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THE RESULTANT OF TWO FOURIER KERNELS

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1. A "Fourier kernel" means here a function $K(x)$ which gives rise to a formula

$$f(x)=\int_0^\infty K(xu)\,du\int_0^\infty K(ut)f(t)\,dt \qquad (1\cdot1)$$

of the Fourier type. Thus

$$\sqrt{\left(\frac{2}{\pi}\right)}\cos x,\quad \sqrt{\left(\frac{2}{\pi}\right)}\sin x,\quad x^{\frac{1}{2}}J_\nu(x),\quad \frac{2}{\pi}\frac{1}{1-x^2},\quad \dots$$

are Fourier kernels.* If $K(x)$ is a Fourier kernel, λ is real, and a positive, then

$$\frac{1}{x}K\left(\frac{1}{x}\right),\quad \lambda x^{\frac{1}{2}(\lambda-1)}K(x^\lambda),\quad a^{\frac{1}{2}}K(ax)$$

are Fourier kernels.

The *resultant*, or *Faltung*, $M(x)$ of $K(x)$ and $L(x)$ is defined by

$$M(x)=\int_0^\infty K(xt)\,L(t)\,dt. \qquad (1\cdot2)$$

If $M(x)$ is the resultant of $K(x)$ and $L(x)$, then

$$\frac{1}{x}M\left(\frac{1}{x}\right)$$

is the resultant of $L(x)$ and $K(x)$.

There are various formal reasons which suggest that *the resultant of two Fourier kernels is a Fourier kernel*. For example, we may argue as follows. Replacing K by M in the integral on the right of $(1\cdot1)$, and substituting from $(1\cdot2)$, we obtain

$$\iint M(xu)\,M(ut)f(t)\,du\,dt=\iiiint K(xuy)\,K(utz)\,L(y)\,L(z)f(t)\,du\,dt\,dy\,dz;$$

and the substitution $t=v/z$, $y=zw$ gives

$$\iint L(z)\,L(zw)\,dz\,dw\iint K(xzwu)\,K(uv)f\left(\frac{v}{z}\right)du\,dv=\iint L(z)\,L(zw)f(xw)\,dz\,dw=f(x).$$

The argument is naturally of a purely formal type, the multiple integrals being

* Further examples are given by Hardy and Titchmarsh (2) and Watson (6).

divergent, and the inversions and substitutions impossible to justify, even in the simplest standard cases.*

We can also appeal to the idea which underlies the recent work of Watson and of Titchmarsh and myself. If

$$k(s)=\int_0^\infty x^{s-1}K(x)\,dx$$

is the Mellin transform of a Fourier kernel $K(x)$, then

$$k(s)\,k(1-s)=1;$$

and this is also a sufficient condition that $K(x)$ should be a Fourier kernel. Now the Mellin transform of $M(x)$ is

$$m(s)=\int_0^\infty x^{s-1}dx\int_0^\infty K(xt)\,L(t)\,dt=\int_0^\infty L(t)\,dt\int_0^\infty x^{s-1}K(xt)\,dx$$

$$=\int_0^\infty t^{-s}L(t)\,dt\int_0^\infty u^{s-1}K(u)\,du=k(s)\,l(1-s);$$

so that $$m(s)\,m(1-s)=k(s)\,k(1-s)\,l(s)\,l(1-s)=1.$$

This argument also is formal, but the transformations are a little nearer to reality than those of the first.

It is plain in any case that we must be prepared for a very liberal interpretation of (1·1) and (1·2). Thus

$$\frac{2}{\pi}\int_0^\infty \cos xt\cos t\,dt$$

is generally summable $(C, 1)$ to 0, but diverges to infinity when $x=1$. The integral is never convergent. Similarly

$$\frac{2}{\pi}\int_0^\infty \cos xt\sin t\,dt=\frac{2}{\pi}\frac{1}{1-x^2}\quad (C,1), \tag{1·3}$$

except for $x=1$, when the value is $1/2\pi$. On the other hand

$$\frac{4}{\pi^2}\int_0^\infty \frac{dt}{(1-x^2t^2)(1-t^2)}$$

converges to 0 in general (as a Cauchy principal value), but diverges to infinity when $x=1$. And a similar freedom of interpretation is necessary in (1·1).

2. It is easy to reduce all this to order by means of Watson's theory.† We start from a function $K_1(x)$ with the properties (i) that $x^{-1}K_1(x)$ is L^2 in $(0,\infty)$, and (ii) that

$$\int_0^\infty \frac{K_1(ax)\,K_1(bx)}{x^2}\,dx=\text{Min}\,(a,b) \tag{2·1}$$

* I have been familiar with these formal ideas for a good many years, but cannot say whence I derived them. Possibly from Ramanujan; but I can refer to nothing in his published work, and it is likely enough that the ideas are much older.

† Watson (6). Considerable simplifications in the theory have been made by Plancherel (3) and Titchmarsh (4).

if a and b are positive. In these circumstances, if $f(x)$ is L^2, and $g(x)$ is defined by

$$\int_0^x g(y)\,dy = \int_0^\infty \frac{K_1(xt)}{t} f(t)\,dt, \tag{2·2}$$

then $g(x)$ is also L^2 and the relationship is reciprocal. We call $f(x)$ and $g(x)$ "K-transforms" of one another. If $F(x)$ and $G(x)$ are also K-transforms of one another, then

$$\int_0^\infty f(x)\,F(x)\,dx = \int_0^\infty g(x)\,G(x)\,dx. \tag{2·3}$$

This is "Parseval's Theorem". In all this there is no direct reference to a function $K(x)$, but, if $K_1(x)$ is the integral of $K(x)$, then the transformation is that envisaged formally in § 1.

Let us now suppose that $K_1(x)$ and $L_1(x)$ satisfy Watson's conditions, and define $M_1(x)$ by

$$\int_0^x M_1\left(\frac{1}{y}\right) dy = \int_0^\infty \frac{K_1(t)}{t}\,\frac{L_1(xt)}{t}\,dt. \tag{2·4}$$

If K_1, L_1, M_1 are the integrals of K, L, M, then two differentiations reduce (2·4) formally to (1·2).

Since $M_1(1/x)$ is the L-transform of $x^{-1}K_1(x)$,

$$\int_0^\infty \frac{M_1^2(x)}{x^2}\,dx = \int_0^\infty M_1^2\left(\frac{1}{x}\right) dx < \infty.$$

Also $M_1(a/x)$, $M_1(b/x)$ are the L-transforms of $x^{-1}K_1(ax)$, $x^{-1}K_1(bx)$; and hence, by Parseval's Theorem,

$$\int_0^\infty \frac{M_1(ax)\,M_1(bx)}{x^2}\,dx = \int_0^\infty M_1\left(\frac{a}{x}\right) M_1\left(\frac{b}{x}\right) dx = \int_0^\infty \frac{K_1(ax)\,K_1(bx)}{x^2}\,dx = \text{Min}\,(a, b).$$

Hence M_1 satisfies the same conditions as K_1 and L_1, and there are formulae in M_1 similar to (2·2) and its reciprocal. When K_1, L_1, M_1 are integrals, then K, L, M are Fourier kernels; and it is natural to call the M-transformation M the resultant of the K- and L-transformations K and L.

If $S_1(x)$ is 0 for $x < 1$, and 1 for $x \geqq 1$, and $K_1(x) = S_1(x)$, then the transformation is

$$g(x) = \frac{1}{x} f\left(\frac{1}{x}\right), \quad f(x) = \frac{1}{x} g\left(\frac{1}{x}\right).$$

We call this transformation S. If $K_1 = L_1$ then

$$\int_0^x M_1\left(\frac{1}{y}\right) dy = \int_0^\infty \frac{K_1(t)\,K_1(xt)\,dt}{t^2} = \text{Min}\,(1, x)$$

and $M_1 \equiv S_1$. If $L_1 = S_1$, then

$$\int_0^x M_1\left(\frac{1}{y}\right) dy = \int_{1/x}^\infty \frac{K_1(t)}{t^2}\,dt = \int_0^x K_1\left(\frac{1}{t}\right) dt,$$

and $M_1 \equiv K_1$. Thus the resultant of K and K is S, and the resultant of K and S is K.

EXAMPLES

3. The interest of the examples which follow is mainly formal, and I allow myself, as in § 1, a certain latitude of expression, speaking in terms of K, L, M when precise expression demands a return to K_1, L_1, M_1.

(1) The equation (1·3) indicates that the resultant of the cosine and sine transformations is that defined by the kernel

$$\frac{2}{\pi}\frac{1}{1-x^2}.$$

Here
$$M_1(x)=\frac{1}{\pi}\log\left|\frac{x-1}{x+1}\right|$$

and is not (in the strict sense) an integral. The M transformation is

$$g(x)=\frac{2}{\pi}\int_0^\infty \frac{f(t)}{1-x^2t^2}\,dt.$$

If we suppose $f(x)$ even, and make some trivial transformations, we obtain

$$\frac{1}{x}g\left(\frac{1}{x}\right)=\frac{1}{\pi}\int_{-\infty}^\infty \frac{f(t)}{x-t}\,dt,$$

the conjugate or "Hilbert transform" of $f(x)$.

If we call this transformation C then the resultant of K and C is defined by

$$M(x)=\frac{2}{\pi}\int_0^\infty \frac{K(xt)}{1-t^2}\,dt;$$

or, regarding $K(x)$ as even, by

$$M(x)=\frac{1}{\pi}\int_{-\infty}^\infty \frac{K(t)}{x-t}\,dt.$$

Thus *the conjugate of a Fourier kernel is a Fourier kernel.*

(2) The function

$$L_1(x)=x\ (x\leq 1),\quad L_1(x)=0\ (x\geqslant 1)$$

satisfies Watson's conditions.* We conclude that, if $K(x)$ is a Fourier kernel, then

$$M(x)=\int_0^\infty K(xt)\,dL_1(t)=\int_0^1 K(xt)\,dt-K(x)=\frac{1}{x}\int_0^x K(u)\,du-K(x)$$

is a Fourier kernel. Or again, taking another of Watson's examples, viz.

$$L_1(x)=0\ (x<1),\quad L_1(x)=\log x-1\ (x\geqq 1),$$

we find that
$$\int_x^\infty \frac{K(u)}{u}\,du-K(x)$$

is a Fourier kernel.

* Watson (6, p. 197).

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(3) Since* $$\int_0^\infty J_\nu(xt)\, J_{\nu-1}\left(\frac{1}{t}\right)\frac{dt}{t} = x^{-\frac{1}{2}} J_{2\nu-1}(2x^{\frac{1}{2}}),$$

the resultant of $t^{\frac{1}{2}} J_\nu(t)$ and $t^{-\frac{3}{2}} J_\nu(1/t)$ is $J_{2\nu-1}(2t^{\frac{1}{2}})$.

(4) Since $$\frac{2}{\pi}\int_0^\infty \frac{\cos}{\sin} xt \frac{\cos}{\sin}\left(\frac{1}{t}\right)\frac{dt}{t} = \frac{2}{\pi} K_0(2x^{\frac{1}{2}}) \mp Y_0(2x^{\frac{1}{2}}),\dagger$$

the functions just written are the resultants of

$$\sqrt{\left(\frac{2}{\pi}\right)} \frac{\cos}{\sin} x, \quad \sqrt{\left(\frac{2}{\pi}\right)} \frac{1}{x} \frac{\cos}{\sin}\left(\frac{1}{x}\right)$$

(the two cosines or the two sines going together). We conclude that the functions

$$x^{\frac{1}{2}}\left\{Y_0(x) \mp \frac{2}{\pi} K_0(x)\right\}$$

are Fourier kernels. The first of them is the kernel which occurs in the theory of Dirichlet's divisor problem. The functions may be generated differently. Thus

$$\frac{2}{\pi}\int_0^\infty \frac{J_0\{2(xt)^{\frac{1}{2}}\}}{1-t^2}\, dt = \frac{4}{\pi}\int_0^\infty \frac{u J_0(2x^{\frac{1}{2}}u)}{1-u^4}\, du = Y_0(2x^{\frac{1}{2}}) + \frac{2}{\pi} K_0(2x^{\frac{1}{2}}),$$

so that this last kernel is the conjugate of $J_0(2x^{\frac{1}{2}})$.

(5) The resultant of $J_0(2x^{\frac{1}{2}})$ and $\cos x$ is $-\sin x$, and that of $J_0(2x^{\frac{1}{2}})$ and $\sin x$ is $\cos x$.

(6) It is easily proved that

$$x^{\frac{1}{2}}\int_0^\infty t J_\mu(xt)\, J_{-\mu}(t)\, dt = -\frac{2\sin\mu\pi}{\pi}\frac{x^{\mu+\frac{1}{2}}}{1-x^2} \quad (C, 1),$$

provided that $x \neq 1$, while when $x=1$ the integral diverges like

$$\frac{\cos\mu\pi}{\pi}\int^\infty dt.$$

This divergence indicates that, when we form the resultant of $x^{\frac{1}{2}} J_\mu(x)$ and $x^{\frac{1}{2}} J_{-\mu}(x)$, there will be a discontinuity in $M_1(x)$ at $x=1$. In fact, in this case,

$$M_1(x) = -\frac{2\sin\mu\pi}{\pi}\int_0^x \frac{t^{\mu+\frac{1}{2}}\, dt}{1-t^2} \quad (x<1), \quad M_1(x) = -\frac{2\sin\mu\pi}{\pi}\int_0^x \frac{t^{\mu+\frac{1}{2}}\, dt}{1-t^2} + \cos\mu\pi \quad (x \geqq 1).$$

The inversion formulae are

$$g(x) = -\frac{2\sin\mu\pi}{\pi}\int_0^\infty \frac{(xt)^{\mu+\frac{1}{2}}}{1-x^2t^2} f(t)\, dt + \cos\mu\pi\, \frac{1}{x} f\left(\frac{1}{x}\right)$$

and the reciprocal formula. The transformation is a generalization of C, to which it reduces when $\mu = -\frac{1}{2}$, the extra term then disappearing.

* The formula is easily deducible from one due to Bateman. See Hardy (1).

† Here, and in (7), K_ν is used as in Watson (5).

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(7) If we form the resultant $M(x)$ of

$$\sqrt{\left(\frac{2}{\pi}\right)}\cos x, \quad J_{\frac{1}{2}}(2x^{\frac{1}{2}}) = \pi^{-\frac{1}{2}}x^{-\frac{1}{4}}\sin 2x^{\frac{1}{2}},$$

and then replace it by

$$\frac{2^{-\frac{1}{2}}}{x} M\left(\frac{1}{2x}\right),$$

we obtain the Fourier kernel

$$(2x)^{\frac{1}{2}}\{\cos(x-\tfrac{1}{8}\pi)\,J_{\frac{1}{4}}(x) + \sin(x-\tfrac{1}{8}\pi)\,J_{-\frac{1}{4}}(x)\}.$$

The analysis involves the calculation of the integrals

$$\int_0^\infty e^{-x^4-4\alpha x^2}\,dx = \tfrac{1}{2}\alpha^{\frac{1}{2}} e^{2\alpha^2} K_{-\frac{1}{4}}(2\alpha^2),$$

$$\int_0^\infty e^{-x^4}\cos 4\alpha x^2\,dx = 2^{-\frac{3}{2}}\pi\alpha^{\frac{1}{2}} e^{-2\alpha^2} I_{-\frac{1}{4}}(2\alpha^2),$$

$$\int_0^\infty \cos x^4 \cos 4\alpha x^2\,dx = 2^{-\frac{3}{2}}\pi\alpha^{\frac{1}{2}}\cos(2\alpha^2-\tfrac{1}{8}\pi)\,J_{-\frac{1}{4}}(2\alpha^2).$$

In all of these α is positive.

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