t)-h_{U_2}(t)$ are plotted in Figures 5(a) and 5(b) for $t=-\ln u$ and $u\in[0,1]$, from which we can see that both difference functions are always non-positive, which is in accordance with the result of Theorem 2.

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(ii) A natural question arises: Does the reversed hazard rate ordering hold in the setting of Theorem 2? Let $\tilde{r}_{U_i}(t)$ be the reversed hazard rate function of U_i, for $i = 1, 2$. However, in the same set-up as (i), the difference function $\tilde{r}_{U_1}(t) - \tilde{r}_{U_2}(t)$ intersects at the line $y = 0$, as displayed in Figure 5(c), which implies that $U_1 \not\leq_{rh} U_2$ and $U_1 \not\geq_{rh} U_2$.

2.3. Allocation of one relevation in series systems: minimal repair

This subsection studies the allocation of one minimal repair in an n-component series system by stochastically optimizing the resulting lifetime by means of the hazard rate order. For the sake of convenience, let us define

$$\beta_{T_i}(t) = \frac{h_i(t)}{1 - \ln \overline{F}_i(t)} \quad \text{for } i = 1, 2,$$

where $\overline{F}_i$ and h_i, respectively, are the survival function and the hazard rate function of T_i for $i = 1, 2$.

Theorem 3. *Let $T_1, T_2, \ldots, T_n$ be the lifetimes of n components with survival functions $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_n$, respectively, in a series system. Let S_1, S_2 be the lifetimes of two minimal repairs with survival functions $\overline{F}_1$ and $\overline{F}_2$. Denote $M_1 = \min\{T_1 \# S_1, T_2, T_3, \ldots, T_n\}$ and $M_2 = \min\{T_1, T_2 \# S_2, T_3, \ldots, T_n\}$. Then $M_1 \geq_{hr} M_2$ if and only if $\beta_{T_1}(t) \geq \beta_{T_2}(t)$ for all $t \in \mathbb{R}_+$.*

Proof. The survival functions of M_1 and M_2 can be expressed as

$$\overline{H}_{M_1}(t) = \prod_{l=3}^{n} \overline{F}_l(t) \left\{ \overline{F}_2(t)\overline{F}_1(t)\left(1 + \int_0^t \frac{f_1(u)}{\overline{F}_1(u)}\, \mathrm{d}u\right)\right\}$$

and

$$\overline{H}_{M_2}(t) = \prod_{l=3}^{n} \overline{F}_l(t) \left\{ \overline{F}_2(t)\overline{F}_1(t)\left(1 + \int_0^t \frac{f_2(u)}{\overline{F}_2(u)}\, \mathrm{d}u\right)\right\},$$

respectively. The desired result boils down to proving the increasing monotonicity of

$$\phi_5(t) := \frac{\overline{H}_{M_1}(t)}{\overline{H}_{M_2}(t)} = \frac{1 - \ln \overline{F}_1(t)}{1 - \ln \overline{F}_2(t)}.$$

Observe that

$$\begin{aligned}\frac{\mathrm{d}\phi_5(t)}{\mathrm{d}t} &\overset{\text{sgn}}{=} \frac{f_1(t)}{\overline{F}_1(t)}(1 - \ln \overline{F}_2(t)) - \frac{f_2(t)}{\overline{F}_2(t)}(1 - \ln \overline{F}_1(t)) \\ &= h_1(t)(1 - \ln \overline{F}_2(t)) - h_2(t)(1 - \ln \overline{F}_1(t)).\end{aligned}$$

Thus the non-negativity of $\mathrm{d}\phi_5(t)/\mathrm{d}t$ is equivalent to $\beta_{T_1}(t) \geq \beta_{T_2}(t)$ for all $t \in \mathbb{R}_+$, which implies the desired result. □

Remark 3. Theorem 3 provides a sufficient and necessary condition for the optimal allocation of a minimal repair in a series system with respect to the hazard rate order, which serves as a nice complement to the usual stochastic ordering result of Lemma 2 in [7].

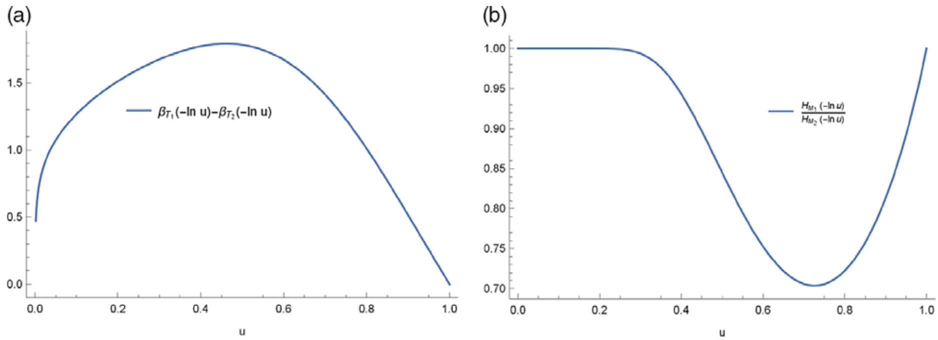

FIGURE 6: (a) Plot of the difference function $\beta_{T_1}(-\ln u) - \beta_{T_2}(-\ln u)$ for all $u \in [0, 1]$. (b) Plot of the ratio function $H_{M_1}(-\ln u)/H_{M_2}(-\ln u)$ for all $u \in [0, 1]$.

One may wonder whether the reversed hazard rate ordering holds in the set-up of Theorem 3. Unfortunately, the following numerical example gives us a negative answer.

Example 4. Suppose a series system is made up of two components with lifetimes T_1 and T_2 having reliability functions

$$\overline{F}_1(t) = \exp\left\{-\int_0^t (2x^3 + 5x)\,\mathrm{d}x\right\} \quad \text{and} \quad \overline{F}_2(t) = \exp\left\{-\left(\frac{t}{8}\right)^6\right\},$$

respectively. Denote $\varphi(t) = H_{M_1}(t)/H_{M_2}(t)$. Although the difference function $\beta_{T_1}(t) - \beta_{T_2}(t)$ is always non-negative in Figure 6(a), for $t = -\ln u$ and $u \in [0, 1]$, the ratio function $\varphi(t)$ is not strictly monotone in $t = -\ln u$ for $u \in [0, 1]$, as shown in Figure 6(b), which indicates that $\beta_{T_1}(t) \geq \beta_{T_2}(t)$ for $t \in \mathbb{R}_+$ implies neither $M_1 \geq_{rh} M_2$ nor $M_1 \leq_{rh} M_2$.

For a parallel system consisting of n independent components, we can develop ordering results similar to those of Theorems 2 and 3 for the load-sharing and minimal repair policies. Define

$$\delta_{T_i}(t) = \frac{\tilde{r}_i(t)}{1 + \ln \overline{F}_i(t)}, \qquad i = 1, 2,$$

where $\overline{F}_i$ and $\tilde{r}_i$, respectively, are the survival function and the reversed hazard rate function of random variable T_i.

Proposition 1. *Consider a parallel system consisting of n independent components with lifetimes $T_1, T_2, \ldots, T_n$. Let $T_{i,L/\alpha}\#T_i$, for $i = 1, 2$, denote the lifetime in a load-sharing model for the component with lifetime T_i. Let*

$$\tilde{U}_1 = \max\{T_{1,L/\alpha}\#T_1, T_2, T_3, \ldots, T_n\} \quad \textit{and} \quad \tilde{U}_2 = \max\{T_1, T_{2,L/\alpha}\#T_2, T_3, \ldots, T_n\}.$$

Then $\tilde{U}_1 \leq_{st} \tilde{U}_2$ if and only if $T_1 \leq_{st} T_2$.

Proposition 2. *Let $T_1, T_2, \ldots, T_n$ be the lifetimes of n components with survival functions $\overline{F}_1, \overline{F}_2, \ldots, \overline{F}_n$, respectively, in a parallel system. Let S_1, S_2 be the lifetimes corresponding to two minimal repairs with survival functions $\overline{F}_1$ and $\overline{F}_2$. Let*

$$\tilde{M}_1 = \max\{T_1\#S_1, T_2, T_3 \ldots, T_n\} \quad \textit{and} \quad \tilde{M}_2 = \max\{T_1, T_2\#S_2, T_3 \ldots, T_n\}.$$

Then $\tilde{M}_1 \leq_{rh} \tilde{M}_2$ if and only if $\delta_{T_1}(t) \geq \delta_{T_2}(t)$ for all $t \in \mathbb{R}_+$.

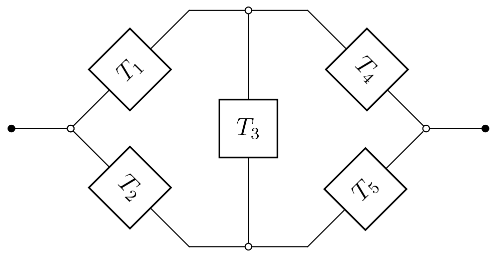

FIGURE 7: Bridge system.

3. Allocation of relevations in coherent systems

In this section we extend the results developed in the previous sections to the case of coherent systems. Let us consider a coherent system made up of n components with lifetimes $T_1, T_2, \ldots, T_n$, structure function Φ, and all minimal path sets $P_1, P_2, \ldots, P_k$, where $1 \le k \le n$. According to [2], the lifetime of this coherent system can be expressed as

$$\tau = \max_{1\le j\le k}\left\{\min_{i\in P_j} T_i\right\}.$$

For example, the bridge system (see Figure 7) has minimal path sets $P_1 = \{1, 4\}$, $P_2 = \{1, 3, 5\}$, $P_3 = \{2, 5\}$, and $P_4 = \{2, 3, 4\}$, which is commonly used in reliability engineering. Then the lifetime of this system can be written as

$$\tau = \max\{\min\{T_1, T_4\}, \min\{T_1, T_3, T_5\}, \min\{T_2, T_5\}, \min\{T_2, T_3, T_4\}\}.$$

For two components located in a common minimal path of a coherent system, the following result establishes the optimal allocation of two relevations in a coherent system in terms of the minimal path sets.

Theorem 4. *For a coherent system with structure function Φ, let $P_1, P_2, \ldots, P_k$ be its minimal path sets. Consider two locations $i, j \in \{1, 2, \ldots, n\}$ such that $i < j$, and either $\{i, j\} \subseteq P_l$ or $\{i, j\} \cap P_l = \emptyset$, for any $l = 1, \ldots, k$. For some minimal path set P_r such that $\{i, j\} \subseteq P_r$, we denote the resulting system lifetimes by τ_1 and τ_2 under two different allocation policies*

$$T_1, \ldots, T_{i-1}, T_i\#S_1, T_{i+1}, \ldots, T_{j-1}, T_j\#S_2, T_{j+1} \ldots, T_n$$

and

$$T_1, \ldots, T_{i-1}, T_i\#S_2, T_{i+1}, \ldots, T_{j-1}, T_j\#S_1, T_{j+1}, \ldots, T_n,$$

respectively. If $T_i \le_{hr} T_j$ and $S_1 \ge_{hr} S_2$, then $\tau_1 \ge_{st} \tau_2$.

Proof. The idea of the proof is borrowed from Theorem 1 of [7]. Let $V_1 = \min\{T_i\#S_1, T_j\#S_2\}$ and $V_2 = \min\{T_i\#S_2, T_j\#S_1\}$. Then, for some minimal path set P_l such that $\{i, j\} \subseteq P_l$, we have

$$\tau_1 = \min\left\{\min_{m\in P_l, m\neq i,j}\{T_m\}, T_i\#S_1, T_j\#S_2\right\} = \min\left\{\min_{m\in P_l, m\neq i,j}\{T_m\}, V_1\right\}$$

and

$$\tau_2 = \min\left\{\min_{m\in P_l, m\neq i,j}\{T_m\}, T_i\#S_2, T_j\#S_1\right\} = \min\left\{\min_{m\in P_l, m\neq i,j}\{T_m\}, V_2\right\}.$$

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By applying Theorem 1, we obtain that $V_1 \geq_{st} V_2$, which further implies that $\tau_1 \geq_{st} \tau_2$. □

Next we present a numerical example to illustrate Theorem 4.

Example 5. Consider a bridge system with components having lifetimes T_i, and let $\overline{F}_i$ be their survival functions, for $i = 1, \ldots, 5$. Assume that

$$\overline{F}_1(t) = \exp\left\{-\int_0^t (5x+4)\,\mathrm{d}x\right\}, \quad \overline{F}_2(t) = \exp\left\{-\int_0^t (4x+2)\,\mathrm{d}x\right\},$$
$$\overline{F}_3(t) = \exp\left\{-\int_0^t (3x+1)\,\mathrm{d}x\right\}, \quad \overline{F}_4(t) = \exp\left\{-\int_0^t (2x+0.4)\,\mathrm{d}x\right\},$$
$$\overline{F}_5(t) = \exp\left\{-\int_0^t (x+0.1)\,\mathrm{d}x\right\}.$$

Assume that S_1 and S_2 have survival functions

$$\overline{G}_1(t) = \exp\left\{-\int_0^t (3x+3)\,\mathrm{d}x\right\}, \quad \overline{G}_2(t) = \exp\left\{-\int_0^t (7x+5)\,\mathrm{d}x\right\},$$

respectively. If we allocate two relevations to the minimal path set $P_4 = \{2, 3, 4\}$, then there are six possible strategies with the resulting reliability functions

$$\overline{H}_1(t) = 1 - [1-\overline{F}_1(t)\overline{F}_4(t)][1-\overline{F}_1(t)\overline{H}_{T_3\#S_2}(t)\overline{F}_5(t)][1-\overline{H}_{T_2\#S_1}(t)\overline{F}_5(t)]$$
$$\times [1-\overline{H}_{T_2\#S_1}(t)\overline{H}_{T_3\#S_2}(t)\overline{F}_4(t)],$$
$$\overline{H}_2(t) = 1 - [1-\overline{F}_1(t)\overline{F}_4(t)][1-\overline{F}_1(t)\overline{H}_{T_3\#S_1}(t)\overline{F}_5(t)][1-\overline{H}_{T_2\#S_2}(t)\overline{F}_5(t)]$$
$$\times [1-\overline{H}_{T_2\#S_2}(t)\overline{H}_{T_3\#S_1}(t)\overline{F}_4(t)],$$
$$\overline{H}_3(t) = 1 - [1-\overline{F}_1(t)\overline{H}_{T_4\#S_2}(t)][1-\overline{F}_1(t)\overline{F}_3(t)\overline{F}_5(t)][1-\overline{H}_{T_2\#S_1}(t)\overline{F}_5(t)]$$
$$\times [1-\overline{H}_{T_2\#S_1}(t)\overline{H}_{T_4\#S_2}(t)\overline{F}_3(t)],$$
$$\overline{H}_4(t) = 1 - [1-\overline{F}_1(t)\overline{H}_{T_4\#S_1}(t)][1-\overline{F}_1(t)\overline{F}_3(t)\overline{F}_5(t)][1-\overline{H}_{T_2\#S_2}(t)\overline{F}_5(t)]$$
$$\times [1-\overline{H}_{T_2\#S_2}(t)\overline{H}_{T_4\#S_1}(t)\overline{F}_3(t)],$$
$$\overline{H}_5(t) = 1 - [1-\overline{F}_1(t)\overline{H}_{T_4\#S_2}(t)][1-\overline{F}_1(t)\overline{H}_{T_3\#S_1}(t)\overline{F}_5(t)][1-\overline{F}_2(t)\overline{F}_5(t)]$$
$$\times [1-\overline{H}_{T_3\#S_1}(t)\overline{H}_{T_4\#S_2}(t)\overline{F}_2(t)],$$
$$\overline{H}_6(t) = 1 - [1-\overline{F}_1(t)\overline{H}_{T_4\#S_1}(t)][1-\overline{F}_1(t)\overline{H}_{T_3\#S_2}(t)\overline{F}_5(t)][1-\overline{F}_2(t)\overline{F}_5(t)]$$
$$\times [1-\overline{H}_{T_3\#S_2}(t)\overline{H}_{T_4\#S_1}(t)\overline{F}_2(t)].$$

From the plots in Figures 8(a) and 8(b) we can see that $T_1 \leq_{hr} T_2 \leq_{hr} T_3 \leq_{hr} T_4 \leq_{hr} T_5$ and $S_1 \geq_{hr} S_2$. Meanwhile, Figure 8(c) plots the difference functions $\overline{H}_i(t) - \overline{H}_{i+1}(t)$, for $i = 1, 3, 5$, where $t = -\ln u$ and $u \in [0, 1]$, from which it is clear that all difference functions are non-negative. Thus the relevation redundancy should be put to the node having the weakest component with lifetime T_2 within the minimal path set $P_4 = \{2, 3, 4\}$, which is in accordance with the theoretical result of Theorem 4.

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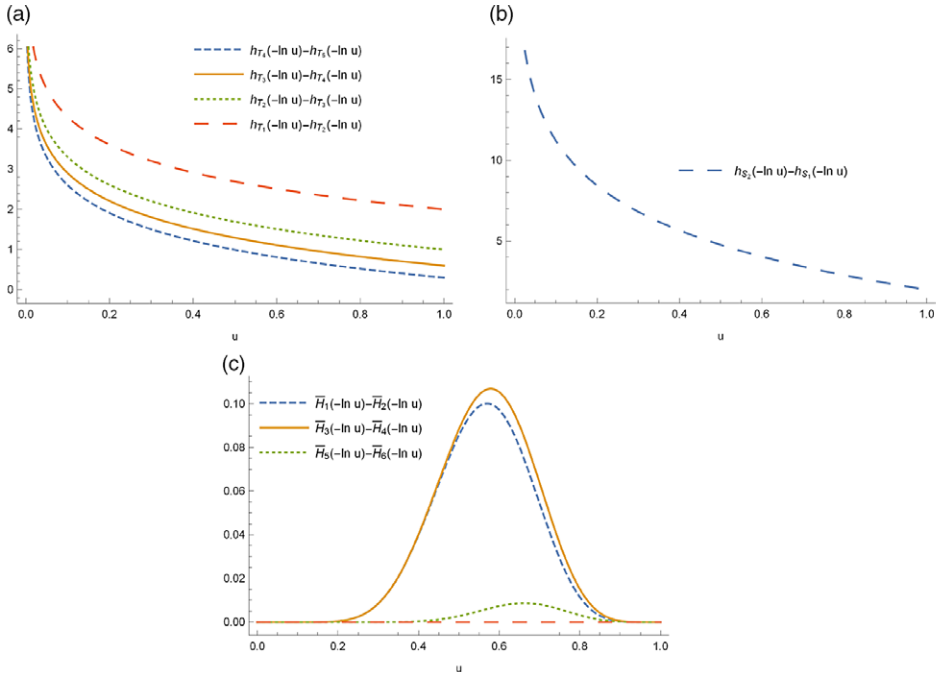

FIGURE 8: (a) Plots of the difference function $h_{T_i}(-\ln u) - h_{T_{i+1}}(-\ln u)$ for $i = 1, \ldots, 4$ and $u \in [0, 1]$. (b) Plot of the difference function $h_{S_2}(-\ln u) - h_{S_1}(-\ln u)$ and $u \in [0, 1]$. (c) Plots of the difference function $\overline{H}_i(-\ln u) - \overline{H}_{i+1}(-\ln u)$ for $i = 1, 3, 5$ and $u \in [0, 1]$.

For the case of load-sharing and minimal repair policies, the following results can be obtained directly from Theorems 2 and 3 for coherent systems.

Proposition 3. *In the set-up of Theorem 4, let* τ_1 *and* τ_2 *be the resulting system lifetimes with common structure function* Φ *under two different load-sharing models*

$$T_1, \ldots, T_{i-1}, T_{i,L/\alpha}\#T_i, T_{i+1}, \ldots, T_j, \ldots, T_n$$

and

$$T_1, \ldots, T_i, \ldots, T_{j-1}, T_{j,L/\alpha}\#T_j, T_{j+1}, \ldots, T_n,$$

respectively. Then the condition $\gamma_{T_i}(t) \leq \gamma_{T_j}(t)$ *for all* $t \in \mathbb{R}_+$ *implies that* $\tau_1 \geq_{hr} \tau_2$.

Proposition 4. *In the set-up of Theorem 4, let* τ_1 *and* τ_2 *be the resulting system lifetimes with common structure function* Φ *under two different minimal repair policies*

$$T_1, \ldots, T_{i-1}, T_i\#S_1, T_{i+1}, \ldots, T_{j-1}, T_j, T_{j+1} \ldots, T_n$$

and

$$T_1, \ldots, T_{i-1}, T_i, T_{i+1}, \ldots, T_{j-1}, T_j\#S_2, T_{j+1}, \ldots, T_n,$$

respectively. Then the condition $\beta_{T_i}(t) \geq \beta_{T_j}(t)$ *for all* $t \in \mathbb{R}_+$ *implies that* $\tau_1 \geq_{hr} \tau_2$.

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4. Conclusion

In this work we have studied the allocations of relevations in coherent systems. In particular, we have studied the problem of allocating one or two relevations to improve the reliabilities of series systems in the sense of the usual stochastic ordering and the hazard rate ordering. These results are further generalized to the case of coherent systems. Further research might focus on the study of allocations of more than two relevations in coherent systems with dependent components. Another problem of interest is to generalize Theorem 4 when the components belong to two different disjoint sets, as mentioned in [7].

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