σ-FRAGMENTABILITY AND ANALYTICITY

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Abstract. We present a new characterization of σ-fragmentability and illustrate its usefulness by proving some results relating analyticity and σ-fragmentability. We show, for instance, that a Banach space with the weak topology is σ-fragmented if, and only if, it is almost Čech-analytic and that an almost Čech-analytic topological space is σ-fragmented by a lower-semicontinuous metric if, and only if, each compact subset of the space is fragmented by the metric.

Introduction. Let (X, τ) be a Hausdorff space and let ρ be a pseudo-metric on X. Given an $\varepsilon > 0$, a non-empty subset A of X is said to be *fragmented by ρ down to ε* if each non-empty subset of A has a relatively τ-open non-empty subset of ρ-diameter less than ε. The set A is said to be *fragmented by ρ* if A is fragmented down to ε for each $\varepsilon > 0$. The set A is said to be *σ-fragmented by ρ* if, for each $\varepsilon > 0$, A can be decomposed as $A = \bigcup_{n=1}^{\infty} A_n$ with each A_n fragmented by ρ down to ε.

The notion of fragmentability has been introduced by Jayne and Rogers [JR] as an abstraction of phenomena often encountered, for example, in Banach spaces with the Radon–Nykodým property, in weakly compact subsets of Banach spaces and in the duals of Asplund spaces. The notion of σ-fragmentability is introduced in [JNR1] in order to extend the study of compact fragmented spaces to non-compact spaces. It turns out that the question of whether a given Banach space with the weak topology is σ-fragmented is closely connected with the question of the existence of an equivalent Kadec (and, in particular, locally uniformly convex) norm. Therefore it seems inevitable that σ-fragmentability will play a central rôle in linear topological characterizations of those Banach spaces that admit Kadec or locally uniformly convex renorming.

In the present note, we offer a new characterization of σ-fragmentability that gives a new perspective on the relation between fragmentability and σ-fragmentability. Roughly speaking, a space X is σ-fragmented by a metric ρ if, and only if, it is the image under the projection onto X of a subset of $X \times \mathbb{N}^{\mathbb{N}}$ that is fragmented by the metric induced by ρ. We then illustrate the new characterization by deriving from it two results. First, we show that a Banach space with the weak topology is σ-fragmented by the norm metric if, and only if, it is "cover-analytic" in the sense of Hansell [Ha2], and this is the case if, and only if, it is "almost Čech-analytic", a notion to be introduced in Section 4. We next prove a generalization of the main theorem in [JNR1]. Our point of view makes the technical part of the proof simpler compared to the original one.

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The paper is organized as follows. The main theorem is formulated and proved in Section 1. Sections 2–3 are preliminaries for the first application of the main theorem, which is given in Section 4. The second application is in Section 5.

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§1. *The main theorem.* The following lemma is a variation of a well-known theorem (see [JR, Lemma 1]). It is recorded here in the form useful for the present paper. The proof follows a familiar pattern and is omitted.

1.1. LEMMA. *Let (X, τ) be a Hausdorff space with a base $\mathscr{B}$, let ρ be a pseudo-metric on X, let S be a non-empty subset of X and let $\varepsilon > 0$. If S is fragmented by ρ down to ε, then there are transfinite sequences $\{S_\alpha : \alpha < \Gamma\}$ and $\{U_\alpha : \alpha < \Gamma\}$ such that:*

(1) *$S = \bigcup\{S_\alpha : \alpha < \Gamma\}$;*
(2) *$U_\alpha \in \mathscr{B}$, $\emptyset \neq S_\alpha \subset U_\alpha$ and ρ-diam $S_\alpha < \varepsilon$ for each $\alpha < \Gamma$;*
(3) *if $\alpha < \beta < \Gamma$, then $U_\alpha \cap S_\beta = \emptyset$; and*
(4) *for each $\gamma < \Gamma$, $\bigcup\{S_\alpha : \alpha < \gamma\} = S \cap \bigcup\{U_\alpha : \alpha < \gamma\}$.*

Conversely, if there are transfinite sequences $\{S_\alpha : \alpha < \Gamma\}$ and $\{U_\alpha : \alpha < \Gamma\}$ satisfying (1)–(3), *then S is fragmented by ρ down to ε.*

The next lemma is proved in [JNR2].

1.2. LEMMA. *Let (X, τ) be a Hausdorff space and let ρ be a metric on X whose topology is stronger than τ. If (X, τ) is σ-fragmented by ρ, then it is σ-fragmented by ρ using ρ-closed sets, i.e., for each $\varepsilon > 0$, X is a countable union of ρ-closed sets each fragmented by ρ down to ε.*

The theorem that follows is the main theorem of the present paper. We denote by $\mathbb{N}^\mathbb{N}$ the countable product of the discrete set $\mathbb{N} = \{1, 2, 3, \ldots\}$ of the natural numbers provided with the product topology. The space $\mathbb{N}^\mathbb{N}$ is metrizable, and we shall fix a metric d so that $(\mathbb{N}^\mathbb{N}, d)$ is complete. For $\sigma, \sigma' \in \mathbb{N}^\mathbb{N}$, let $d(\sigma, \sigma') = 0$ if $\sigma = \sigma'$. Otherwise let $d(\sigma, \sigma') = n^{-1}$, where n is the least natural number such that $\sigma(n) \neq \sigma'(n)$. The open n^{-1}-ball around σ is $\{\sigma' \in \mathbb{N}^\mathbb{N} : \sigma(i) = \sigma'(i) \text{ for } i \leqslant n\}$. This set is denoted by $[\sigma(1), \ldots, \sigma(n)]$.

1.3. THEOREM. *Let (X, τ) be a Hausdorff space and let ρ be a metric for X. Then the following statements are equivalent.*

(a) *(X, τ) is σ-fragmented by ρ.*
(b) *There is a subset M of $X \times \mathbb{N}^\mathbb{N}$ that projects onto X and is fragmented by the metric $\bar{d}$, where $\bar{d}((x, \sigma), (x', \sigma')) = \max(\rho(x, x'), d(\sigma, \sigma'))$.*

(c) *There is a subset* M *of* $X \times \mathbb{N}^{\mathbb{N}}$ *that projects onto* X *and is fragmented by the pseudometric* $\bar{\rho}$ *where* $\bar{\rho}((x, \sigma), (x', \sigma')) = \rho(x, x')$.

(d) *The same as* (c) *with* $\mathbb{N}^{\mathbb{N}}$ *replaced with an arbitrary second countable Hausdorff space.*

Furthermore, if the topology of ρ *is stronger than* τ, *then in* (b) *and* (c) M *can be chosen to be closed in* $(X, \rho) \times \mathbb{N}^{\mathbb{N}}$, *i.e.,* $\bar{d}$*-closed.*

Proof. (a)$\Rightarrow$(b). Let p_1 and p_2 be the projections of $X \times \mathbb{N}^{\mathbb{N}}$ onto X and $\mathbb{N}^{\mathbb{N}}$ respectively. For each n, we can write $X = \bigcup_{i=1}^{\infty} X_i^n$, where each X_i^n is fragmented by ρ down to n^{-1}. Let

$$M = \{(x, \sigma) \in X \times \mathbb{N}^{\mathbb{N}} : x \in X^n_{\sigma(n)} \text{ for each } n \in \mathbb{N}\}.$$

Then clearly $p_1(M) = X$. (If the topology of ρ is stronger than τ, then, by Lemma 1.2, each X_i^n can be chosen to be ρ-closed. Then, as easily seen, M is closed in $(X, \rho) \times \mathbb{N}^{\mathbb{N}}$.)

We must show that M is fragmented by $\bar{d}$. Let $\varnothing \neq A \subset M$ and $\varepsilon > 0$. Choose an $N \in \mathbb{N}$ so that $(N+1)^{-1} < \varepsilon$, and let $\sigma \in p_2(A)$ and $V = [\sigma(1), \ldots, \sigma(N)]$. Then V is an open subset of $\mathbb{N}^{\mathbb{N}}$ such that $B \overset{\text{def}}{=} A \cap (X \times V) \neq \varnothing$ and $p_1(B) \subset X^n_{\sigma(N)}$. Hence there is a τ-open subset U of X such that $U \cap p_1(B) \neq \varnothing$ and ρ-diam $(U \cap p_1(B)) < \varepsilon$. Then $(U \times V) \cap A \neq \varnothing$, and $(x, \sigma) \in (U \times V) \cap A$ implies that $\sigma \in V$ and $x \in U \cap p_1(B)$. Hence if $(x, \sigma), (x', \sigma') \in (U \times V) \cap A$ then $\rho(x, x') < \varepsilon$ and $d(\sigma, \sigma') \leqslant (N+1)^{-1} < \varepsilon$, whence $\bar{d}((x, \sigma), (x', \sigma')) < \varepsilon$.

(b)$\Rightarrow$(c). This is trivial since $\bar{\rho} \leqslant \bar{d}$.

(c)$\Rightarrow$(d). Also trivial.

(d)$\Rightarrow$(a). Let Y be a second countable Hausdorff space, and let M be a subset of $X \times Y$ that projects onto X and is fragmented by $\bar{\rho}$. Let $\mathscr{V}$ be a countable base for the topology of Y and let $\mathscr{B}$ be a base for $X \times Y$ given by

$$\mathscr{B} = \{U \times V : U \text{ is open in } X \text{ and } V \in \mathscr{V}\}.$$

Fix $\varepsilon > 0$. Then by Lemma 1.1, there are transfinite sequences $\{M_\alpha : \alpha < \Gamma\}$ and $\{U_\alpha \times V_\alpha : \alpha < \Gamma\}$ satisfying:

(1) $M = \bigcup \{M_\alpha : \alpha < \Gamma\}$;

(2) $U_\alpha \times V_\alpha \in \mathscr{B}$, $\varnothing \neq M_\alpha \subset U_\alpha \times V_\alpha$ and $\bar{\rho}$-diam $M_\alpha < \varepsilon$ for each $\alpha < \Gamma$; and

(3) $(U_\alpha \times V_\alpha) \cap M_\beta = \varnothing$ whenever $\alpha < \beta < \Gamma$.

For each $V \in \mathscr{V}$, let $\Gamma_V = \{\alpha : \alpha < \Gamma \text{ and } V_\alpha = V\}$. Also let $p : X \times Y \to X$ be the projection map. Then $\{p(M_\alpha) : \alpha \in \Gamma_V\}$ and $\{U_\alpha : \alpha \in \Gamma_V\}$ are transfinite sequences of subsets of X which satisfy condition (2) of Lemma 1.1. If $\alpha, \beta \in \Gamma_V$ and $\alpha < \beta$, then $U_\alpha \cap p(M_\beta) = \varnothing$ since $M_\beta \subset U_\beta \times V_\beta \subset X \times V$ and $(U_\alpha \times V) \cap M_\beta = \varnothing$ by (3) above. Hence by Lemma 1, $X_V \overset{\text{def}}{=} \bigcup \{p(M_\alpha) : \alpha \in \Gamma_V\}$ is fragmented by ρ down to ε, and by (1), $X = p(M) = \bigcup \{p(M_\alpha) : \alpha < \Gamma\} = \bigcup \{X_V : V \in \mathscr{V}\}$. This proves (a) because $\mathscr{V}$ is countable.

§2. *Cover-complete and almost Čech-complete spaces.* A cover $\mathscr{U}$ of a topological space is called *exhaustive* if, whenever A is a non-empty subset of X, there exists a $U \in \mathscr{U}$ such that $U \cap A$ is non-empty and relatively open in A. The following lemma is due to Michael [Mi1].

2.1. LEMMA. *A cover $\mathscr{U}$ of a topological space X is exhaustive if, and only if, $\mathscr{U}$ can be well-ordered in such a way that $\bigcup\{V\in\mathscr{U}: V\leqslant U\}$ is open in X for each $U\in\mathscr{U}$.*

A sequence $\{\mathscr{U}_n : n\in\mathbb{N}\}$ of families of subsets of a topological space X is called *complete* if, whenever $\mathscr{A}$ is a family of subsets of X with the f.i.p. (the finite intersection property) that contains an $\mathscr{U}_n$-small set for each $n\in\mathbb{N}$, then $\bigcap\{\bar{A}: A\in\mathscr{A}\}\neq\varnothing$. A Hausdorff space X is said to be *cover-complete* [Ha2] (or *partition-complete* [TW]) if there exists a complete sequence of exhaustive covers.

A Tychonoff space X is called *Čech-complete* if X is a G_δ-subset of βX, and this is the case if, and only if, X is a G_δ-subset in an arbitrary compactification. It is shown by Frolík [Fro] that a Tychonoff space X is Čech-complete if, and only if, there exists a complete sequence of open covers of X. Since an open cover is an exhaustive cover, *each Čech-complete space is cover-complete.*

A family $\mathscr{A}$ of subsets of a topological space is an *almost cover* if $\bigcup\mathscr{A}$ is dense in X. After [Mi2], a Hausdorff space is called *almost complete* (or *almost Čech-complete* in [AL]) if there exists a complete sequence of open almost covers of X. It is easy to see that if $\mathscr{U}$ is an exhaustive cover of X, then the family $\{U^\circ: U\in\mathscr{U}\}$ of interiors U° of U is an open almost-cover of X [Mi2]. Hence *each cover-complete space is almost complete.* The next theorem is due to Frolík [Fro].

2.2. THEOREM. *A Tychonoff space is almost complete if and only if it contains a dense Čech-complete subspace.*

Finally, we note that a closed subspace of a cover-complete (resp. Čech-complete) space is again cover-complete (resp. Čech-complete). We call a Tychonoff space X *hereditarily almost complete* if each closed subspace of X is almost complete. By remarks made above, we see the following:

2.3. LEMMA. *A cover-complete Tychonoff space is hereditarily almost complete.*

2.4. *Remarks.* (1) It is proved in [Mi1] that for a metrizable space X the following conditions are equivalent: (a) X is completely metrizable, (b) X is Čech-complete and (c) X is cover-complete. Also note that a scattered Hausdorff space is always cover-complete (hence almost complete), since the set of all singletons is an exhaustive cover of the space.

(2) Let X be an uncountable Polish space. A subset A of X is called *perfectly meager* (*always of first category* in [Ku, Section 40]) if, for each perfect set P in X, $P\cap A$ is of the first category (*i.e.*, meager) in P. We show that if A is a perfectly meager subset of X, then $B\overset{\text{def}}{=}X\backslash A$ is hereditarily almost complete. Let F be a closed subset of X and let $F=P\cup S$ where P is perfect and S is scattered. Then for $F\cap B$ to be almost complete it is sufficient that $P\cap B=P\backslash(P\cap A)$ be almost-complete since, being scattered, $S\cap B$ is almost complete. By hypothesis, $P\cap A$ is of the first category in P and P is completely metrizable. Hence, $P\backslash(P\cap A)=P\cap B$ contains a dense G_δ-subset of P, and therefore, by

Theorem 2.2, $P \cap B$ is almost complete. This proves the claim. Now there are $2^{\aleph_1}$ perfectly meager subsets of X [Ku, Section 40 III] which exceed the number of analytic (hence coanalytic) subsets of X if the Continuum Hypothesis (CH) is assumed. Thus we have seen that, under (CH), *each uncountable Polish space contains a hereditarily almost complete set that is not analytic* (*and hence not cover-complete*). This shows that the converse of Lemma 2.3 is false under (CH).

§3. *Property* $\mathcal{N}$. A topological space X is said to have *property* $\mathcal{N}$, if for each compact Hausdorff space K and each continuous map $\varphi: X \to (C(K)$, pointwise), the set of all points of continuity of the map $\varphi: X \to (C(K)$, norm) is a dense (and, necessarily, a G_δ-) subset of X.

3.1. LEMMA. *If a Tychonoff space* X *contains a dense subset* Y *with property* $\mathcal{N}$, *then* X *has property* $\mathcal{N}$.

Proof. By [StR], Y (and hence X) is a Baire space. Let K be a compact Hausdorff space, and let $\varphi: X \to (C(K), \tau_p)$ be a continuous map, where τ_p is the pointwise topology.

For an $\varepsilon > 0$, let

$$O_\varepsilon = \bigcup \{U: U \text{ is open in } X \text{ and diam } \varphi(U) \leqslant \varepsilon\}.$$

Here, the diameter is relative to the norm. As in [N], it suffices to prove that O_ε is dense in X. Let W be a non-empty open subset of X. Then by hypothesis, $\varphi | Y: Y \to (C(K)$, norm) is continuous at some point $p \in Y \cap W$. Then there is an open neighborhood V of p in X such that $V \subset W$ and diam $\varphi(Y \cap V) \leqslant \varepsilon$. Since φ is τ_p-continuous, $\varphi(V) \subset \overline{\varphi(Y \cap V)}^{\tau_p}$. Because the norm in $C(K)$ is τ_p-lower-semicontinuous, the τ_p-closure does not increase the diameter. Therefore,

$$\text{diam } \varphi(V) \leqslant \text{diam } \overline{\varphi(Y \cap V)}^{\tau_p} = \text{diam } \varphi(Y \cap V) \leqslant \varepsilon.$$

Hence $V \subset O_\varepsilon$ and $W \cap O_\varepsilon \supset V \neq \varnothing$.

As shown in [N], each Čech-complete space has property $\mathcal{N}$. Hence by Theorem 2.2 and the above, we have the following corollary.

3.2. COROLLARY. *Each almost complete Tychonoff space* (*in particular, each cover-complete Tychonoff space*) *has property* $\mathcal{N}$.

The definition of property $\mathcal{N}$ immediately implies the following theorem (*cf.* [N, Theorem 2.3]).

3.3. THEOREM. *Let* X *be a space with property* $\mathcal{N}$, *let* E *be a Banach space, and let* $\varphi: X \to (E$, *weak*) *be a continuous map. Then the set of all points of continuity of the map* $\varphi: X \to (E$, *norm*) *is a dense* G_δ-*subset of* X.

§4. σ-*fragmentability and analyticity in Banach spaces.* We begin with the following observation.

4.1. LEMMA. *Let (X, τ) be a Hausdorff space, and let ρ be a metric on X such that (X, ρ) is complete and the ρ-topology is stronger than τ. If (X, τ) is fragmented by ρ, then (X, τ) is cover-complete.*

Proof. Let $n \in \mathbb{N}$. Then since (X, τ) is fragmented by ρ down to n^{-1}, by Lemma 1.1, there is a transfinite sequence $\{X_\alpha : \alpha < \Gamma_n\}$ of non-empty subsets of X such that $X = \bigcup\{X_\alpha : \alpha < \Gamma_n\}$, $X_\alpha \cap X_\beta = \varnothing$ whenever $\alpha < \beta < \Gamma_n$, ρ-diam $X_\alpha < n^{-1}$ for each α and $\bigcup\{X_\alpha : \alpha < \gamma\}$ is open for each $\gamma < \Gamma_n$. If we let $\mathcal{U}_n = \{X_\alpha : \alpha < \Gamma_n\}$, then by Lemma 2.1, $\mathcal{U}_n$ is an exhaustive cover of X. We show that $\{\mathcal{U}_n : n \in \mathbb{N}\}$ is complete. Let $\mathcal{A}$ be a family of subsets of X with the f.i.p. that contains an $\mathcal{U}_n$-small set for each $n \in \mathbb{N}$. Then by completeness of (X, ρ), $\bigcap\{\bar{A}^\rho : A \in \mathcal{A}\} \neq \varnothing$. By hypothesis $\bar{A}^\rho \subset \bar{A}^\tau$ for each $A \in \mathcal{A}$, and therefore $\bigcap\{\bar{A}^\tau : A \in \mathcal{A}\} \neq \varnothing$.

A Tychonoff space X is said to be *Čech-analytic* if there is a Čech-complete subspace of $X \times \mathbb{N}^\mathbb{N}$ that projects onto X. This definition is due to Fremlin [Fre], and, for basic properties of such spaces, refer [Fre], [JNR1] or [Ha3]. Analogously, Hansell [Ha2] defines the space X to be *cover-analytic* if there is a cover-complete subspace of $X \times \mathbb{N}^\mathbb{N}$ that projects onto X. Clearly Čech-analytic spaces are cover-analytic (see Section 2). Similarly, the space X is called *almost Čech-analytic* if there is a hereditarily almost complete subset of $X \times \mathbb{N}^\mathbb{N}$ that projects onto X. By Lemma 2.3, a cover-analytic space is almost Čech-analytic, but the converse is not true as seen from the example in Remarks 2.4(2). (Note that, by Remarks 2.4(1), a cover-analytic subset of a Polish space is analytic.)

4.2. THEOREM. *Let (X, τ) be a Hausdorff space and let ρ be a metric on X such that (X, ρ) is complete and the ρ-topology is stronger than τ. If (X, τ) is σ-fragmented by ρ, then (X, τ) is cover-analytic.*

Proof. Let $\bar{d}$ be the metric on $X \times \mathbb{N}^\mathbb{N}$ as in Theorem 1.3, *viz.* $\bar{d}((x, \sigma), (x', \sigma')) = \max(\rho(x, x'), d(\sigma, \sigma'))$. Then by Theorem 1.3, there exists a subspace $M \subset (X, \tau) \times \mathbb{N}^\mathbb{N}$ that is $\bar{d}$-closed and fragmented by $\bar{d}$ and that projects onto X. Since (X, ρ) is complete, $(X \times \mathbb{N}^\mathbb{N}, \bar{d})$ is complete and so is $(M, \bar{d})$. Also $\bar{d}$-topology is stronger than that of $(X, \tau) \times \mathbb{N}^\mathbb{N}$. It follows from Lemma 4.1 that M is cover-complete, and hence (X, τ) is cover-analytic.

In certain situations, the converse of the above is true as seen in the next Theorem. If $\mathcal{A}$ is a family of subsets of a topological space X, a *Souslin-$\mathcal{A}$* set is a subset L of X of the form

$$L = \bigcup\{\bigcap\{A(\sigma|n) : n \in \mathbb{N}\} : \sigma \in \mathbb{N}^\mathbb{N}\}$$

with each set $A(\sigma|n)$ in $\mathcal{A}$, where $\sigma|n$ denotes the finite sequence $\sigma(1), \sigma(2), \ldots, \sigma(n)$ for each $\sigma \in \mathbb{N}^\mathbb{N}$. Given such a representation of L, one can associate to it a subset Λ of $X \times \mathbb{N}^\mathbb{N}$ defined by

$$\Lambda = \{(x, \sigma) \in X \times \mathbb{N}^\mathbb{N} : x \in A(\sigma|n) \text{ for each } n \in \mathbb{N}\}.$$

Clearly Λ projects onto L, and it can be readily seen that Λ is closed in $X \times \mathbb{N}^\mathbb{N}$ whenever each $A(\sigma|n)$ is closed in X.

4.3. THEOREM. *Let S be a subset of a Banach space E. Then the implications* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *hold among the following conditions:*

(a) *(S, weak) is cover-analytic;*

(b) *(S, weak) is almost Čech-analytic; and*

(c) *(S, weak) is σ-fragmented by the norm-metric ρ.*

If S is a Souslin-$\mathscr{F}$ set, where $\mathscr{F}$ is the family of norm closed sets in E, then (a), (b) *and* (c) *are equivalent.*

Proof. (a) $\Rightarrow$ (b) is remarked above. Assume (b). Then there is a hereditarily almost complete subspace M of (S, weak) $\times \mathbb{N}^{\mathbb{N}}$ such that $p(M) = S$, where p denotes the projection $S \times \mathbb{N}^{\mathbb{N}} \to S$. By Theorem 1.3, for (c) to hold it is sufficient that M be fragmented by $\bar{\rho}$ where $\bar{\rho}((x, \sigma), (x', \sigma')) = \rho(x, x')$. Let $\varepsilon > 0$ and let A be a non-empty subset of M. Then the closure $\bar{A}$ of A in M is almost complete in the relative topology. Hence by Corollary 3.2 and Theorem 3.3, the map $p|\bar{A}: \bar{A} \to (E, \text{norm})$ has a point of continuity. It follows that there exists an open subset U of (S, weak) $\times \mathbb{N}^{\mathbb{N}}$ such that $U \cap \bar{A} \neq \varnothing$ and ρ-diam $p(U \cap \bar{A}) = \bar{\rho}$-diam $(U \cap \bar{A}) < \varepsilon$. Since $U \cap A \neq \varnothing$ whenever $U \cap \bar{A} \neq \varnothing$, M is fragmented by $\bar{\rho}$ down to ε. This proves (b) $\Rightarrow$ (c).

If $S \in \mathscr{F}$, then (S, ρ) is complete, and (c) $\Rightarrow$ (a) follows directly from Theorem 4.2. For the case of Souslin-$\mathscr{F}$ sets, an additional argument is necessary. So let S be a Souslin-$\mathscr{F}$ set that satisfies (c). Then by the remark preceding the theorem, there is a closed subset Σ of (E, norm) $\times \mathbb{N}^{\mathbb{N}}$ such that $p_1(\Sigma) = S$, where $p_1 : E \times \mathbb{N}^{\mathbb{N}} \to E$ is the projection. Let $\bar{d}$ be the product metric on $E \times \mathbb{N}^{\mathbb{N}}$ as in Theorem 1.3. Then $(\Sigma, \bar{d})$ is complete. Denoting by τ the topology of the product (E, weak) $\times \mathbb{N}^{\mathbb{N}}$, we show that (Σ, τ) is σ-fragmented by $\bar{d}$. Clearly, it is sufficient to see that $(S \times \mathbb{N}^{\mathbb{N}}, \tau)$ is σ-fragmented by $\bar{d}$. Let $\varepsilon > 0$. Then it is possible to write S and $\mathbb{N}^{\mathbb{N}}$ as $S = \bigcup\{S_m : m \in \mathbb{N}\}$ and $\mathbb{N}^{\mathbb{N}} = \bigcup\{V_n : n \in \mathbb{N}\}$ where (S_m, weak) is fragmented by ρ down to ε for each m and d-diam $(V_n) < \varepsilon$ for each n. Then it is easy to check that $(S_m \times V_n, \tau)$ is fragmented by $\bar{d}$ down to ε. Since $S \times \mathbb{N}^{\mathbb{N}} = \bigcup\{S_m \times V_n : (m, n) \in \mathbb{N} \times \mathbb{N}\}$ and ε is arbitrary, $(S \times \mathbb{N}^{\mathbb{N}}, \tau)$ is σ-fragmented by $\bar{d}$.

Clearly $\bar{d}$-topology is stronger than τ. Hence by Lemma 4.2 the space (Σ, τ) is cover-analytic, which means that, for some cover-complete subset $M \subset (E, \text{weak}) \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$, $p_2(M) = \Sigma$ where $p_2: E \times \mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}} \to E \times \mathbb{N}^{\mathbb{N}}$ is the projection given by $p_2(x, \sigma, \sigma') = (x, \sigma)$. Then $p_1 p_2(M) = p_1(\Sigma) = S$ and, if one identifies $\mathbb{N}^{\mathbb{N}} \times \mathbb{N}^{\mathbb{N}}$ with $\mathbb{N}^{\mathbb{N}}$, then $p_1 p_2$ is just the projection $E \times \mathbb{N}^{\mathbb{N}} \to E$. Hence ($S$, weak) is cover-analytic.

4.4. *Remarks.* (1) In Theorem 4.3, if $E = C(K)$ for some compact Hausdorff space, all the conclusions are still valid when the weak topology is replaced with the topology of pointwise convergence. The proof is identical except that Theorem 3.3 is not needed.

(2) In Theorem 4.3, consider the additional condition:

(a′) (S, weak) *is Čech-analytic.*

Since, as noted above, (a′) $\Rightarrow$ (a), we obtain (a′) $\Rightarrow$ (c) for each subset S of the Banach space E. This result is essentially obtained in [JNR1] although only the case $S = E$ is enunciated there [JNR1, Proposition 6.3]. Even in the case of $S = E$, we do not know if the converse (c) $\Rightarrow$ (a′) holds. However, in view of the theorem above, this question can be restated as follows. *Is* (E, weak) *Čech-analytic whenever it is cover-analytic*?

(3) If a Tychonoff space M is almost complete then M contains a dense Čech-complete subspace by Theorem 2.2. It follows that each almost Čech-analytic space contains a dense Čech-analytic subspace. Now suppose that E is a Banach space such that (E, weak) is σ-fragmented by the norm-metric. Then by Theorem 4.3 each Souslin-$\mathscr{F}$ subset of E is almost Čech-analytic in the weak topology and therefore contains a weakly dense Čech-analytic subspace. We do not know if the converse is true or not.

(4) Professor P. Holický informed us that (a)$\Leftrightarrow$(c) can be derived from his Theorem 6 in [Ho]. This theorem depends on [Ha1], and, as pointed out by Hansell (private communication), it applies only to Souslin-$\mathscr{F}$ sets. Also Holický's argument relies on as yet unpublished fact that being "almost K-descriptive" is equivalent to being cover-analytic. Our proof *via* the new characterization of σ-fragmentability (Theorem 1.3) seems more direct and, possibly, much simpler.

§5. *Fragmentability and σ-fragmentability.* In [JNR1], the result mentioned in Remarks 4.4(2) is a consequence of the main theorem, *vis.* [JNR1, Theorem 4.1], the most difficult part of which states that a Čech-analytic space X is σ-fragmented by a lower-semicontinuous metric ρ if, and only if, each compact subset of X is fragmented by ρ. In this section we give a simpler proof of this theorem assuming somewhat less—almost Čech-analyticity of X rather than Čech-analyticity. We begin with a lemma which is a non-metrizable variant of a special case of Theorem 1 in Mycielski [My].

5.1. LEMMA. *Let Z be a non-empty Čech-complete space and let R be a closed reflexive relation on Z, i.e., R is a closed subset of $Z\times Z$ that includes the diagonal Δ. If no point of Δ is in the interior of R, then there exists a compact subset K of Z and a continuous surjective map $\varphi: K\to 2^{\mathbb{N}}$ such that $(\varphi^{-1}(\xi)\times\varphi^{-1}(\xi'))\cap R=\emptyset$ whenever ξ and ξ' are distinct points in $2^{\mathbb{N}}$.*

Proof. Let $\{G_n : n\in\mathbb{N}\}$ be a sequence of open subsets of βZ such that $Z=\bigcap\{G_n : n\in\mathbb{N}\}$. By hypothesis, for each non-empty open subsets U of βZ, there are $x, y\in U\cap Z$ such that $(x, y)\notin\bar{R}$, where $\bar{R}$ is the closure of R in $\beta Z\times\beta Z$. Let T be the set of all finite sequences in $2=\{0, 1\}$, and, for each $t\in T$, let $l(t)$ be the length of t. By induction in $l(t)$, we choose a family $\{U(t): t\in T\}$ of non-empty open subsets of βZ such that, for each $t\in T$,

(i) $\overline{U(\emptyset)}=\beta Z$,
(ii) $\overline{U(t)}\subset G_{l(t)}$,
(iii) $U(t0)\cup U(t1)\subset U(t)$ and $\overline{U(t0)}\cap\overline{U(t1)}=\emptyset$, and
(iv) $(\overline{U(t0)}\times\overline{U(t1)})\cap R=\emptyset$.

(i) Starts the induction with $l(\emptyset)=0$. Suppose $U(t)\neq\emptyset$ has been chosen. Then there are x_0, $x_1\in U(t)\cap Z$ such that $(x_0, x_1)\notin\bar{R}$. Then one can choose open neighbourhoods $U(t0)$ and $U(t1)$ of x_0 and x_1 respectively satisfying (iii) and (iv). Since x_0, $x_1 \in Z \cap G_{l(t)+1}$, we can also satisfy $\overline{U(ti)} \subset G_{l(t)+1}$ for $i = 0, 1$. This concludes the inductive step.

Let $K = \bigcap_{n=1}^{\infty} \bigcup \{\overline{U(t)} : l(t) = n\}$. Then K is a compact subset of Z. Define $\varphi: K \to 2^{\mathbb{N}}$ so that $\varphi^{-1}(\xi) = \bigcap \{\overline{U(\xi|n)} : n \in \mathbb{N}\}$ for each $\xi = (\xi_1, \xi_2, \ldots) \in 2^{\mathbb{N}}$. Here $\xi|n = (\xi_1, \ldots, \xi_n)$. Then φ satisfies all the conditions of the lemma.

5.2. THEOREM. *Let ρ be a lower-semicontinuous metric on a Tychonoff space X. Then the implications* (a) $\Rightarrow$ (b) $\Rightarrow$ (c) *hold among the following conditions. Further, if X is almost Čech-analytic then all the conditions are equivalent.*

(a) *X is σ-fragmented by ρ.*

(b) *Each compact subset of X is fragmented by ρ.*

(c) *Whenever φ is a continuous map of a compact subset L of X onto $2^{\mathbb{N}}$, the ρ-distances of the inverse images of distinct points of $2^{\mathbb{N}}$ cannot be bounded away from zero.*

Proof. By [JNR, Theorem 4.1], it is sufficient to prove (c) $\Rightarrow$ (a) assuming that X is almost Čech-analytic, *i.e.*, there exists a hereditarily almost complete subset M of $X \times \mathbb{N}^{\mathbb{N}}$ with $p(M) = X$, where p is the projection $X \times \mathbb{N}^{\mathbb{N}} \to X$. Suppose (a) is false. Then by Theorem 1.3, M is not fragmented by $\bar{\rho}$ where $\bar{\rho}((x, \sigma), (x', \sigma')) = \rho(x, x')$. This means that there is a closed non-empty subspace N of M and an $\varepsilon > 0$ with the following property:

(∗) for each open set $U \subset X \times \mathbb{N}^{\mathbb{N}}$ with $N \cap U \neq \varnothing$, $\bar{\rho}$-diam $(N \cap U) \geqslant \varepsilon$.

Being closed in M, N is almost complete, and hence by Theorem 2.2 N contains a dense Čech-complete subspace Z. The property (∗) again holds with N replaced with Z, because the lower-semicontinuity of ρ (hence, of $\bar{\rho}$) implies

$$\bar{\rho}\text{-diam}\,(Z \cap U) = \bar{\rho}\text{-diam}\,(\overline{Z \cap U}) = \bar{\rho}\text{-diam}\,(\overline{N \cap U}) = \bar{\rho}\text{-diam}\,(N \cap U)$$

whenever U is open in $X \times \mathbb{N}^{\mathbb{N}}$ and $Z \cap U \neq \varnothing$. Therefore, if R is the relation on Z defined by

$$R = \{((x, \sigma), (x', \sigma')) \in Z \times Z : \bar{\rho}((x, \sigma), (x', \sigma')) = \rho(x, x') \leqslant \varepsilon/2\},$$

then R satisfies the conditions of Lemma 4.5.

It follows that there is a compact subset K of Z and a continuous map φ of K onto $2^{\mathbb{N}}$ such that, for $(x, \sigma), (x', \sigma') \in K$, $\varphi(x, \sigma) \neq \varphi(x', \sigma')$ implies that $\bar{\rho}((x, \sigma), (x', \sigma')) = \rho(x, x') > \varepsilon/2$. Let $L = p(K)$. Then L is a compact subset of X. Define a map $f: L \to 2^{\mathbb{N}}$ by $f(x) = \varphi(K \cap p^{-1}(x))$. This map is well-defined, continuous and surjective by the property of φ stated above. It is also clear that if ξ and ξ' are distinct points of $2^{\mathbb{N}}$, then $f^{-1}(\xi)$ and $f^{-1}(\xi')$ are separated by ρ-distance at least $\varepsilon/2$. This contradicts (c), and hence (c) $\Rightarrow$ (a) holds.

By the usual trick of choosing a minimal map, one immediately obtains the following corollary.

5.3. COROLLARY. *Let ρ be a lower-semicontinuous metric on an almost Čech-analytic space X. If X is not σ-fragmented by ρ, then there exists a compact perfect subset K of X and a continuous map f of K onto $2^{\mathbb{N}}$ such that, for some $\varepsilon > 0$, ρ-dist $(f^{-1}(\xi), f^{-1}(\xi')) \geqslant \varepsilon$ whenever ξ and ξ' are distinct points of $2^{\mathbb{N}}$.*

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By letting ρ to be the trivial 2-valued metric, as in the proof of [JNR1, Corollary 4.5.1], we obtain the following (compare [Ha2, Corollary 1.2]).

5.4. COROLLARY. *An almost Čech-analytic space is either σ-scattered (i.e., a countable union of scattered subsets) or contains a non-empty compact perfect set.*

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