Let $C \to 0$ in (3). Then

$$\prod_{n=1}^{P} \frac{(1-Eq^n)(1-Dq^n)}{(1-q^{2N+n})(1-DEq^n)}\,{}_3\Psi_3\left[\begin{matrix} q^{2P+N+1}, D, E\,;\, q \\ q^{N+1}, DEq^{P+1}, q^{P+1}\end{matrix}\right]_N$$
= the same expression with P and N interchanged.

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SOLUTION OF LINEAR ALGEBRAIC AND DIFFERENTIAL EQUATIONS BY THE LONG-DIVISION ALGORITHM

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1. *Solution of* $Lx = b$. If L is a non-singular lower triangular matrix with any number of rows and columns, and x, b are column vectors, the equation

$$Lx = b$$

can be solved by a process resembling long division, or, more exactly, resembling the division of polynomials by the method of detached coefficients. This is best shown by an example:

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{array}{cccccccc}
1 & 2 & 3) & 3 & 2 & 1 & (& 3 \\
 & & & 3 & 6 & 9 & & \\ \cline{4-6}
 & 4 & 5) & & -4 & -8 & (& -1 \\
 & & & & -4 & -5 & & \\ \cline{4-6}
 & & 6) & & & -3 & (& -\tfrac{1}{2} \\
 & & & & & -3 & & \\ \cline{4-6}
\end{array}$$

Thus

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ -\frac{1}{2} \end{bmatrix}.$$

In this process the successive 'divisors' are the successive columns of L. The first 'division' corresponds to the determination of x_1 from the first equation, and the elimination of x_1 from the others.

A check on the accuracy of the arithmetic can be obtained by noting that the scalar product of x and the vector, found by summing the columns of L, should be equal to the sum of the elements of b. In the case of the example just given, we have

$$(6,\ 9,\ 6).(3,\ -1,\ -\tfrac{1}{2}) = 6.$$

The equation $Ux = y$, where U is an upper triangular matrix, can be solved in a very similar way, as also can the equations $x'L = y'$, $x'U = y'$; in the latter cases, it is necessary to work with the rows of the matrices L and U instead of the columns.

2. *Resolution of square matrix.* The long-division algorithm can also be used to express a general non-singular square matrix as the product of a lower triangular and an upper triangular matrix in the manner required for Choleski's method of solving a set of linear algebraic equations. Take, for example, the matrix:

$$\begin{bmatrix} 9 & -2 & 1 \\ 1 & 5 & -3 \\ -2 & 2 & 7 \end{bmatrix}.$$

We begin with

$$\begin{array}{ccccc} A & = & L & & U \\ \begin{bmatrix} 9 & -2 & 1 \\ 1 & 5 & -3 \\ -2 & 2 & 7 \end{bmatrix} & = & \begin{bmatrix} 9 & 0 & 0 \\ 1 & & 0 \\ -2 & & \end{bmatrix} & \times & \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}, \end{array}$$

where the blank places have to be filled in. First, 'divide' the second column of A by the first column of L:

$$\begin{array}{rrrl|rrrl} 9 & 1 & -2) & & -2 & 5 & 2 & (-0{\cdot}222 \\ & & & & -2 & -0{\cdot}222 & 0{\cdot}444 & \\ \hline & & & & & 5{\cdot}222 & 1{\cdot}556 & \end{array}$$

We can now fill the blank spaces in the second columns of L and U, obtaining

$$\begin{bmatrix} 9 & -2 & 1 \\ 1 & 5 & -3 \\ -2 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 1 & 5{\cdot}222 & 0 \\ -2 & 1{\cdot}556 & \end{bmatrix} \begin{bmatrix} 1 & -0{\cdot}222 & \\ 0 & 1 & \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, 'divide' the third column of A by the first and second columns of L:

$$\begin{array}{rrrl|rrrl} 9 & 1 & -2 &) & 1 & -3 & 7 & (\ 0{\cdot}111 \\ & & & & 1 & 0{\cdot}111 & -0{\cdot}222 & \\ \hline & 5{\cdot}222 & 1{\cdot}556 &) & & -3{\cdot}111 & 7{\cdot}222 & (-0{\cdot}596 \\ & & & & & -3{\cdot}111 & -0{\cdot}927 & \\ \hline & & & & & & 8{\cdot}149 & \end{array}$$

If we now fill in the blanks in the third columns of L and U we obtain

$$\begin{bmatrix} 9 & -2 & 1 \\ 1 & 5 & -3 \\ -2 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 9 & 0 & 0 \\ 1 & 5{\cdot}222 & 0 \\ -2 & 1{\cdot}556 & 8{\cdot}149 \end{bmatrix} \begin{bmatrix} 1 & -0{\cdot}222 & 0{\cdot}111 \\ 0 & 1 & -0{\cdot}596 \\ 0 & 0 & 1 \end{bmatrix}.$$

When the matrix has been factorized in this way, the solution of the equations $Ax = b$ can be obtained (according to Choleski's method) by solving first the set of equations $Ly = b$, and then the set $Ux = y$, using the procedure described in § 1.

The methods described in this and the preceding section are not limited in application to matrices with three rows and columns only.

3. *Solution of* $LX = B$. The method of § 1 may easily be extended to obtain the unknown square matrix X from the equation

$$LX = B.$$

It is only necessary to regard the equation as split up into the set of equations

$$Lx_i = b_i,$$

where x_i and b_i are the ith columns of X and B respectively. The equation $UX = B$ may be similarly treated.

If B is taken to be the unit matrix I, a method is obtained for inverting an upper triangular or lower triangular matrix, and hence for inverting any non-singular matrix which has been split up into a product of a pair of such matrices.

4. *Solution of ordinary differential equations.** The application of the methods discussed here to the solution of ordinary differential equations by means of standard recurrence formulae will be illustrated by taking as an example the following equation:

$$\frac{d^2y}{dx^2} + xy = 0.$$

To the accuracy with which d^2y/dx^2 may be replaced by δ^2y/h^2 (where h is the interval in x) this equation is equivalent to the recurrence equation

$$y_i - (2 - h^2x_{i+1})\,y_{i+1} + y_{i+2} = 0.$$

If h is taken to be 0·2, the values of $y_0, y_1, y_2, \ldots, y_n$ are given by

$$\begin{bmatrix} 1 & -1{\cdot}992 & 1 & 0 & 0 & 0 & . \\ 0 & 1 & -1{\cdot}984 & 1 & 0 & 0 & . \\ 0 & 0 & 1 & -1{\cdot}976 & 1 & 0 & . \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \end{bmatrix}.$$

Note that the matrix elements are given exactly by the three decimal places used. The matrix on the left-hand side has $n+1$ rows and $n+3$ columns, where n is the number of intervals into which the range of integration is divided. If the boundary conditions are such that the values of y at the two ends of the range are given, the methods described in this paper are of no assistance. If, however, y_0 and y_1 are given, the solution may be found by dividing the right-hand side by the various columns of the matrix in turn. The work is set out in Example 1. Since y_0 and y_1 are given, the first two terms of the quotient are known, and the division proper does not start until the third term is reached. In the example the integration is carried as far as $x = 2{\cdot}0$, but could be carried further if desired; it is convenient to regard the matrices as having an indefinite number of rows, as many being used as are required in a particular case.

* Cf. Tustin (1).

The check described in § 1 may be applied in this case, but, since the division does not terminate, the remainders after division must be taken into account. The details of the check, applied to Example 1, are as follows:

Sum of columns of matrix = (1, − 0·992, 0·016, 0·024, 0·032, 0·040, 0·048, 0·056, 0·064, 0·072, 0·080)

$(y_0, y_1, \ldots, y_{10})$ = (0·35503, 0·40628, ..., 0·34010, 0·22592)

Scalar product of above vectors	=	0·13224
Remainders after division	=	{ 0·09367 − 0·22592
		− 0·00001
Sum of elements of vector on right-hand side	=	0
		− 0·00001

The difference is here sufficiently small to be ascribed to rounding-off errors.

When $x = 2{\cdot}0$, the integration gives $y = 0{\cdot}22592$; this may be compared with the true value $y = 0{\cdot}227407$ given in (2). If greater accuracy is required, a smaller interval in x may be used, or, alternatively, the method of Fox and Goodwin (3) may be applied. This consists in estimating the truncation error from the approximate solution obtained in the manner described above, and repeating the calculation with this estimate subtracted from the vector on the right-hand side. In the present example, the truncation error is given approximately by $-\frac{1}{12}\delta^4 y_{i+1}$ and was found to be (0·00006, 0·00005, etc.); one step of integration was performed in a backward direction in order that the value of $\delta^4 y_1$ might be obtainable. The working of the second integration is given in Example 2. The value of y for $x = 2{\cdot}0$ is now 0·22741, in much better agreement with the true value.

The method is also applicable if the equation includes a term in dy/dx and if the right-hand side is a function of x. If the equation is

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)\,y = f(x),$$

the corresponding recurrence relation is

$$y_i - 2y_{i+1} + y_{i+2} + \tfrac{1}{2}hp(x_{i+1})\,(y_{i+2} - y_i) + h^2 q(x_{i+1})\,y_{i+1} = h^2 f(x_{i+1}).$$

From this the corresponding matrix equation may be readily derived. The estimate of the truncation error required for refining the solution can be obtained by evaluating $-\frac{1}{12}\{hp(x_{i+1})\,(\delta^3 y_{i+\frac{1}{2}} + \delta^3 y_{i+\frac{3}{2}}) + \delta^4 y_{i+1}\}$ (see Method VI of Fox and Goodwin (3)). The method can also be applied to equations of higher order than the second.

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Example 1

	0·35503	*0·40628*	0·45428	0·49501	0·52386	0·53595	0·52660	0·49197	0·42979	0·34010	0·22592
1)	0	0	0	0	0	0	0	0	0	0	0
	0·35503										
−1·992 1)	−0·35503	0									
	−0·80931	0·40628									
1 −1·984 1)	0·45428	−0·40628	0								
	0·45428	−0·90129	0·45428								
1 −1·976 1)		0·49501	−0·45428	0							
		0·49501	−0·97814	0·49501							
1 −1·968 1)			0·52386	−0·49501	0						
			0·52386	−1·03096	0·52386						
1 −1·960 1)				0·53595	−0·52386	0					
				0·53595	−1·05046	0·53595					
1 −1·952 1)					0·52660	−0·53595	0				
					0·52660	−1·02792	0·52660				
1 −1·944 1)						0·49197	−0·52660	0			
						0·49197	−0·95639	0·49197			
1 −1·936 1)							0·42979	−0·49197	0		
							0·42979	−0·83207	0·42979		
1 −1·928 1)								0·34010	−0·42979	0	
								0·34010	−0·65571	0·34010	
1 −1·920 1)									0·22592	−0·34010	0
									0·22592	−0·43377	0·22592
										0·09367	−0·22592

Notes. (1) The values shown in italics (namely, *0·35503* and *0·40628*) are given in advance.
(2) The 'quotient' is written above the 'dividend'.

Example 2

	0·35503	*0·40628*	0·45422	0·49484	0·52356	0·53555	0·52619	0·49170	0·42986	0·34076	0·22741
1)	−0·00006	−0·00005	−0·00002	0·00002	0·00007	0·00013	0·00019	0·00025	0·00028		
	0·35503										
−1·992 1)	−0·35509	−0·00005									
	−0·80931	0·40628									
1 −1·984 1)	0·45422	−0·40633	−0·00002								
	0·45422	−0·90117	0·45422								
1 −1·976 1)		0·49484	−0·45424	0·00002							
		0·49484	−0·97780	0·49484							
1 −1·968 1)			0·52356	−0·49482	0·00007						
			0·52356	−1·03037	0·52356						
1 −1·960 1)				0·53555	−0·52349	0·00013					
				0·53555	−1·04968	0·53555					
1 −1·952 1)					0·52619	−0·53542	0·00019				
					0·52619	−1·02712	0·52619				
1 −1·944 1)						0·49170	−0·52600	0·00025			
						0·49170	−0·95586	0·49170			
1 −1·936 1)							0·42986	−0·49145	0·00028		
							0·42986	−0·83221	0·42986		
1 −1·928 1)								0·34076	−0·42958		
								0·34076	−0·65699		
									0·22741		