LIFTING PROPERTIES AND UNIFORM REGULARITY OF LEBESGUE MEASURES ON TOPOLOGICAL SPACES

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§1. *Introduction.* Let $(X, \mathscr{F}, \mu)$ be a topological measure space with X a completely regular Hausdorff space and $\mathscr{F}$, the σ-algebra of all μ-measurable sets, containing all the Baire sets of X. Consider the following two conditions on $(X, \mathscr{F}, \mu)$.

(a) Every lifting of $\mathscr{F}$ is almost strong.

(b) The measure μ is uniformly regular relative to some admissible uniformity of X.

When X is compact and metrizable, conditions (a) and (b) trivially hold. In this note, we show, in the case when μ is non-atomic and inner regular for the compact sets, that every Lebesgue space satisfies (a) and that measure spaces satisfying both (a) and (b) are necessarily Lebesgue. If we further assume that X is Čech-complete and μ is a Baire measure, *i.e.* $\mathscr{F}$ is the μ-completion of the Baire sets, then (a) and (b) characterize the topologically Lebesgue spaces: essentially those in which the measure is carried by a separable metric subspace.

§2. *Preliminary material.* Throughout, $(X, \mathscr{F}, \mu)$ denotes a complete measure space such that X is a completely regular Hausdorff topological space, $\mathscr{F}$ a σ-algebra of subsets of X containing all the Baire sets and μ a totally finite non-negative σ-additive measure defined on $\mathscr{F}$. Unexplained topological notions follow the usage in [**6**] and [**10**]; for the measure-theoretic ones we adopt the terminologies of [**8**] and [**15**].

We call the measure μ

(a) a *Baire measure*, if $\mathscr{F}$ is the μ-completion of the Baire sets;

(b) a *Borel measure*, if $\mathscr{F}$ is the μ-completion of the Borel sets and μ is inner regular for the closed sets;

(c) a *Radon measure*, if μ is a Borel measure which is inner regular for the compact sets.

We observe that a Baire measure is always inner regular for the zero sets, *i.e.* sets of the form $f^{-1}(0)$ for some continuous function f on X.

The measure space $(X, \mathscr{F}, \mu)$ is called a *Lebesgue space* and μ a *Lebesgue measure*, if there exist a μ-null set $N \subset X$, a Lebesgue measurable set $D \subset I$ the closed interval $[0, \mu(X)]$ with Lebesgue measure $m(D) = 0$ and a bimeasurable bijection $\phi : X \setminus N \to I \setminus D$ such that $\phi(\mu) = m$. Thus our Lebesgue measures are the non-atomic Lebesgue measures of [**13**]. If ϕ is a homeomorphism, the interval I being given the usual topology, then $(X, \mathscr{F}, \mu)$ is called a *topologically Lebesgue*

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measure space and μ a topologically Lebesgue measure [**3**]. It is well known that every non-atomic measure on a compact metric space is topologically Lebesgue [**4**; p. 85].

A map $\rho : \mathscr{F} \to \mathscr{F}$ is called a *lifting of* $\mathscr{F}$, if, for all A, $B \in \mathscr{F}$, the following requirements hold.

(i) $\mu(\rho(A) \triangle A) = 0$.

(ii) $\mu(A \triangle B) = 0 \Rightarrow \rho(A) = \rho(B)$.

(iii) $\rho(A \cap B) = \rho(A) \cap \rho(B)$.

(iv) $\rho(A \cup B) = \rho(A) \cup \rho(B)$.

(v) $\rho(X) = X$ and $\rho(\emptyset) = \emptyset$.

A lifting ρ is said to be

(a) *strong*, if $G \subset \rho(G)$ for all open $G \in \mathscr{F}$;

(b) *almost strong*, if there is a μ-null set N such that $G \subset \rho(G) \cup N$ for all open $G \in \mathscr{F}$.

It follows that ρ is almost strong, if, and only if,

$$\mu\left(\bigcup_G (G \setminus \rho(G))\right) = 0,$$

where the union is taken over all open $G \in \mathscr{F}$. The problem of the existence of strong liftings received the attention of a number of authors, *e.g.* [**5**], [**7**], [**9**] and [**12**]. If X is locally compact and metrizable and μ is inner regular for the compact sets, then every lifting of $\mathscr{F}$ is almost strong [**9**; Ch. viii]. On the other hand, Losert, [**12**], has provided an example of a compact measure space admitting no strong liftings.

Now, let $\mathscr{U}$ be an admissible uniformity of X, *i.e.* a uniformity inducing the topology of X. We say that μ is *uniformly regular relative to* $\mathscr{U}$, if there is a sequence $\{U_n\}$ of elements $\mathscr{U}$ such that, for any closed set $C \in \mathscr{F}$, we have $U_n(C) = \{y : (x, y) \in U_n, \text{ for some } x \in C\} \in \mathscr{F}$ and $\lim_n \mu[U_n(C)] = \mu(C)$. The notion of uniform regularity was introduced in [**2**] in the case when X is compact; then X has a unique admissible uniformity, so that uniform regularity is a purely topological property of μ. We shall make use of the following representation of uniformly regular measures, the proof of which was given in [**2**] in the case when X is compact.

2.1 THEOREM *Suppose that μ is uniformly regular for some admissible uniformity of X. Then there is a measure space $(T, \mathscr{F}'v)$, where T is a metric space, and a continuous surjection $p : X \to T$ such that, for any $A \in \mathscr{F}$ which is either closed or a Baire set, we have $p(A) \in \mathscr{F}'$ and $v[p(A)] = \mu(A)$.*

Proof. Suppose that $\{U_n\}$ is a sequence of elements of an admissible uniformity $\mathscr{U}$ such that for any closed $C \in \mathscr{F}$, $U_n(C) \in \mathscr{F}$ and $\mu(C) = \lim_n \mu[U_n(C)]$. Let T be the Hausdorff quotient of X given the pseudometrizable topology provided by the subuniformity of $\mathscr{U}$ generated by the sequence $\{U_n\}$, and $p : X \to T$, the quotient map. Define $\mathscr{F}' = \{E \subset T : p^{-1}(E) \in \mathscr{F}\}$, and let v be the image of μ under p, *i.e.*

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$\nu(E) = \mu[p^{-1}(E)]$, for all $E \in \mathscr{F}'$.

For $x \in X$, $F \subset X$, write

$$\tilde{x} = p^{-1}p(x) = \bigcap_n U_n(x) = \{y : (x, y) \in U_n, \text{ for all } n\}; \ \tilde{F} = \bigcup_{x \in F} \tilde{x}.$$

Let $A \in \mathscr{F}$. If A is closed then

$$A \subset p^{-1}p(A) = \tilde{A} \subset \bigcap_n U_n(A),$$

and, by hypothesis,

$$\mu(A) = \mu\left[\bigcap_n U_n(A)\right].$$

Therefore $\tilde{A} \in \mathscr{F}$ and $\mu(\tilde{A}) = \mu(A)$; so that $p(A) \in \mathscr{F}'$ and $\nu[p(A)] = \mu(A)$.

The same conclusion is easily seen to follow when A is an F_σ set and, in particular, when A is a cozero set. If A is any Baire set, then by approximating A from inside and outside by a zero set and a cozero set respectively, we see that $p(A) \in \mathscr{F}'$ and $\nu[p(A)] = \mu(A)$.

One natural uniformity to consider, for completely regular spaces, is the uniformity $\mathscr{C}^*$ generated by the bounded continuous functions on X. This is the uniformity inherited by X from the unique admissible uniformity of βX, the Stone–Čech compactification of X. The proof of the following proposition is straightforward and we omit it.

2.2. PROPOSITION. *Let μ be a tight Baire measure on X and suppose that the induced Baire measure $\tilde{\mu}$ on βX is uniformly regular. Then μ is $\mathscr{C}^*$-uniformly regular.*

§3. *Main results.* In this section we relate lifting properties of the topological measure space $(X, \mathscr{F}, \mu)$ to certain regularity conditions on μ. We shall assume throughout this section that μ is inner regular for the compact sets, *i.e.*, for any $E \in \mathscr{F}$, we have $\mu(E) = \sup\{\mu(K) : K \subset E, K \in \mathscr{F}, K \text{ compact}\}$.

3.1. THEOREM. *If μ is inner regular for the metrizable sets, then it is a Radon measure and every lifting of $\mathscr{F}$ is almost strong.*

Proof. By hypothesis, we can find a sequence $\{K_n\}$ of mutually disjoint compact metrizable μ-measurable sets such that $\sum_n \mu(K_n) = \mu(X)$. If B is any Borel subset of X then $B_n = K_n \cap B$ is a Baire subset of K_n and, since K_n is C^*-embedded in X, B_n is the trace in K_n of a Baire subset of X, *i.e.* $B_n \in \mathscr{F}$. It follows that $E = \bigcup_n B_n \in \mathscr{F}$, $E \subset B$ and $\mu(E) = \nu(B)$, where ν is the unique regular extension of μ to the Borel sets. Hence B is μ-measurable. The inner regularity of μ for the compact sets shows that $\mathscr{F}$ is the completion w.r.t.μ of the Borel sets, *i.e.* μ is a Radon measure.

Let ρ be a lifting of $\mathscr{F}$. For each n, let $C_n = K_n \cap \rho(K_n)$, $\mathscr{F}_n = \{E \in \mathscr{F} : E \subset C_n\}$, $\mu_n = \mu \mid \mathscr{F}_n$ and define $\rho_n : \mathscr{F}_n \to \mathscr{F}_n$ by:

$$\rho_n(E) = C_n \cap \rho(E).$$

Clearly, $\{C_n\}$ is a disjoint sequence of second countable subspaces of $X, \mu(\bigcup_n C_n) = \mu(X)$ and ρ_n is a lifting of $(C_n, \mathscr{F}_n, \mu_n)$. It follows that ρ_n is almost strong.

Let $F_n = \bigcup\{G \setminus \rho_n(G) : G$ is relatively open in $C_n\}$, and

$$N = \left(X \Big\backslash \bigcup_n C_n\right) \cup \left(\bigcup_n F_n\right).$$

Then $\mu(N) = 0$ and for any open subset U of $X, \rho(U) \supset U \setminus N$, *i.e.* ρ is almost strong.

I do not know whether the converse of Theorem (3.1) holds in general. We establish the converse under the additional assumption of uniform regularity of μ.

3.2. Theorem. *If μ is uniformly regular for any admissible uniformity of X and every lifting of $\mathscr{F}$ is almost strong, then μ is inner regular for the metrizable sets.*

Proof. The uniform regularity of μ gives, by (2.1), a continuous surjection $p : X \to T$, where T is metrizable, such that for any Baire set A we have, $\mu(A) = p(\mu)[p(A)]$. Moreover, the inner regularity of μ for the compact sets implies that, for any $E \in \mathscr{F}$, there is a Baire set A such that $\mu(E \triangle A) = 0$.

Denote by $\mathscr{F}'$ the σ-algebra of all $p(\mu)$-measurable subsets of T and let θ' be any lifting of $\mathscr{F}'$. For each $E \in \mathscr{F}$, let A be a Baire set such that $\mu(E \triangle A) = 0$, and define

$$\theta(E) = p^{-1}[\theta'(p(A))].$$

Here θ is well defined, since, if A_1 and A_2 are two Baire sets such that $\mu(A_1 \triangle A_2) = 0$, we have,

$$p(\mu)[p(A_1) \triangle p(A_2)] = \mu[p^{-1}p(A_1) \triangle p^{-1}p(A_2)].$$

As $\mu[p^{-1}p(A_i) \setminus A_i] = 0$, $i = 1, 2$, it follows that

$$p(\mu)[p(A_1) \triangle p(A_2)] = \mu(A_1 \triangle A_2) = 0,$$

and thus

$$p^{-1}[\theta'(p(A_1))] = p^{-1}[\theta'(p(A_2))].$$

It is easily verifiable that θ is a lifting of $\mathscr{F}$.

By hypothesis θ is almost strong. Let $N \in \mathscr{F}$ be such that $\mu(N) = 0$ and, for any open $G \in \mathscr{F}$, we have $G \subset \theta(G) \cup N$. We show that the map p is one-to-one on $Y = X \setminus N$.

Suppose that x and y are two distinct points in Y. Let V and W be two disjoint open Baire neighbourhoods of x and y respectively. Then, since $V \subset p^{-1}[\theta'(p(V))] \cup N$ and $x \notin N$, we have $x \in p^{-1}[\theta'(p(V))]$, *i.e.* $p(x) \in \theta'(p(V))$. Similarly $p(y) \in \theta'(p(W))$.

As $V \cap W = \varnothing$, $\mu[p^{-1}p(V)] = \mu(V)$ and $\mu[p^{-1}p(W)] = \mu(W)$, we have $p(\mu)[p(V) \triangle p(W)] = 0$. Thus $\theta'(p(V)) \cap \theta'(p(W)) = \varnothing$, and so $p(x) \neq p(y)$, *i.e.* $p|Y$ is injective.

It follows from the injectivity of $p|Y$ that every compact set $K \subset Y$ is homeomorphic to $p(K)$, and so K is metrizable. Since μ is inner regular for the compact sets and $\mu(Y) = \mu(X)$, μ is inner regular for the metrizable sets.

The next theorem gives sufficient conditions under which a measure, which is inner regular for the compact metrizable sets, is carried by a metrizable subspace. Let us recall that a completely regular space X is Čech-complete, if it is a G_δ in one of (or, equivalently, in all of) its compactifications.

3.3. THEOREM. *If X is Čech-complete and μ is inner regular for the metrizable compact G_δ sets, then there is a metrizable subset Y of X such that $\mu(Y) = \mu(X)$.*

Proof. Let $\{K_n\}$ be a sequence of disjoint compact metrizable G_δ subsets of X such that $\sum_n \mu(K_n) = \mu(X)$. Embed X as a G_δ set in a compactification X^*. Then each K_n is a G_δ in X^*. For each n there are two sequences $\{U_{n,m}\}_{m=1,2,\ldots}$ and $\{V_{n,r}\}_{r=1,2,\ldots}$ of open subsets of X^* such that, for each n, $\{U_{n,m} \cap K_n\}_{m=1,2,\ldots}$ is a base for the topology of K_n, and $\bigcap_r V_{n,r} = K_n$.

Let T be the Hausdorff quotient of X^* endowed with the pseudometrizable subtopology generated by the family $\{U_{n,m}, V_{n,r}\,; n, m, r = 1, 2, \ldots\}$ of open sets, and $p : X^* \to T$ the quotient map. Then T is compact and metrizable, and clearly, for any $y \in Y = \bigcup_n K_n$, we have, $p^{-1}p(y) = \{y\}$, so that $p|Y$, the restriction of p to Y, is injective, and $p(E \cap Y) = p(E) \cap p(Y)$ for any $E \subset X^*$.

Now, if C is a relatively closed subset of Y, there is a compact subset K of X^* such that $C = K \cap Y$. Thus, $p(C) = p(K \cap Y) = p(K) \cap p(Y)$ is relatively closed in $p(Y)$. Therefore $p|Y$ is a homeomorphism and so Y is metrizable.

§4. *Remarks, examples and applications.* Throughout §3 we assumed that μ is inner regular for the compact sets. This is not satisfied by all tight Baire measures. For, let X be a P-space without isolated points [6; p. 13], and fix $x \in X$. Then the Baire measure μ induced by a unit-point-mass at x is clearly tight. If μ is inner regular for the compact sets then μ is inner regular for the compact G_δ sets. But any compact G_δ is clopen and finite and hence consists of a finite number of isolated points, which is a contradiction. However, I do not know whether a non-atomic tight Baire measure is necessarily inner regular for the compact sets.

Throughout the rest of this section we shall assume that the measure space $(X, \mathscr{F}, \mu)$ is non-atomic. We shall make use of the following characterizations of Lebesgue and topologically Lebesgue measures.

4.1. THEOREM. (a) *Suppose that μ is inner regular for the compact sets. Then, μ is Lebesgue, if, and only if, it is inner regular for the metrizable sets.*

(b) *μ is topologically Lebesgue, if, and only if, μ is inner regular for the compact sets and there is a μ-measurable set Y such that Y is metrizable and $\mu(Y) = \mu(X)$.*

Proof. (a) If μ is Lebesgue, then there is a bimeasurable bijection $\phi : Y \to L$, where $Y \in \mathscr{F}$, $\mu(Y) = \mu(X)$ and L is a Lebesgue measurable subset with full Lebesgue measure in an interval of the real line. The inner regularity of μ for the metrizable sets follows from the inner regularity of μ for the compact sets and the Lusin measurability of ϕ [**14**; p. 25].

Conversely, suppose that μ is inner regular for the metrizable sets. Let $\{K_n\}$ be a sequence of disjoint compact metrizable sets such that $\sum_n \mu(K_n) = \mu(X)$, and put $Y = \bigcup_n K_n$. Let τ_1 be the relative topology of Y and τ_2 the disjoint union topology on Y, so that $G \in \tau_2$, if, and only if, for each n, $G \cap K_n$ is relatively open in K_n. The identity map $i:(Y, \tau_1) \to (Y, \tau_2)$ is clearly Borel bimeasurable. The image measure $i(\mu)$ is a Radon measure on the separable metric space (Y, τ_2) and hence is a Lebesgue measure. We observe that the inner regularity of μ for the compact sets implies that $\mathscr{F}$ is contained in the μ-completion of the Borel subsets of (Y, τ_1). It follows that μ is Lebesgue [**13**].

(b) Suppose that μ is inner regular for the compact sets and there is a metrizable μ-measurable set of full measure. Let $Y \in \mathscr{F}$ be σ-compact and metrizable with $\mu(Y) = \mu(X)$. Then Y has a metrizable compactification Y^* and the Radon measure μ' on Y^*, defined on each Borel set $B \subset Y^*$ by $\mu'(B) = \mu(B \cap Y)$, is topologically Lebesgue [**4**; p. 85]. *I.e.* there is a Borel set $Z \subset Y^*$ with full μ'-measure and a homeomorphism $\phi: Z \to L$, where L is a subset of the interval $[0, \mu(X)]$ with full Lebesgue measure in this interval. The restriction of ϕ to $Z \cap Y$ gives the required homeomorphism and so μ is topologically Lebesgue. As the converse is obvious, the proof is complete.

It follows from (3.1) and (4.1) that, under the assumption of inner regularity for the compact sets, every lifting for a Lebesgue measure space is almost strong. This is not the case without the regularity assumption. Since we are dealing with finite complete measure spaces where liftings always exist, the property that every lifting is almost strong implies the existence of strong liftings. One necessary condition for the existence of strong liftings is the τ-additivity of μ, *i.e.* if $\{G_\alpha\}$ is an increasing net of open μ-measurable sets with $G_\alpha \uparrow X$, then $\mu(G_\alpha) \to \mu(X)$. Thus the non-τ-additive Lebesgue space of [**3**; ex. 4.2] shows that a Lebesgue space need not admit any almost strong lifting. Under the assumption of τ-additivity, an application of a result of [**5**] gives under the continuum hypothesis the existence of strong liftings for Lebesgue spaces. However, as the following example shows, even in this case liftings which are not almost strong may exist.

4.2. *Example.* Let X be the unit interval $[0, 1]$ with the topology generated by the family of all left-closed right-open intervals $[a, b)$. It can be easily shown that the Borel subsets of X are precisely the usual Borel subsets of the interval $[0, 1]$. Thus the standard Lebesgue measure induces a Borel measure μ on X which is clearly τ-additive and Lebesgue. The class $\mathscr{F}$ of all μ-measurable sets coincides with the class of Lebesgue subsets of $[0, 1]$. There is a lifting ρ of $\mathscr{F}$ such that for any left-open right-closed subinterval $(a, b]$, $\rho\{(a, b]\} = (a, b]$, [**9**; Ch. viii, Th. 6]. Clearly for this lifting, we have

$$\bigcup\{(G \setminus \rho(G)) : G \text{ is open}\} = X,$$

and so ρ is not almost strong.

The proof of the following corollary follows immediately from Theorems (3.1), (3.2) and (4.1).

4.3. Corollary. *Suppose that μ is a Radon measure which is uniformly regular for some admissible uniformity of X. Then μ is Lebesgue, if, and only if, every lifting of $\mathscr{F}$ is almost strong.*

For Baire measures, we have,

4.4. Corollary. *Suppose that μ is a Baire measure and X is Čech-complete. Then μ is topologically Lebesgue, if, and only if, μ is $\mathscr{C}^*$-uniformly regular and every lifting of $\mathscr{F}$ is almost strong.*

Proof. Suppose that μ is $\mathscr{C}^*$-uniformly regular and every lifting of $\mathscr{F}$ is almost strong. Since μ is finite and complete there exists a strong lifting of $\mathscr{F}$. It follows that μ is τ-additive, and, as X is Čech-complete, μ is tight. We show that μ is inner regular for the compact G_δ sets. Embed X in its Stone–Čech compactification βX and let $\tilde{\mu}$ be the induced Baire measure on βX and $\tilde{\nu}$ the Borel extension of $\tilde{\mu}$. Then $\tilde{\nu}(\beta X \setminus X) = 0$, [**11**]. Since each compact subset of βX can be covered by a compact Baire set of the same measure, and $\beta X \setminus X$ is σ-compact and $\tilde{\nu}$-null, there is a Baire set B in βX such that $B \subset X$ and $\mu(B) = \mu(X)$. It follows that μ is inner regular for the compact G_δ sets. By Theorem (3.2), μ is inner regular for the metrizable sets and so inner regular for the metrizable compact G_δ sets. By (3.3) and (4.1), μ is topologically Lebesgue.

Conversely, if μ is topologically Lebesgue, then, by (3.1), every lifting of $\mathscr{F}$ is almost strong. Furthermore, $\tilde{\mu}$ is topologically Lebesgue on βX and so uniformly regular. It follows from (2.2) that μ is $\mathscr{C}^*$-uniformly regular.

We remark that (4.4) does not hold for Borel measures. A privately communicated example by Fremlin shows that there is a Lebesgue Borel measure μ on a compact Hausdorff space which is uniformly regular but not topologically Lebesgue. Theorem (3.1) implies that for this measure every lifting is almost strong. However, by (4.4), not every lifting of its Baire restriction is almost strong for the Baire restriction.

Finally we give an example showing that the assumption of Čech-completeness in (3.3) and (4.4) cannot be omitted.

4.5. *Example.* Let I be the closed unit interval $[0, 1]$ and $Z = I^I$ the space of all functions $z : I \to I$ with the pointwise convergence topology. The space Z is separable compact and Hausdorff, [**10**; p. 103]. Let $\{z_n\}$ be a dense sequence in Z. Put $k_0 = 1, k_1 = 2$ and define

$$I_1 = \{z : z = \alpha z_1 + (1-\alpha) z_2\,;\text{for some } \alpha \in I\}\,.$$

If I_n is constructed so that $I_n = \{z : z = \alpha z_{k_{n-1}} + (1-\alpha) z_{k_n}\,; \alpha \in I\}$, let k_{n+1} be the smallest positive integer such that

$$z_{k_{n+1}} \notin \bigcup_{j=1}^{n} I_j$$

and define,

$$I_{n+1} = \{z : z = \alpha z_{k_n} + (1-\alpha) z_{k_{n+1}}\,; \alpha \in I\}\,.$$

Each I_n is homeomorphic to I.

Let $X = \bigcup_n I_n$ and define the Baire measure μ on X by $\mu(B) = \sum_n \lambda_n(B \cap I_n)$, where λ_n is the uniform distribution of mass 2^{-n} over I_n. Now μ is a non-atomic tight Baire measure on X and the Baire measure $\tilde{\mu}$ induced by μ on Z was shown not to be uniformly regular [2].

We show that μ is inner regular for the metrizable compact G_δ sets but μ is not topologically Lebesgue. We show that each point $x \in X$ is a G_δ. For each n find a sequence G_{nm} of neighbourhoods of x in X such that $\bigcap_m G_{nm} \cap I_n$ is either empty or $= \{x\}$. Then $\bigcap_{n,m} G_{n,m} = \{x\}$. For each n, let B_n be a Baire set in X such that $B_n \supset I_n$ and $\mu(B_n) = \nu(I_n)$, where ν is the extension of μ to the Borel sets. Since each point is a G_δ and, for $m \neq n, I_n \cap I_m$ is at most a singleton set, we have $\mu(B_n \cap B_m) = 0$. Thus, for any n, the set $C_n = (X \setminus \bigcup_{m \neq n} B_m)$ is a Baire set with $C_n \subset I_n$ and

$$\mu(C_n) = 1 - \sum_{m \neq n} \mu(B_m) = \nu(I_n),$$

i.e. I_n is μ-measurable. The inner regularity of μ for the metrizable compact G_δ sets follows from the simple observation that any zero set Z such that $Z \subset I_n$ is a metrizable compact G_δ set. Moreover, since $\tilde{\mu}$ is not uniformly regular in Z, $\tilde{\mu}$ is not topologically Lebesgue, and so μ is not topologically Lebesgue.

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