THE TWO-BALL PROPERTY: TRANSITIVITY AND EXAMPLES

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Abstract. The 2-ball property is shown to be transitive. Combining this with some results on the decomposability of convex bodies, we produce new examples of Banach spaces which contain proper semi-M-ideals. These semi-M-ideals are not hyperplanes, nor are they the direct sums of examples which are hyperplanes.

We shall be concerned here with subspaces of Banach spaces which have the n-ball property for some $n \in \mathbf{N} \cup \{1\frac{1}{2}\}$. Recall that a closed subspace M of a Banach space X is said to have the n-ball property ($n \in \mathbf{N}$) if, whenever $B(a_1, r_1), \ldots, B(a_n, r_n)$ are closed balls in X, with $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$, and $M \cap B(a_i, r_i) \neq \emptyset$ for each i, then $M \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$. If the conditions $B(a_1, r_1) \cap B(a_2, r_2) \neq \emptyset$, $M \cap B(a_1, r_1) \neq \emptyset$ and $a_2 \in M$ imply that $M \cap B(a_1, r_1 + \varepsilon) \cap B(a_2, r_2 + \varepsilon) \neq \emptyset$ for every $\varepsilon > 0$, then we say that M has the $1\frac{1}{2}$-ball property in X. If we can take $\varepsilon = 0$ in these definitions, then we speak of the strong n-ball property ($n \in \mathbf{N} \cup \{1\frac{1}{2}\}$). These concepts were first studied by Alfsen and Effros, who showed, amongst other things, that a subspace with the 3-ball property already has the n-ball property for all n [**1**, Theorem I.5.9]. For further work along these lines, see [**2**, Ch. 2], [**9**] and [**11**]. We refer to [**17**] for an account of the duality theory of such subspaces, elementary approximation theory, and for most definitions omitted in this paper.

Examples of M-ideals (*i.e.*, subspaces with the 3-ball property) are well known these days, predominantly from Banach algebras and operator theory. They include:

any ideal in a C^*-algebra;
certain ideals in uniform algebras;
the compact operators in $B(l_p)$, for $1 < p < \infty$;
the compact operators in $B(X, c_0)$, for any Banach space X.

Examples of subspaces which have the $1\frac{1}{2}$-ball property but not the 2-ball property (*i.e.*, are not semi-M-ideals) are also well known now. They include:

subalgebras (other than ideals) in $C_R(K)$, for any compact Hausdorff space K;
the compact operators in $B(L_1(\mu), l_1)$, for almost any measure μ;
the compact operators in $B(C(K_1), C(K_2))$, when K_1 is dispersed and K_2 is extremally disconnected.

Again the examples have a somewhat algebraic flavor.

[MATHEMATIKA, **35** (1988), 190–197]

However, when we search for some subspaces, which have the 2-ball property but not the 3-ball property, examples are rather thin on the ground. In complex Banach spaces, not even one example is known. (There is good reason to believe that, in complex Banach spaces, the 2-ball property is equivalent to the 3-ball property [**18**].) In real Banach spaces, a typical example is $X = L_1(\mu)$ and $M = \{f \in X: \int f d\mu = 0\}$, where μ is any measure for which the dimension of $L_1(\mu)$ is at least three. For the case when $L_1(\mu)$ is 3-dimensional, this was the first counterexample, produced originally by Alfsen and Effros [**1**, Theorem I.5.9]. More generally, let us recall that a Banach space X is a base norm space if its unit ball U has a closed proper face F, known as the base, such that the convex hull of $-F \cup F$ contains the interior of U. This is easily shown to be equivalent to the usual definition in terms of vector orderings [**16**, Ch. 9]. Lima [**9**, Section 3] showed that a real Banach space is a base norm space, if, and only if, it contains a hyperplane with the 2-ball property. We digress to give a short proof of this, *via* a similar result for the $1\frac{1}{2}$-ball property. For any set S in a Banach space X we denote by S_1 the intersection of S with the closed unit ball of X. If M is a subspace of X, then $M^\perp$ will denote its metric complement, $M^\perp = \{x \in X: \|x\| = d(x, M)\}$.

PROPOSITION 1. *Let M be a hyperplane in a real Banach space X. Then*
(i) *M has the $1\frac{1}{2}$-ball property in X, if, and only if,* $\operatorname{int} X_1 \subset \operatorname{co}(M_1 \cup M_1^\perp)$; *and*
(ii) *M has the strong $1\frac{1}{2}$-ball property in X, if, and only if,*

$$X_1 = \operatorname{co}(M_1 \cup M_1^\perp).$$

Proof. We only prove (i), since the proof of (ii) is quite similar. ($\Rightarrow$) Choose a in the interior of X_1 and set $d = d(a, M)$. If $d = 0$ then $a \in M_1$, so we assume that $0 < d < 1$. Courtesy of the $1\frac{1}{2}$-ball property, there is an $x \in M \cap B(a, d) \cap B(0, 1-d)$. It follows immediately that

$$a = x + (a - x) \in (1-d)M_1 + dM_1^\perp.$$

($\Leftarrow$) Suppose $M = \ker f$ where $f \in X^*$ and $\|f\| = 1$. We set

$$M^\pm = \{x \in X_1: f(x) = \pm 1\}.$$

Then M^+ and M^- are closed, convex sets and, by hypothesis,

$$\operatorname{int} X_1 \subseteq \operatorname{co}(M_1 \cup M^+ \cup M^-) = \operatorname{co}(M_1 \cup M^+) \cup \operatorname{co}(M_1 \cup M^-).$$

Now consider $a \in X$ and $r > 0$ such that $M \cap B(a, r) \neq \emptyset$ and $\|a\| < r+1$. We assume without loss of generality that $f(a) > 0$, so $(r+1)^{-1}a \in \operatorname{co}(M_1 \cup M^+)$. Then

$$a = (r+1-f(a))x + f(a)b$$

for some $x \in M_1$, $b \in M^+$. Now $M \cap B(a, r) \neq \emptyset$ forces $|f(a)| = d(a, M) \leq r$. Thus $\|a - x\| = \|(r - f(a))x + f(a)b\| \leq r$, and so $x \in M \cap B(0, 1) \cap B(a, r)$, as required.

The assumption that M is a hyperplane is not needed in the first half of the preceding proof. However, it is essential in the second half. For if $X = l_\infty(3)$

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and M is the linear span of $(0, 1, 2)$, then $X_1 = \text{co}\,(M_1 \cup M_1^{\perp})$, but M fails the $1\frac{1}{2}$-ball property. Nonetheless a subspace M has the $1\frac{1}{2}$-ball property in X, if, and only if, it has the $1\frac{1}{2}$-ball property in N for every 1-dimensional extension N of M. Thus Proposition 1 can be interpreted as a characterization of the $1\frac{1}{2}$-ball property.

Garkavi [**4**, p. 477] was the first to give an example showing that the $1\frac{1}{2}$-ball property is strictly weaker than the strong $1\frac{1}{2}$-ball property. In fact, he showed that any nonreflexive space can be renormed so that a given hyperplane has the $1\frac{1}{2}$-ball property but not the strong $1\frac{1}{2}$-ball property.

PROPOSITION 2 [**9**]. *A real Banach space X is a base norm space, if, and only if, it contains a hyperplane with the* 2-*ball property.*

Proof. ($\Rightarrow$) Let M be a hyperplane parallel to the base. By Proposition 1, M has the $1\frac{1}{2}$-ball property. According to [**17**, Theorem 4], it suffices to show that M has the unique extension property. Given $f \in M^*$ with $\|f\| = 1$, we must show that

$$\sup\{f(y) - \|y - a\| : y \in M\} \geqslant \inf\{f(z) + \|z - a\| : z \in M\}, \quad \text{for all } a \in X.$$

We may assume that $d(a, M) = 1$. The base may then be taken to be $(a - M)_1$.

Given $\varepsilon > 0$, we can find $x \in \text{int}\, M_1$ with $f(x) > 1 - \varepsilon$. Now

$$x = \lambda(a - z) - (1 - \lambda)(a - y)$$

for some $y, z \in M \cap B(a, 1)$ and some $\lambda \in [0, 1]$. But $x \in M$, so $\lambda = \frac{1}{2}$. Thus

$$(f(y) - \|y - a\|) - (f(z) + \|z - a\|) = f(2x) - 2 > -2\varepsilon,$$

as required.

($\Leftarrow$) Let M be a hyperplane with the 2-ball property. Then $M = \ker f$ for some $f \in M^*$ with $\|f\| = 1$. Since M is proximinal in X [**17**, Theorem 3], $F = X_1 \cap f^{-1}(1)$ is a nonempty, closed proper face of X_1. To show that $\text{co}\,(-F \cup F)$ contains the interior of X_1, first fix $a \in F$.

Now let $x \in X$ be given, with $\|x\| < 1$. Put $\lambda = \frac{1}{2}(1 + f(x))$; then $0 < \lambda < 1$. Now $M \cap B(a, 1) \neq \varnothing$ and $d(a - \lambda^{-1}x, M) = |f(a - \lambda^{-1}x)| = \lambda^{-1}(1 - \lambda)$ so that the 2-ball property gives us some

$$b \in M \cap B(a, 1) \cap B(a - \lambda^{-1}x, \lambda^{-1}(1 - \lambda)).$$

Put $y = a - b$ and $z = (1 - \lambda)^{-1}(\lambda y - x)$. Clearly $y \in F$. Also

$$f(z) = (1 - \lambda)^{-1}(\lambda - f(x)) = 1$$

and $\|z\| = \lambda(1 - \lambda)^{-1}\|a - \lambda^{-1}x - b\| \leqslant 1$, whence $z \in F$. Thus

$$x = \lambda y - (1 - \lambda)z \in \text{co}\,(-F \cup F).$$

Recall that a real Banach space is an order unit space, if, and only if, it contains a 1-dimensional semi-L-summand [**10**, Theorem 4.7]. Given this, and Proposition 2, the duality between order unit spaces and base norm spaces [**16**, Chapter 9] follows immediately from the duality between semi-L-summands and the 2-ball property [**17**].

So, the best known examples of subspaces with the 2-ball property but not the 3-ball property are all hyperplanes. Of course, examples which are not hyperplanes can easily be constructed by taking direct sums, or even injective tensor products. If M (respectively N) has the 2-ball property in X (respectively Y), then $M \oplus N$ has the 2-ball property in $X \oplus Y$, provided we take the l_∞-norm on the direct sums. If M fails the 3-ball property in X and N is a proper subspace of Y, then $M \oplus N$ has codimension at least two in $X \oplus Y$, and the 3-ball property fails. (We allow the possibility that $N = \{0\}$.) One of our purposes here is to exhibit examples of codimension two which do not admit such trivial decompositions.

Our technique is to first show that the 2-ball property is transitive, and then use some ideas from Shephard [**14**]. In the process we study transitivity of the n-ball properties in general. Note that we are only concerned with isometric problems in this paper. As remarked after [**18**, Proposition 4], any real Banach space (of dimension at least three) can be renormed so that a given hyperplane has the 2-ball property but not the 3-ball property.

Recall that M is said to be L-proximinal in X [**12**] if it is proximinal and $\|x\| = d(x, M) + d(0, P_M(x))$ for all $x \in X$. Part (i) of the next result was first proved by Godini [**5**, Corollary 4].

PROPOSITION 3. (i) *A closed subspace M of a Banach space X has the $1\frac{1}{2}$-ball property, if, and only if, it is L-proximinal.*

(ii) *Furthermore, it has the strong $1\frac{1}{2}$-ball property, if, and only if, it is L-proximinal and the infimum defining $d(0, P_M(x))$ is attained, for all $x \in X$.*

Proof. As before, it is only necessary to prove part (i) in detail. ($\Rightarrow$) For any $x \in X$, $\varepsilon > 0$, the $1\frac{1}{2}$-ball property gives us a point

$$y \in M \cap B(x, d(x, M)) \cap B(0, \|x\| - d(x, M) + \varepsilon)$$
$$= P_M(x) \cap B(0, \|x\| - d(x, M) + \varepsilon).$$

Letting $\varepsilon \to 0$ we obtain $d(0, P_M(x)) \leq \|x\| - d(x, M)$. This is sufficient, as the reverse inequality is trivial.

($\Leftarrow$) Given any $x \in \text{int } M_1$, we set $\varepsilon = 1 - \|x\|$. Choose $y \in P_M(x)$ so that $\|y\| < d(0, P_M(x)) + \varepsilon = 1 - d(x, M)$. Then $x = y + (x - y) \in \text{co}(M_1 \cup M_1^\perp)$. By Proposition 1, M has the $1\frac{1}{2}$-ball property.

THEOREM 4. *Let X, Y and Z be Banach spaces satisfying $X \subset Y \subset Z$. Suppose that X is a semi-L-summand in Z.*

(i) *If Y/X has the $1\frac{1}{2}$-ball property in Z/X, then Y has the $1\frac{1}{2}$-ball property in Z.*
(ii) If *Y/X has the strong $1\frac{1}{2}$-ball property in Z/X, then Y has the strong $1\frac{1}{2}$-ball property in Z.*
(iii) *If Y/X is a semi-L-summand in Z/X, then Y is a semi-L-summand in Z.*
(iv) *If Y/X is an L-summand in Z/X, and also X is an L-summand in Z, then Y is an L-summand in Z.*

Proof. For any $z \in Z$, we note for later use that $d(z + X, Y/X) = d(z, Y)$. (This is true whether or not X is a semi-L-summand.) Also, let $\pi: Z \to X$ be the semi-L-projection.

(i) Let $z \in Z$. We must show that $\|z\| = d(z, Y) + d(0, P_Y(z))$. Choose $\varepsilon > 0$. We know that $\|z+X\| = d(z+X, Y/X) + d(0, P_{Y/X}(z+X))$. Choosing $x+X \in P_{Y/X}(z+X)$ so that $\|x+X\| < d(0, P_{Y/X}(z+X)) + \varepsilon$, we obtain $\|z+X\| > \|(z+X)-(x+X)\| + \|x+X\| - \varepsilon$. Then

$$\begin{aligned}\|z-\pi(z)\| + \|\pi(z)-\pi(x)-\pi(z-x)\| &= \|z-\pi(x)-\pi(z-x)\| \\ &\leqslant \|x-\pi(x)\| + \|z-x-\pi(z-x)\| \\ &= \|x+X\| + \|z-x+X\| \\ &< \|z+X\| + \varepsilon \\ &= \|z-\pi(z)\| + \varepsilon.\end{aligned}$$

It follows that $\|\pi(z)-\pi(x)-\pi(z-x)\| < \varepsilon$. Now set $y = x + \pi(z-x) \in Y$. Then $y \in P_Y(z)$, because $\|z-y\| = \|z-x+X\| = d(z+X, Y/X) = d(z, Y)$. Furthermore,

$$\begin{aligned}\|z\| &= \|\pi(z)\| + \|z+X\| \\ &> \|\pi(z)\| + \|x+X\| + \|z-x+X\| - \varepsilon \\ &> \|\pi(x)+\pi(z-x)\| - \varepsilon + \|x-\pi(x)\| + d(z, Y) - \varepsilon \\ &= \|x+\pi(z-x)\| + d(z, Y) - 2\varepsilon \\ &= \|y\| + d(z, Y) - 2\varepsilon.\end{aligned}$$

Since ε was arbitrary, $\|z\| \geqslant d(0, P_Y(z)) + d(z, Y)$. The reverse inequality is trivial, so Y has the $1\frac{1}{2}$-ball property in Z.

(ii) Similar to the proof of part (i), but with $\varepsilon = 0$.

(iii) By [**17**, Theorem 5], it suffices to show that Y is a Chebyshev subspace of Z. This follows from [**3**, Theorem 9], but we would rather give a direct argument. Note that if $y \in P_Y(z)$, then $y + X \in P_{Y/X}(z+X)$ and $\pi(z-y) = 0$. (Since $\|(y+X)-(z+X)\| \leqslant \|y-z\| = d(z, Y) = d(z+X, Y/X)$ and $\|y-z\| = d(z+X, Y/X) \leqslant \|(y+X)-(z+X)\| = d(y-z, X)$.) Now suppose that $y_1, y_2 \in P_Y(z)$. Since Y/X is Chebyshev in Z/X, we must have

$$y_1 + X = y_2 + X.$$

Hence $y_1 = y_1 + \pi(z - y_2 + y_2 - y_1) = y_1 + y_2 - y_1 + \pi(z - y_2) = y_2$. This shows that $P_Y(z)$ is a singleton.

(iv) This part can be established by a simple direct argument. Alternatively, the semi-L-projection given by part (iii) is easily verified to be linear.

THEOREM 5. *Let $X \subset Y \subset Z$ be Banach spaces such that X has the m-ball property in Y, and Y has the n-ball property in Z, where $n \geqslant 2$. Then X has the* min $\{m, n\}$*-ball property in Z.*

Proof. We assume that $m \geqslant 1\frac{1}{2}$, otherwise there is nothing to prove. By X^0 and Y^0 we denote the annihilators of X and Y when considered as subspaces of Z. When Y^* is identified with Z^*/Y^0, the annihilator of X in Y^* becomes X^0/Y^0. Thus X^0/Y^0 has the $1\frac{1}{2}$-ball property (or is a semi-L-summand, or an L-summand, depending on the value of m) in Z^*/Y^0. Since

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Y^0 is a semi-L-summand in Z^*, the various cases of Theorem 4 tell us that X^0 has the $1\frac{1}{2}$-ball property (or is a semi-L-summand, or an L-summand) in Z^*. The conclusion then follows from standard duality results [**17**].

Thus the 2-ball property is transitive. This was already known for the 3-ball property [**1**, p. 138]. Our next result shows that the $1\frac{1}{2}$-ball property is not transitive.

EXAMPLE 6. *There is no analogue of Theorem 5 for $n = 1\frac{1}{2}$.*

Proof. Take $Z = \mathbf{R}^3$, equipped with the l_1-norm, $Y = \{(\alpha, \alpha, \beta) : \alpha, \beta \in \mathbf{R}\}$ and $X = \{(\alpha, \alpha, 2\alpha) : \alpha \in \mathbf{R}\}$. Routine calculations show that X is an M-summand in Y, and so has the n-ball property for every n. Note that every extreme point of Z_1 lies in Y or $Y^\perp$. Proposition 1 then tells us that Y has the $1\frac{1}{2}$-ball property in Z. However, Proposition 1 also shows us that X^0 does not have the $1\frac{1}{2}$-ball property in Z^*. Thus X fails to have even the $1\frac{1}{2}$-ball property in Z.

For the remainder of this paper, we will consider only Minkowski spaces, *i.e.*, finite dimensional real Banach spaces. In this situation, every M-ideal is automatically an M-summand. As in [**18**], we will call a Banach space a proper semi-M-ideal if it is a semi-M-ideal, without being an M-summand, in some larger Banach space. Following Shephard [**14**], a convex body P is said to be reducible if $P = \frac{1}{2}(Q - Q)$ for some convex body Q which is not a translate of P. It is clear from Proposition 2 that a Banach space is a proper semi-M-ideal, if, and only if, its unit ball is reducible, if, and only if, it contains nontrivial sets of constant width [**8**]. Grünbaum [**6**] showed that a parallelogram is not reducible. Thus $\mathbf{R}^2$, equipped with the l_1 (or l_∞) norm is not a proper semi-M-ideal. Conversely, Straus and Asplund [**6**] showed that any symmetric 2-dimensional convex body, other than a parallelogram, is reducible. Thus every 2-dimensional Banach space, not isometric to $l_\infty(2)$, is a proper semi-M-ideal. Even the 2-dimensional Hilbert space is a proper semi-M-ideal.

Determining which n-dimensional Banach spaces are proper semi-M-ideals is not so easy when $n \geq 3$. The existence [**8**, p. 621] of nontrivial sets of constant width in $\mathbf{R}^n$ shows that its unit ball is reducible. (Consider a complete set of unit diameter, containing a regular simplex with sides of unit length.) Hence the n-dimensional Hilbert space is a proper semi-M-ideal. Various authors, ranging for example from [7] to [**15**], have studied properties of sets of constant width in Minkowski spaces. To the best of our knowledge, only Shephard [**14**] has attempted to determine which Minkowski spaces have nontrivial sets of constant width. He examined the problem of reducibility for polytopes. From [**14**] it follows, for example, that cubes, octahedrons and icosahedrons are not reducible, but that dodecahedrons are reducible. Thus $l_1(3)$ and $l_\infty(3)$ are not proper semi-M-ideals, but the Banach space whose unit ball is a regular dodecahedron is a proper semi-M-ideal. For completeness, let us mention the positive results of Shephard.

THEOREM 7 [**14**]. *Let P be a symmetric polytope in $\mathbf{R}^n$.*

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(i) *Let F be an $(n-1)$-face of P, whose absolutely convex hull is not all of P. Suppose that exactly n edges of P meet at every vertex of F. Then P is reducible.*

(ii) *Suppose P is the vector sum of line segments. Then P is reducible, if, and only if, some three of the segments are coplanar.*

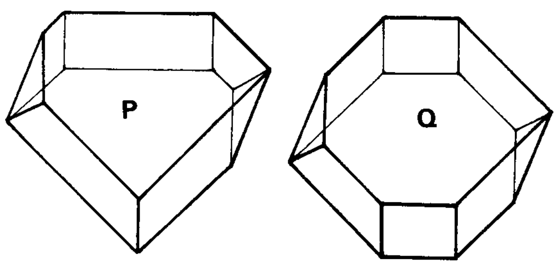
(iii) *Suppose $P = Q + R$, where the symmetric polytopes Q and R lie in two subspaces whose intersection is a singleton. Then P is reducible, if, and only if, at least one of Q, R is reducible.*

We will not explicitly make use of Theorem 7. However, the next result derives from a careful study of the work in [**13**] and [**14**].

THEOREM 8. *There is a 4-dimensional Banach space Z, containing no M-summands, but which contains a 2-dimensional proper semi-M-ideal, X. Moreover, the unit ball of X is an octagon, so X has no M-summands.*

Proof. In $\mathbf{R}^3$, let the polytope P be the convex hull of the twelve points

$$(\pm 4, 3, \pm 1),\ (-5, \pm 2, 1),\ \pm(7, 0, 1),\ (5, \pm 2, -1) \text{ and } (0, -7, \pm 1).$$



Let the polytope Q be the convex hull of the fourteen points

$$(\pm 2, \pm 5, \pm 1),\ (-5, \pm 2, 1),\ (5, \pm 2, -1) \text{ and } \pm(7, 0, 1).$$

Straightforward calculations show that $Q = \frac{1}{2}(P - P)$.

Put $Z = \mathbf{R}^4$, and let $F = \{(x, y, z, 1) : (x, y, z) \in P\}$. We make Z into a Banach space by giving it the norm whose unit ball is co $(-F \cup F)$. It is clear that Z is a base norm space, and that the hyperplane $Y = \mathbf{R}^3 \oplus \{0\}$ has the 2-ball property in Z. A careful inspection of the 24 extreme points of its unit ball shows that Z has no nontrivial M-summand.

The unit ball of Y is $\frac{1}{2}(F - F) = Q \oplus \{0\}$, which is obviously the convex hull of two opposite heptagonal faces. Thus Y is also a base norm space, and $X = \mathbf{R}^2 \oplus \{0\}^2$ has the 2-ball property in Y.

By Theorem 5, X has the 2-ball property in Z, notwithstanding the fact that it has codimension two. Since X_1 is an octagon, X contains no semi-M-ideals, let alone M-summands.

Arguments like this, inspired by Theorems 5 and 7, can obviously be used to exhibit many more examples of proper semi-M-ideals. We will give only one more.

EXAMPLE 9. *There is a 4-dimensional Banach space containing no M-summands, but which does contain a 2-dimensional proper semi-M-ideal, whose unit ball is a hexagon.*

Proof. Let $Z=\mathbf{R}^4$, equipped with the norm

$$\|(w,x,y,z)\| = \max(|w|,|x-w|,|y-w|,|z-w|,|x+y-w|).$$

Any extreme point (w,x,y,z) of Z_1 must satisfy $w=\pm 1$, and so $Y=\{0\}\oplus\mathbf{R}^3$ is a semi-M-ideal in Z. Routine calculations then show that $X=\{0\}\oplus\mathbf{R}^2\oplus\{0\}$ is a semi-M-ideal in Z, and that X_1 is a hexagon. One might argue that this example is slightly degenerate, since X is actually an M-summand in Y.

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