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A multiplicative property of R-sequences and H_1-sets

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Introduction. Let R be a commutative ring which may not contain a multiplicative identity. A set of elements $a_1, \ldots, a_k$ in R will be called an H_1-*set* (this notation is explained in section 1) if for each relation $r_1 a_1 + \ldots + r_k a_k = 0$ ($r_i \in R$) there exist elements $s_{ij} \in R$ such that

$$r_1 X_1 + \ldots + r_k X_k = \sum_{i,j} s_{ij}(a_j X_i - a_i X_j),$$

where $X_1, \ldots, X_k$ are indeterminates. Any R-sequence is an H_1-set, but there do exist H_1-sets which are not R-sequences (see section 1). Throughout this note we consider an H_1-set $a_1, \ldots, a_k$ which we suppose to be partitioned into two non-empty sets $b_1, \ldots, b_r$ and $c_1, \ldots, c_s$. Our main purpose is to show that the ideals $B = Rb_1 + \ldots + Rb_r$ and $C = Rc_1 + \ldots + Rc_s$ satisfy $B^m \cap C^n = B^m C^n$ for all positive integers m and n (Corollary 1). This generalizes Lemma 2 of Caruth (2) where the result is proved when $a_1, \ldots, a_k$ is a permutable R-sequence. Our proof involves more detail than is necessary just for this, and we obtain various other properties of H_1-sets. In particular we extend the main results of Corsini (3) concerning the symmetric and Rees algebras of a power of the ideal $Ra_1 + \ldots + Ra_k$ (Corollary 3).

1. *Examples of H_1-sets.* In (5), p. 363, Kabele called elements $x_1, \ldots, x_k$ an H_1-*regular sequence* if the first homology module of the associated Koszul complex is zero. It is clear that an H_1-regular sequence is the same as an H_1-set (cf. 1·3 of (5)).

That an R-sequence is an H_1-set was shown explicitly by Micali (8), Lemma 2, p. 42. A very short proof is given in (6), p. 122. The property of an R-sequence $y_1, \ldots, y_k$ required for this is that, for each i, if $ry_{i+1} \in Ry_1 + \ldots + Ry_i$ then $r \in Ry_1 + \ldots + Ry_i$.

We now mention some examples of H_1-sets which are not R-sequences. The simplest way of obtaining such an example is to take a suitable rearrangement of any non-permutable R-sequence. However we recall that inside the Jacobson radical of a

Noetherian ring R, R-sequences are permutable and, also, any H_1-set is an R-sequence (see pp. 90 and 122 of (6)). Examples in (5) show that for neither conclusion may the Noetherian condition be omitted. We now give an example of a Noetherian ring R with an H_1-set consisting of two zero-divisors x and y. Thus neither x, y nor y, x is an R-sequence, although, by Theorem 2 below, the Noetherian condition does imply that the ideal (x, y) can be generated by an R-sequence.

Example. *Take* $R = F[X, Y, Z, W]/(ZX, Z(1-Y), WY, W(1-X))$ *and* $x = \bar{X}, y = \bar{Y}$, *where* F *is a field and* $^-$ *denotes the natural image in* R.

Clearly x and y are zero-divisors. We require to prove that if f and g are polynomials in $F[X, Y, Z, W]$, $= T$ say, such that $\bar{f}\bar{X} + \bar{g}\bar{Y} = 0_R$ then there exists $\bar{h} \in R$ such that $\bar{f} = \bar{h}\bar{Y}$ and $\bar{g} = -\bar{h}\bar{X}$. Evidently $fX + gY \in TZ + TW$ and, since X, Y, Z, W is an H_1-set in T, it is easy to see that there exists $k \in T$ such that, modulo $TZ + TW, f = kY$ and $g = -kX$. Thus it will be sufficient to consider the case in which $f, g \in TZ + TW$. Since the pair $\bar{Z}, 0_R$ may be written as $\bar{Z}\bar{Y}, -\bar{Z}\bar{X}$ one sees that we may assume that f does not involve Z, and, furthermore, since $\bar{W}\bar{X} = \bar{W}$ and $\bar{W}\bar{Y} = 0_R$ we may even assume that $f = Wf_1$ where f_1 involves only W. Similarly we may assume $g = Zg_1$ where g_1 involves Z only. We then have

$$Wf_1X + Zg_1Y = aZX + bZ(1-Y) + cWY + dW(1-X),$$

where $a, b, c, d \in T$. Putting $Y = Z = 0$ and $X = 1$ we obtain $Wf_1 = 0$. Thus $f = 0$. Similarly $g = 0$, and the required conclusion follows.

2. *The basic theorem and corollaries.* Let $f(X)$ and $g(Y)$ be polynomial forms over R (i.e. with coefficients in R) respectively of degrees m in $X_1, \ldots, X_r$ and n in $Y_1, \ldots, Y_s$, and denote by $f(b)$ and $g(c)$ the results of the substitutions $X_i \to b_i$ and $Y_i \to c_i$. Our basic theorem shows that when $f(b) + g(c) = 0$ the coefficients in $f(X) + g(Y)$ satisfy the natural relations.

Theorem 1. *If* $f(b) + g(c) = 0$ *then* $f(X) + g(Y)$ *is in the* R*-module generated by all forms of the type*

$$\nu(c)\,\mu(X) - \mu(b)\,\nu(Y), \quad (b_jX_i - b_iX_j)\,\mu'(X) \quad \text{or} \quad (c_jY_i - c_iY_j)\,\nu'(Y),$$

where $\mu(X)$ *and* $\mu'(X)$ *are monomials in* $X_1, \ldots, X_r$ *of degrees* m *and* $m-1$ *respectively, and* $\nu(Y)$ *and* $\nu'(Y)$ *are monomials in* $Y_1, \ldots, Y_s$ *of degrees* n *and* $n-1$ *respectively.*

Proof. Use induction on $m+n$. The result is clear when $m = 1 = n$, and so, using symmetry, it will be sufficient to prove the result in the case when $f(X)$ has degree $m \geqslant 1$ and $g(Y)$ has degree $n \geqslant 2$. We may write

$$g(Y) = \sum_{\lambda} h_\lambda(Y)\,\lambda(Y), \tag{1}$$

where the summation is over all monomials λ of degree $n-1$ in $Y_1, \ldots, Y_s$ and where each h_λ is linear. By induction there exist forms $p_{ij}(X)$ of degree $m-1$, $q_{ij}(Y)$ of degree $n-2$ and elements $s_{\mu\lambda}$ such that

$$f(X) + \sum_{\lambda} h_\lambda(c)\,\lambda(Y) = \sum_{\mu,\lambda} s_{\mu\lambda}(\lambda(c)\,\mu(X) - \mu(b)\,\lambda(Y)) + \sum_{i,j}(b_jX_i - b_iX_j)\,p_{ij}(X) + \sum_{i,j}(c_jY_i - c_iY_j)\,q_{ij}(Y). \tag{2}$$

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Equating the coefficients of each monomial $\lambda(Y)$ gives linear forms $k_\lambda(Y)$ such that

$$h_\lambda(c) = -\sum_\mu s_{\mu\lambda}\mu(b) + k_\lambda(c) \tag{3}$$

and

$$\sum_\lambda k_\lambda(Y)\,\lambda(Y) = 0. \tag{4}$$

By induction (3) implies the existence of elements $t_{\mu\xi}, q'_{ij} \in R$ and forms $p'_{ij}(X)$ of degree $m-1$ such that, for each λ,

$$h_\lambda(Y) - k_\lambda(Y) + \sum_\mu s_{\mu\lambda}\mu(X)$$
$$= \sum_{\mu,\xi} t_{\mu\xi}(c_\xi \mu(X) - \mu(b)\,Y_\xi) + \sum_{i,j}(b_j X_i - b_i X_j)\,p'_{ij}(X) + \sum_{i,j}(c_j Y_i - c_i Y_j)\,q'_{ij}.$$

Equating the terms in Y gives

$$h_\lambda(Y) - k_\lambda(Y) = -\sum_{\mu,\xi} t_{\mu\xi}\mu(b)\,Y_\xi + \sum_{i,j}(c_j Y_i - c_i Y_j)\,q'_{ij}, \tag{5}$$

and equating the terms in X gives, for each λ,

$$\sum_\mu s_{\mu\lambda}\mu(X) = \sum_{\mu,\xi} t_{\mu\xi}c_\xi\mu(X) + \sum_{i,j}(b_j X_i - b_i X_j)\,p'_{ij}(X). \tag{6}$$

But, by (2),

$$f(X) = \sum_{\mu,\lambda} s_{\mu\lambda}\lambda(c)\,\mu(X) + \sum_{i,j}(b_j X_i - b_i X_j)\,p_{ij}(X). \tag{7}$$

Hence, by (1) and (5),

$$f(X) + g(Y) = \sum_\lambda \Big(\sum_\mu s_{\mu\lambda}\mu(X)\Big)\lambda(c) + \sum_{i,j}(b_j X_i - b_i X_j)\,p_{ij}(X)$$
$$+ \sum_\lambda \Big(k_\lambda(Y) - \sum_{\mu,\xi} t_{\mu\xi}\mu(b)\,Y_\xi + \sum_{i,j}(c_j Y_i - c_i Y_j)\,q'_{ij}\Big)\lambda(Y),$$

which, by (4) and (6),

$$= \sum_\lambda \sum_{\mu,\xi} t_{\mu\xi}(c_\xi\lambda(c)\,\mu(X) - \mu(b)\,Y_\xi\lambda(Y)) + \sum_\lambda\sum_{i,j}(b_j X_i - b_i X_j)\,p'_{ij}(X)\,\lambda(c)$$
$$+ \sum_{i,j}(b_j X_i - b_i X_j)\,p_{ij}(X) + \sum_\lambda\sum_{i,j}(c_j Y_i - c_i Y_j)\,q'_{ij}\lambda(Y),$$

and this is a polynomial of the required type.

Corollary 1. *For all positive integers m and n,*

$$B^m \cap C^n = B^m C^n.$$

Proof. An element of $B^m \cap C^n$ may be written as either $f(b)$ or $-g(c)$ where $f(X)$ and $g(Y)$ are forms of respective degrees m and n. Then $f(b)+g(c) = 0$ and so we may express $f(X)+g(Y)$ as in Theorem 1. Equating terms in X then gives an equation for $f(X)$ of the form (7) above (with λ of degree n) and, hence, $f(b)$ takes the form

$$\sum_{\mu,\lambda} s_{\mu\lambda}\lambda(c)\,\mu(b) \quad \text{which is in} \quad B^m C^n.$$

Corollary 1 implies that, in (2), Theorem 1 and hence also the concluding remark about the Gorenstein property are true for an H_1-set generating a proper ideal in any commutative ring. (In the proof of (2), Theorem 1, replace a_i^t by ra_i^t where r is an arbitrary element of R, and deduce the contradiction that $R \subseteq a_{J_3}$.)

There is a natural interpretation of our Theorem 1 and its proof when the set $b_1, \ldots, b_r$ is empty and $c_1 = a_1, \ldots, c_k = a_k$.

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Corollary 2. *If $g(Y)$ is a form of degree n in $Y_1, \ldots, Y_k$ such that $g(a) = 0$ then $g(Y)$ is in the R-module generated by all forms of the type $(a_jY_i - a_iY_j)\,\nu(Y)$ where $\nu(Y)$ is a monomial of degree $n-1$.*

Proof. Simplify the proof of Theorem 1 by replacing each term involving X or μ by zero.

In the case when $a_1, \ldots, a_k$ is an R-sequence, Corollary 2 is implicit in (8), Chapitre 1. It implies Corollary 3 which was proved for an R-sequence by Corsini in (3).

Let n be a fixed positive integer and let S be the ring of polynomials over R in the set Z of all indeterminates $Z_{\mu_1 \ldots \mu_k}$ where each μ_i is a non-negative integer and $\mu_1 + \ldots + \mu_k = n$. When $f(Z) \in S$ let $f(a)$ denote the result of the substitutions

$$Z_{\mu_1 \ldots \mu_k} \to a_1^{\mu_1} \ldots a_k^{\mu_k}.$$

Denote by $\mathbf{q}_\infty$ (resp. $\mathbf{q}$) the ideal of S generated by the set of all forms (resp. linear forms) $f(Z)$ such that $f(a) = 0$. To avoid a possible ambiguity when R has no identity we emphasize that the ideal generated by a set Γ in the ring S means here the set $\sum_{\gamma \in \Gamma} S\gamma$ which we shall write as $S\Gamma$.

Corollary 3. (i) *The ideal $\mathbf{q}$ is generated by the set Γ_1 of all forms of the type*

$$a_i Z_{\nu_1 \ldots \nu_k} - a_j Z_{\mu_1 \ldots \mu_k},$$

where $\nu_w = \mu_w$ for $w \neq i$ or j, $\nu_i + 1 = \mu_i$ and $\nu_j = \mu_j + 1$.

(ii) *The ideal $\mathbf{q}_\infty$ is generated by Γ_1 together with the set Γ_2 of all forms of the type*

$$Z_{\nu_1 \ldots \nu_k} Z_{\mu_1 \ldots \mu_k} - Z_{\nu'_1 \ldots \nu'_k} Z_{\mu'_1 \ldots \mu'_k},$$

where $\nu_w + \mu_w = \nu'_w + \mu'_w$ for all w.

Proof. Denote by ψ the homomorphism from S to $R[X_1, \ldots, X_k]$ determined by the substitutions $Z_{\nu_1 \ldots \nu_k} \to X_1^{\nu_1} \ldots X_k^{\nu_k}$. Let $f(Z)$ be a form in S of degree m such that $f(a) = 0$. By Corollary 2, writing $\psi(f(Z)) = f(X)$, we have $f(X) \in \sum_{\kappa, i, j} R(a_jX_i - a_iX_j)\,\kappa(X)$ where each $\kappa(X)$ is a monomial in $X_1, \ldots, X_k$ of degree $mn-1$. Clearly there exist integers $\kappa_1, \ldots, \kappa_k$ with sum $n-1$ and a monomial $\xi_\kappa(Z)$ of degree $(m-1)\,n$ in the Z's such that, when $i \neq j$,

$$(a_jX_i - a_iX_j)\,\kappa(X) = \psi((a_jZ_{\kappa_1 \ldots \kappa_i+1 \ldots \kappa_j \ldots \kappa_k} - a_iZ_{\kappa_1 \ldots \kappa_i \ldots \kappa_j+1 \ldots \kappa_k})\,\xi_\kappa(Z)).$$

Thus $\psi(f(Z)) = \psi(g(Z))$ where $g(Z) \in S\Gamma_1$. Put $\ker \psi = H$. Since 0 is the only linear form in H, (i) follows.

To prove (ii) it will suffice to show that $H = S\Gamma_2$. A form $h(Z)$ in S of degree m may be written $\sum_\mu r_\mu \mu(Z)$ where $\mu(Z)$ runs over the distinct monomials of degree m in the Z's. Then

$$h(X) = \sum_\lambda s_\lambda \lambda(X),$$

where $\lambda(X)$ runs over all the distinct monomials of degree mn in $X_1, \ldots, X_k$ and $s_\lambda, = r_{\lambda_1} + r_{\lambda_2} + \ldots + r_{\lambda_{t(\lambda)}}$ say, is the sum of all r_μ such that $\mu(X) = \lambda(X)$. Thus

$$h(Z) = \sum_\lambda (r_{\lambda_1}\lambda_1(Z) + r_{\lambda_2}\lambda_2(Z) + \ldots + r_{\lambda_{t(\lambda)}}\lambda_{t(\lambda)}(Z)),$$

but if $h(Z) \in H$, i.e. $h(X) = 0$, then $s_\lambda = 0$ for all λ and so

$$h(Z) = \sum_\lambda (r_{\lambda_2}(-\lambda_1(Z) + \lambda_2(Z)) + \ldots + r_{\lambda_{t(\lambda)}}(-\lambda_1(Z) + \lambda_{t(\lambda)}(Z))).$$

Therefore, since ψ is homogeneous, H is generated by forms of the type $\mu(Z)-\mu'(Z)$ where $\mu(Z)$ and $\mu'(Z)$ are monomials. (ii) now follows from (3) section 4 where it may be seen that $S\Gamma_2$ contains every form $r(\mu(Z)-\mu'(Z))$ where $r \in R$ and $\mu(X) = \mu'(X)$.

As mentioned in (8), (3) and (1), $S/\mathbf{q}$ and $S/\mathbf{q}_\infty$ are, respectively, the symmetric and Rees algebras of the ideal $(Ra_1 + \ldots + Ra_k)^n$. It is easy to deduce from Theorem 1 that the symmetric algebra of the ideal $B^m + C^n$ may be represented as the quotient of the polynomial ring

$$R[\{Z_{\mu_1\ldots\mu_r} | \mu_1 + \ldots + \mu_r = m\} \cup \{W_{\nu_1\ldots\nu_s} | \nu_1 + \ldots + \nu_s = n\}]$$

by the ideal generated by forms of the type

$$c_1^{\nu_1}\ldots c_s^{\nu_s} Z_{\mu_1\ldots\mu_r} - b_1^{\mu_1}\ldots b_r^{\mu_r} W_{\nu_1\ldots\nu_s},$$

$$b_i Z_{\mu_1\ldots\mu_i\ldots\mu_j+1\ldots\mu_r} - b_j Z_{\mu_1\ldots\mu_i+1\ldots\mu_j\ldots\mu_r}$$

or

$$c_i W_{\nu_1\ldots\nu_i\ldots\nu_j+1\ldots\nu_s} - c_j W_{\nu_1\ldots\nu_i+1\ldots\nu_j\ldots\nu_s}.$$

Following Davis (4) we call elements $x_1, \ldots, x_k$ in R *strongly analytically independent* provided that whenever a form $f(X)$ in $R[X_1, \ldots, X_k]$, $= T$ say, is such that $f(x) = 0$ then $f(X) \in x_1 T + \ldots + x_k T$. Another, immediate, consequence of Corollary 2 is the following generalization of Theorem 2·1(i) of (10) and of Theorem 1·6 in (5).

COROLLARY 4. *The H_1-set $a_1, \ldots, a_k$ is strongly analytically independent.*

For the rest of this section R is assumed to contain an identity.

We now bring together some known results to show, in particular, that in the Noetherian case the converse of Corollary 4 holds (cf. (10) 2·2).

THEOREM 2. *For a proper ideal $I = (x_1, \ldots, x_k)$ in a Noetherian ring R, the following are equivalent*:

(1) *The set $x_1, \ldots, x_k$ is strongly analytically independent.*
(2) *$x_1, \ldots, x_k$ is an H_1-set.*
(3) *Any base for I with k elements is an H_1-set.*
(4) *I is generated by an R-sequence of length k.*
(5) *The grade of I is k.*

Proof. The equivalence of (1), (4) and (5) is contained in (4), Theorem, p. 202. (2) implies (1) by Corollary 4, and (5) implies (3) by (9), Theorem 6, p. 371.

Theorem 2 breaks down when R is not Noetherian; in particular, (1) does not imply (2) by (5), Example 1. However, in the case when R is local the following result was given in (5), 1·5, p. 365. The general case is an open problem.

PROPOSITION. *For any R, conditions* (2) *and* (3) *of Theorem* 2 *are equivalent when I is contained in the Jacobson radical of R.*

Proof. Proposition 3 in (7) and the remarks preceding it imply the result in a (more general) graded situation. (Alternatively, one may use directly a matrix argument starting from the fact that $x_1, \ldots, x_k$ is an H_1-set if, and only if, (in matrix notation) $[r_i][x_i]^t = 0$ implies $[r_i]$ is in the R-submodule of R^k generated by elements of the form $[y_i]$ where $y_i = 0$ if $i \neq u$ or v, $y_u = x_v$ and $y_v = -x_u$.)

3. *A generalization of the basic theorem.* Theorem 1 and Corollary 1 may be generalized as follows. Suppose the H_1-set $a_1, \ldots, a_k$ to be partitioned into p non-empty sets $\{a_{w1}, \ldots, a_{wq_w}\}$ $(w = 1, \ldots, p)$ and, for each w, let f_w be a form of degree m_w in indeterminates $X_{w1}, \ldots, X_{wq_w}$.

THEOREM 1*a*. *If* $\sum_{w=1}^{p} f_w(a) = 0$ *then* $\sum_{w=1}^{p} f_w(X)$ *is in the R-module generated by all forms of the type*

$$\mu_v(a)\,\mu_u(X) - \mu_u(a)\,\mu_v(X) \quad \text{or} \quad (a_{wj}X_{wi} - a_{wi}X_{wj})\,\mu'_w(X),$$

where μ_w *and* μ'_w *are monomials in* $X_{w1}, \ldots, X_{wq_w}$ *of respective degrees* m_w *and* $m_w - 1$.

Proof. Use induction on $\sum_w m_w$. The case $m_1 = \ldots = m_p = 1$ is easy, and it is sufficient to take a case in which $m_p \geqslant 2$. Then proceed by analogy with the proof of Theorem 1, letting f_p take the role of g and $\sum_{w=1}^{p-1} f_w$ the role of f.

Write $Ra_{w1} + \ldots + Ra_{wq_w} = A_w$ for $w = 1, \ldots, p$.

COROLLARY 1*a*. *If* $1 \leqslant i < p$ *then, for all positive integers* $m_1, \ldots, m_p$, *the intersection of the ideals*

$$A_1^{m_1} + \ldots + A_i^{m_i} \quad \text{and} \quad A_{i+1}^{m_{i+1}} + \ldots + A_p^{m_p}$$

is the same as their product.

Proof is analogous to the proof of Corollary 1.

COROLLARY 1*b*. *Let* $m_1, \ldots, m_p$ *and* $n_1, \ldots, n_p$ *be positive integers, and let* T *denote the set of numbers* t *between* 1 *and* p *for which* $m_t \leqslant n_t$. *Then*

$$(A_1^{m_1} + \ldots + A_p^{m_p}) \cap (A_1^{n_1} + \ldots + A_p^{n_p}) = \sum_{t \in T} A_t^{n_t} + \sum_{s \notin T} A_s^{m_s} + (\sum_{t \in T} A_t^{m_t})(\sum_{s \notin T} A_s^{n_s}).$$

Proof. Observe that $\sum_{t \in T} A_t^{n_t} \subseteq \sum_{t \in T} A_t^{m_t}$ and $\sum_{s \notin T} A_s^{m_s} \subseteq \sum_{s \notin T} A_s^{n_s}$, and use the modular law twice.

Our final result is the following immediate consequence of Theorem 1*a*.

COROLLARY 5. *For any positive integers* $m_1, \ldots, m_{p-1}$, *the images of the elements* $a_{p1}, \ldots, a_{pq_p}$ *in the ring* $R/(A_1^{m_1} + \ldots + A_{p-1}^{m_{p-1}})$ *constitute an* H_1*-set.*

Remark. An analogous result for R-sequences follows from (4), Remark, p. 202.

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