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Integrals involving classical polynomials

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1. *Introduction.* In a recent paper the author (2) has derived a number of integrals with the help of results ((5), p. 343)

$$\int_0^\infty \int_0^\infty F(\alpha x+\beta y,\, \alpha' x+\beta' y)\,dx\,dy = \frac{1}{K}\int_0^\infty \int_0^\infty F(u,v)\,du\,dv, \tag{1·1}$$

where α, β, α', β' are parameters such that

$$K = \begin{vmatrix} \alpha & \alpha' \\ \beta & \beta' \end{vmatrix} \neq 0 \tag{1·2}$$

and
$$\int_{-\infty}^\infty \int_{-\infty}^\infty F(\alpha x+\beta y, \alpha' x+\beta' y)\,dx\,dy = \frac{1}{K}\int_{-\infty}^\infty \int_{-\infty}^\infty F(u,v)\,du\,dv, \tag{1·3}$$

where K has the same meaning as above.

The purpose of this paper is to state a theorem for orthogonality and to generalize the integrals (1·1) and (1·3) to n-ples integrals.

2. THEOREM. *Let $\{f_n(x)\}$ be a set of orthogonal functions with the weight function $\phi(x)$ in the range (a, b), then we have*

$$\int_a^b \int_a^b \phi(\alpha x+\beta y)\,\phi(\alpha' x+\beta' y) f_n(\alpha x+\beta y) f_m(\alpha x+\beta y) f_\mu(\alpha' x+\beta' y) f_\nu(\alpha' x+\beta' y)\,dx\,dy$$
$$= (1/K)\,\delta_{(m,n)(\mu,\nu)}, \tag{2·1}$$

where
$$\delta_{(m,n)(\mu,\nu)} = 0 \quad \textit{if} \quad m \neq n \quad \textit{or} \quad \mu \neq \nu$$
$$= \textit{constant} \quad \textit{if} \quad m = n \quad \textit{and} \quad \mu = \nu.$$

Similarly,

'If we have two sets of orthogonal polynomials $\{f'_n(x)\}$ and $\{f''_n(x)\}$ with weight functions $\phi'(x)$ and $\phi''(x)$ respectively in the range (a,b), then we have

$$\int_a^b \int_a^b \phi'(\alpha x+\beta y)\,\phi''(\alpha' x+\beta' y) f'_m(\alpha x+\beta y) f'_n(\alpha x+\beta y) f''_\mu(\alpha' x+\beta' y) f''_\nu(\alpha' x+\beta' y)\,dx\,dy$$
$$= (1/K)\,\delta_{(m\ n)}\delta_{(\mu,\nu)}, \tag{2·2}$$

which is
$$= 0 \quad \text{if} \quad m \neq n \quad \text{or} \quad \mu \neq \nu$$
$$= \text{constant} \quad \text{if} \quad m = n \quad \text{and} \quad \mu = \nu.\text{'}$$

Example. For Laguerre polynomials $L_n^\alpha(x,\gamma,p)$, we have (4)

$$\int_0^\infty x^\alpha e^{-px^\gamma} L_n^\alpha(x,\gamma,p)\, L_m^\alpha(x,s,q)\,dx$$
$$= 0 \quad \text{for} \quad n > ms$$
$$= \frac{(-1)^{m(s+1)}(qs)^m}{m!\,\gamma(p)^{(\alpha+ms+1)/\gamma}}\,\Gamma\left(\frac{\alpha+ms+1}{\gamma}\right) \quad \text{for} \quad n = ms \quad \text{and} \quad Re(\alpha) > -1. \qquad (2\cdot3)$$

Therefore,

$$\int_0^\infty\int_0^\infty (\alpha x+\beta y)^\lambda (\alpha' x+\beta' y)^\mu \exp[-p_1(\alpha x+\beta y)^{\gamma_1} - p_2(\alpha' x+\beta' y)^{\gamma_2}]\, L_n^\lambda(\alpha x+\beta y,\gamma_1,p_1)$$
$$\times L_m^\lambda(\alpha x+\beta y,s,q_1)\, L_\nu^\mu(\alpha' x+\beta' y,\gamma_2,p_2)\, L_\omega^\mu(\alpha' x+\beta' y,t,q_2)\,dx\,dy$$
$$= 0 \quad \text{if} \quad n > ms \quad \text{or} \quad \nu > \omega t$$
$$= \frac{(-1)^{m(s+1)+\omega(t+1)}(q_1 s)^m (q_2 t)^\omega}{Km!\,\omega!\,\gamma_1\gamma_2(p_1)^{(\lambda+ms+1)/\gamma_1}(p_2)^{(\mu+\omega t+1)/\gamma_2}}\,\Gamma\left(\frac{\lambda+ms+1}{\gamma_1}\right)\Gamma\left(\frac{\mu+wt+1}{\gamma_2}\right)$$
$$\text{for} \quad n = ms \quad \text{and} \quad \gamma = \omega t, \quad \text{and} \quad Re(\lambda) > 0, Re(\mu) > 0, \qquad (2\cdot4)$$

which for $\gamma_1 = \gamma_2 = s = t = p_1 = p_2 = q_1 = q_2 = 1$, reduces to

$$\int_0^\infty\int_0^\infty (\alpha x+\beta y)^\lambda (\alpha' x+\beta' y)^\mu \exp[-(\alpha x+\beta y)-(\alpha' x+\beta' y)].\, L_n^\lambda(\alpha x+\beta y)\, L_m^\lambda(\alpha x+\beta y)$$
$$\times L^\mu(\alpha' x+\beta' y)\, L_\omega^\mu(\alpha' x+\beta' y)\,dx\,dy$$
$$= 0 \quad \text{if} \quad n \neq m \quad \text{or} \quad \nu \neq \omega,$$
$$= \frac{\Gamma(\lambda+m+1)\,\Gamma(\mu+\omega+1)}{Km!\,\omega!} \quad \text{for} \quad n = m, \quad \nu = \omega \quad \text{and} \quad Re(\lambda) > 0, \quad Re(\mu) > 0. \qquad (2\cdot5)$$

3. *Some generalizations.* If we have a set of n variables (x_n) and a set of parameters $\{\alpha_{ij}\}$; $i = 1, 2, 3, \ldots, n$ and $j = 1, 2, 3, \ldots, n$, then (1·1) can directly be generalized as follows:

$$\int_0^\infty \cdots \int_0^\infty F[(\alpha_{11}x_1 + \ldots + \alpha_{1n}x_n), \ldots, (\alpha_{n1}x_1 + \ldots + \alpha_{nn}x_n)]\,dx_1\,dx_2\ldots dx_n$$
$$= \frac{1}{K}\int_0^\infty \cdots \int_0^\infty F(u_1, \ldots, u_n)\,du_1\,du_2\ldots du_n, \qquad (3\cdot1)$$

where
$$K = \begin{vmatrix} a_{11} & \alpha_{12} & & \ldots\alpha_{1n} \\ \alpha_{21} & \alpha_{22} & \ldots & \alpha_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ \alpha_{n1} & \alpha_{n2} & \ldots & \alpha_{nn} \end{vmatrix} \neq 0 \qquad (3\cdot2)$$

and in particular

$$\int_0^\infty \cdots \int_0^\infty f_1(\alpha_{11}x_1 + \ldots + \alpha_{1n}x_n) f_2(\alpha_{21}x_1 + \ldots + \alpha_{2n}x_n) \ldots f_n(\alpha_{n1}x_1 + \ldots + \alpha_{nn}x_n)$$
$$\times dx_1\,dx_2\ldots dx_n$$
$$= \frac{1}{K}\int_0^\infty f_1(u_1)\,du_1 \int_0^\infty f_2(u_2)\,du_2 \ldots \int_0^\infty f_n(u_n)\,du_n. \qquad (3\cdot3)$$

Similar generalizations can be given to (1·3), (2·1) and (2·2).

Now we give below certain integrals derived with the help of the above generalizations. For brevity I consider only triple integral but the results can be extended to n-ples integrals.

Example 1. We know for Hermite polynomials (3)

$$\int_{-\infty}^{\infty} \exp(-x^2)\, H_n(x)\, H_m(x)\, dx = 0 \quad \text{if} \quad n \neq m,$$
$$= 2^n\sqrt{\pi}\, n! \quad \text{if} \quad n = m.$$

Therefore,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-(\alpha_1 x+\beta_1 y+\gamma_1 z)^2-(\alpha_2 x+\beta_2 y+\gamma_2 z)^2-(\alpha_3 x+\beta_3 y+\gamma_3 z)^2]$$
$$\times \prod_{j=1}^{3} [H_{n_j}(\alpha_j x+\beta_j y+\gamma_j z)\,.\,H_{m_j}(\alpha_j x+\beta_j y+\gamma_j z)]\, dx\, dy\, dz \qquad (3\cdot 4)$$
$$= 0 \quad \text{if} \quad n_j \neq m_j \quad (j = 1, 2, 3),$$
$$= 2^{n_1+n_2+n_3}(\sqrt{\pi})^3\, (n_1)!\, (n_2)!\, (n_3)! \quad \text{if} \quad n_j = m_j \quad (j = 1, 2, 3).$$

Example 2. We know ((3); Ex (4), p. 199)

$$P_n(x) = \frac{2}{n!\sqrt{\pi}}\int_0^{\infty} \exp(-t^2)\, t^n H_n(xt)\, dt.$$

Therefore, we have

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \exp[-(\alpha_1 x+\beta_1 y+\gamma_1 z)-(\alpha_2 x+\beta_2 y+\gamma_2 z)-(\alpha_3 x+\beta_3 y+\gamma_3 z)]$$
$$\times \prod_{j=1}^{3} [H_{m_j}(u_j(\alpha_j x+\beta_j y+\gamma_j z))]\, dx\, dy\, dz$$
$$= \frac{(\sqrt{\pi})^3\, m_1!\, m_2!\, m_3!}{8K} P_{m_1}(u_1)\, P_{m_2}(u_2)\, P_{m_3}(u_3). \qquad (3\cdot 5)$$

Example 3. Again since (1)

$$\int_0^{\infty} e^{-x} x^k L_m^{\alpha}(x)\, L_n^{\beta}(x)\, dx = \frac{(1+\alpha)_m (\beta-k)_n\, \Gamma(k+1)}{m!\, n!}\, {}_3F_2\left[\begin{matrix}-m, k+1, 1-\beta+k;\\ 1+\alpha, 1-\beta+k-n;\end{matrix}\right],$$

therefore,

$$\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} \exp[-(\alpha_1+\alpha_2+\alpha_3)x-(\beta_1+\beta_2+\beta_3)y-(\gamma_1+\gamma_2+\gamma_3)z]\, (\alpha_1 x+\beta_1 y+\gamma_1 z)^{p_1}$$
$$\times (\alpha_2 x+\beta_2 y+\gamma_2 z)^{p_2}\, (\alpha_3 x+\beta_3 y+\gamma_3 z)^{p_3}$$
$$\times \prod_{j=1}^{3} L_{m_j}^{a_j}(\alpha_j x+\beta_j y+\gamma_j z)\, L_{n_j}^{b_j}(\alpha_j x+\beta_j y+\gamma_j z)\, dx\, dy\, dz$$
$$= \prod_{j=1}^{3}\left[\frac{(1+\alpha_j)_{m_j}(b_j-p_j)_{n_j}\, \Gamma(p_j+1)}{(m_j)!\,(n_j)!}\, {}_3F_2\left[\begin{matrix}-m_j,\ p_j+1,\ 1-b_j+p_j;\\ 1+a_j,\ 1-b_j+p_j-n_j\end{matrix}\right]\right]. \qquad (3\cdot 6)$$

When $p_j = a_j$ $(j = 1, 2, 3)$, integral (3·6) reduces to

$$\Gamma(p_1+1)\, \Gamma(p_2+1)\, \Gamma(p_3+1) \quad (n_j \geqslant m_j) \qquad (3\cdot 7)$$

and for $p_j = a_j + b_j$ $(j = 1, 2, 3)$, we get

$$\prod_{j=1}^{3} (-1)^{m_j+n_j} \binom{a_j+m_j}{n_j} \binom{b_j+n_j}{m_j} \Gamma(a_j+b_j+1). \tag{3·8}$$

Since, for $p_j = a_j + b_j$ $(j = 1, 2, 3)$, ${}_3F_2$ of (3·6) reduces to

$$\frac{(-\beta_j - n_j)_{m_j}}{(1-a_j-n_j)_{m_j}} = \frac{(-a_j)_{n_j-m_j}(1+b_j)_{n_j}}{(-a_j)_{n_j}(1+b_j)_{n_j-m_j}}$$

and
$$\frac{(1+a_j)(-a_j)_{n_j}}{m_j!\,n_j!} \frac{(-a_j)_{n_j-m_j}(1+b_j)_{n_j}}{(-a_j)_{n_j}(1+b_j)_{n_j-m_j}} = (-1)^{m_j+n_j}\binom{a_j+m_j}{n_j}\binom{b_j+n_j}{m_j}.$$

Example 4. Lastly, consider

$$f_1(\alpha_1 x+\beta_1 y+\gamma_1 z) = (\alpha_1 x+\beta_1 y+\gamma_1 z)^{-\frac{1}{2}} e^{-(\alpha_1 x+\beta_1 y+\gamma_1 z)} f_l \left[\begin{matrix}(a_p);\ (\alpha_2 x+\beta_2 y+\gamma_2 z)\ (\alpha_3 x+\beta_3 y+\gamma_3 z \\ \frac{1}{2},\ (b_q)\end{matrix}\right],$$

$$f_2(\alpha_2 x+\beta_2 y+\gamma_2 z) = (\alpha_2 x+\beta_2 y+\gamma_2 z)^{-\frac{1}{2}} e^{-(\alpha_2 x+\beta_2 y+\gamma_2 z)} f_m \left[\begin{matrix}(a_p);\ (\alpha_1 x+\beta_1 y+\gamma_1 z)\ (\alpha_3 x+\beta_3 y+\gamma_3 z) \\ \frac{1}{2},\ (b_q)\end{matrix}\right],$$

$$f_3(\alpha_3 x+\beta_3 y+\gamma_3 z) = (\alpha_3 x+\beta_3 y+\gamma_3 z)^{-\frac{1}{2}} e^{-(\alpha_3 x+\beta_3 y+\gamma_3 z)} f_n \left[\begin{matrix}(a_p);\ (\alpha_1 x+\beta_1 y+\gamma_1 z)\ (\alpha_2 x+\beta_2 y+\gamma_2 z) \\ \frac{1}{2},\ (b_q)\end{matrix}\right].$$

where (a_p) means $a_1, a_2, a_3, \ldots, a_p$ and (b_q) means $b_1, b_2, b_3, \ldots, b_q$ and $f_n(x)$ is Sister Celine's polynomial (3).

Therefore, with the help of ((3); p. 291)

$$\int_0^\infty \int_0^\infty \int_0^\infty \prod_{j=1}^{3} (\alpha_j x+\beta_j y+\gamma_j z)^{-\frac{1}{2}} \exp\left[-\sum_{j=1}^{3}(\alpha_j x+\beta_j y+\gamma_j z)\right] f_l[\ ]\, f_m[\ ]\, f_n[\ ]\, dx\, dy\, dz$$
$$= \frac{(\pi)^{\frac{3}{2}}}{K} f_l\left[\begin{matrix}(a_p);\ u \\ (b_q)\end{matrix}\right] f_m\left[\begin{matrix}(a_p);\ v \\ (b_q)\end{matrix}\right] f_n\left[\begin{matrix}(a_p);\ w \\ (b_q)\end{matrix}\right]. \tag{3·9}$$

Particular cases of (3·9). (*a*) Let $p = 1$, $a_1 = \frac{1}{2}$, $q = 1$, $b_1 = \frac{1}{2}$ in (3·9), to get

$$\int_0^\infty \int_0^\infty \int_0^\infty \prod_{j=1}^{3} (\alpha_j x+\beta_j y+\gamma_j z)^{-\frac{1}{2}} \exp\left[-\sum_{j=1}^{3}(\alpha_j x+\beta_j y+\gamma_j z)\right]$$
$$\times f_l[-;\ 1;\ (\alpha_2 x+\beta_2 y+\gamma_2 z)(\alpha_3 x+\beta_3 y+\gamma_3 z)]$$
$$\times f_m[-;\ 1;\ (\alpha_1 x+\beta_1 y+\gamma_1 z)(\alpha_3 x+\beta_3 y+\gamma_3 z)]$$
$$\times f_n[-;\ 1;\ (\alpha_1 x+\beta_1 y+\gamma_1 z)(\alpha_2 x+\beta_2 y+\gamma_2 z)]\, dx\, dy\, dz$$
$$= (\pi^{\frac{3}{2}}/K)\, Z_l(u)\, Z_m(v)\, Z_n(w), \tag{3·10}$$

where $Z_n(x)$ is Bateman's polynomial ((3); p. 285).

(*b*) If $p = 1$, $a_1 = \frac{1}{2}$, $q = 0$, we have

$$\int_0^\infty \int_0^\infty \int_0^\infty \prod_{j=1}^{3} (\alpha_j x + \beta_j y + \gamma_j z)^{-\frac{1}{2}} \exp\left[-\sum_{j=1}^{3} (\alpha_j x + \beta_j y + \gamma_j z)\right]$$

$$\times f_l[-;-;(\alpha_2 x + \beta_2 y + \gamma_2 z)(\alpha_3 x + \beta_3 y + \gamma_3 z)]$$

$$\times f_m[-;-;(\alpha_1 x + \beta_1 y + \gamma_1 z)(\alpha_3 x + \beta_3 y + \gamma_3 z)]$$

$$\times f_n[-;-;(\alpha_1 x + \beta_1 y + \gamma_1 z)(\alpha_2 x + \beta_2 y + \gamma_2 z)]\, dx\, dy\, dz$$

$$= (\pi^{\frac{3}{2}}/K)\, P_l(1-2u)\, P_m(1-2v)\, P_n(1-2w), \tag{3·11}$$

where $P_n(x)$ is Legendre's polynomials ((3); p. 157).

(*c*) If $p = 2$, $a_1 = \frac{1}{2}$, $a_2 = \xi$, $q = 1$, $b_1 = p$, we get from (3·9)

$$\int_0^\infty \int_0^\infty \int_0^\infty \prod_{j=1}^{3} (\alpha_j x + \beta_j y + \gamma_j z)^{-\frac{1}{2}} \exp\left[-\sum_{j=1}^{3} (\alpha_j x + \beta_j y + \gamma_j z)\right]$$

$$\times f_l[\xi;p;(\alpha_2 x + \beta_2 y + \gamma_2 z)(\alpha_3 x + \beta_3 y + \gamma_3 z)]$$

$$\times f_m[\xi;p;(\alpha_1 x + \beta_1 y + \gamma_1 z)(\alpha_3 x + \beta_3 y + \gamma_3 z)]$$

$$\times f_n[\xi;p;(\alpha_1 x + \beta_1 y + \gamma_1 z)(\alpha_2 x + \beta_2 y + \gamma_2 z)]\, dx\, dy\, dz$$

$$= (\pi^{\frac{3}{2}}/K)\, H_l(\xi,p,u)\, H_m(\xi,p,v)\, H_n(\xi,p,w). \tag{3·12}$$

where $H_n(\xi, p, x)$ is Rice's polynomial ((3); p. 28).

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