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A finite evaluation of a special exponential sum

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Let $f(x)$ and $g(x)$ be rational functions of x with integer coefficients, p an odd prime, $e(x) = e(2\pi ix/p)$, $e(u/v) = e(w)$, where $u \equiv vw \pmod p$† and $\chi(x)$ is a character mod p. It is well known that very abstruse methods are required to find an estimate for sums such as

$$S_1 = \sum_{x=0}^{p-1} \chi(g(x))\, e(f(x)),$$

where in the summation,‡ the values of x are omitted for which either $f(x)$ or $g(x)$ is not defined.

There are not many instances apart from classical ones when such a sum can be expressed in elementary terms. An interesting one found by Salié(1) about forty years ago, states in a slightly different form that if

$$S_2 = \Sigma e\left(ax + \frac{b}{x}\right)\left(\frac{x}{p}\right) \quad (ab \not\equiv 0), \tag{1}$$

where (x/p) is the Legendre symbol, then

$$S_2 = \epsilon\left(\frac{a}{p}\right)\sqrt{p}\sum_h e(h), \quad \epsilon = i^{\{\frac{1}{2}(p-1)\}^2}, \tag{2}$$

where $\sum_h$ refers to the solutions of

$$h^2 \equiv 4ab, \tag{3}$$

and the sum is zero if no solutions exist.

The result still holds if $b \equiv 0$ since

$$\Sigma\, e(ax)\left(\frac{x}{p}\right) = \epsilon\sqrt{p}\left(\frac{a}{p}\right). \tag{4}$$

We now show that the series

$$S = \Sigma'\, e\left(\frac{Ax+B}{x^2+C}\right)\left(\frac{x^2+C}{p}\right) \tag{5}$$

can be summed very simply. The Σ' denotes that a factor $\frac{1}{2}$ occurs when $x = 0$. Let a, b, c, d, e, f be any integers with $acd \not\equiv 0$. Write

$$T = \Sigma e(ax)\left(\frac{x}{p}\right), \tag{6}$$

† Hereafter mod p will be omitted in congruences.

‡ Hereafter the limits of summation will be omitted if they are 0 to $p-1$ unless otherwise specified.

where the summation is extended over the solutions mod p of the congruence

$$cx+\frac{d}{x}\equiv f. \tag{7}$$

The sum is zero if no solutions exist. We shall show in (10) that S can be expressed in terms of T with $a=-A/2C$, $d=-C$, $f=2B/A$, $c=1$. Replace (6), (7) by

$$\begin{aligned} pT &= \sum_{x,t} e\left(ax+t\left(cx+\frac{d}{x}-f\right)\right)\left(\frac{x}{p}\right) \\ &= \sum_{x,t} e\left((a+ct)\,x+\frac{dt}{x}-ft\right)\left(\frac{x}{p}\right). \end{aligned}$$

Then from (2), if $a+ct\not\equiv 0$, we have a contribution

$$\epsilon\sqrt{p}\sum_{h,t} e(h-ft)\left(\frac{a+ct}{p}\right)$$

taken over the values of h mod p where

$$h^2\equiv 4(a+ct)\,dt. \tag{8}$$

If $a+ct\equiv 0$, we have a contribution

$$\sum_x e\left(\frac{dt}{x}-ft\right)\left(\frac{x}{p}\right),$$

which from (4) gives, since $dt\not\equiv 0$,

$$\epsilon\sqrt{p}\, e\left(\frac{af}{c}\right)\left(\frac{-acd}{p}\right).$$

Hence
$$\sqrt{p}\,T=\epsilon\sum_{h,t} e(h-ft)\left(\frac{a+ct}{p}\right)+\epsilon\, e\left(\frac{af}{c}\right)\left(\frac{-acd}{p}\right).$$

In (8), put $dt\equiv(a+ct)\,y^2$. Then

$$t\equiv\frac{-ay^2}{cy^2-d},\quad a+ct\equiv\frac{-ad}{cy^2-d},\quad h\equiv\frac{\pm 2ady}{cy^2-d}.$$

Hence
$$\sqrt{p}\,T=\epsilon\sum_y e\left(\frac{\pm 2ady+afy^2}{cy^2-d}\right)\left(\frac{-ad}{p}\right)\left(\frac{cy^2-d}{p}\right)+\epsilon\, e\left(\frac{af}{c}\right)\left(\frac{-acd}{p}\right). \tag{9}$$

In the first term on the right-hand side of (9), write

$$ay^2=\frac{a}{c}(cy^2-d)+\frac{ad}{c}.$$

We dispense with the $\pm$ sign on writing $-y$ for y and have

$$\sqrt{p}\,T=2\epsilon{\sum_y}'\left(\frac{-ad}{p}\right)e\left(\frac{af}{c}\right)\left(\frac{2ady+afd/c}{cy^2-d}\right)\left(\frac{cy^2-d}{p}\right)+\epsilon\left(\frac{af}{c}\right)\left(\frac{-acd}{p}\right),$$

when the Σ' denotes a factor $\frac{1}{2}$ when $y=0$. Hence from (5), we have the value of S on putting

$$c=1,\quad d=-C,\quad 2ad=A,\quad adf=B,$$

and so $$2a = -A/C, \quad f = 2B/A.$$

Then $$T\sqrt{p} = 2\epsilon\left(\frac{-A/2}{p}\right)e\left(\frac{-B}{C}\right)S + \epsilon\left(\frac{-A/2}{p}\right)e\left(\frac{-B}{C}\right). \tag{10}$$

The value $A = 0$ has been excluded. But then a substitution $x^2 + c \equiv 1/y$ reduces S at once to a Gauss' sum. It may be remarked that if $(-C/p) = 1$, say $C \equiv -D^2$, then S reduces at once to a Salié sum by the substitution $x \to D + 1/x$.

A similar method can be applied to some exponential sums in n variables, for example, to
$$S = \sum_{(x)} e\left(\frac{a_1}{x_1} + \ldots + \frac{a_n}{x_n}\right)\left(\frac{x_1 \ldots x_n}{p}\right), \tag{11}$$

where the summation is extended over the (x) satisfying the congruence
$$b_1 x_1 + \ldots + b_n x_n + f \equiv 0. \tag{12}$$

If $b_1 \equiv 0$, S can be summed for x_1 and then becomes a similar sum in $n-1$ variables, and so it may be supposed that $b_1 \ldots b_n \not\equiv 0$. Let us first take the case when $a_1 \ldots a_n \not\equiv 0$. From (11) and (12), we have
$$pS = \sum_{(x),\, t} e\left(\frac{a_1}{x_1} + \ldots + \frac{a_n}{x_n} + t(b_1 x_1 + \ldots + b_n x_n + f)\right)\left(\frac{x_1 \ldots x_n}{p}\right). \tag{13}$$

The sum when $t \equiv 0$ gives a contribution
$$\epsilon^n p^{\frac{1}{2}n}\left(\frac{a_1 \ldots a_n}{p}\right).$$

Next on summing for $x_1, \ldots, x_n$, we have
$$pS = \epsilon^n p^{\frac{1}{2}n}\left(\frac{b_1 \ldots b_n}{p}\right)\sum_{t \neq 0} e(2h_1 + \ldots + 2h_n + ft)\left(\frac{t}{p}\right)^n + \epsilon^n p^{\frac{1}{2}n}\left(\frac{a_1 \ldots a_n}{p}\right), \tag{14}$$

where $$h_1^2 \equiv a_1 b_1 t, \ldots, h_n^2 \equiv a_n b_n t. \tag{15}$$

Hence the sum is zero unless with integers d, c, T
$$a_1 b_1 \equiv dc_1^2, \ldots, a_n b_n \equiv dc_n^2, \quad t \equiv dT^2.$$

Then $$h_1 \equiv \pm dc_1 T, \ldots, h_n \equiv \pm dc_n T, \tag{16}$$

where the signs are independent of each other. Then
$$pS = \epsilon^n p^{\frac{1}{2}n}\left(\frac{b_1 \ldots b_n}{p}\right)\sum_{T=0} e(\pm 2dc_1 T \ldots \pm 2dc_n T + fdT^2)\left(\frac{d}{p}\right)^n\left(\frac{T^{2n}}{p}\right) + \epsilon^n p^{\frac{1}{2}n}\left(\frac{a_1 \ldots a_n}{p}\right). \tag{17}$$

This series is easily summed for T. Rewrite it as
$$pS = \epsilon^n p^{\frac{1}{2}n}\left(\frac{b_1 \ldots b_n}{p}\right)\sum_{T=0} e(\pm 2dc_1 T \ldots \pm 2dc_n T + fdT^2)\left(\frac{d}{p}\right)^n$$
$$+ \epsilon^n p^{\frac{1}{2}n}\left(\frac{a_1 \ldots a_n}{p}\right) - 2^n \epsilon^n p^{\frac{1}{2}n}\left(\frac{b_1 \ldots b_n}{p}\right)\left(\frac{d}{p}\right)^n.$$

Since $a_1 b_1 \not\equiv 0$, then $d \not\equiv 0$. Hence if, for example, $f \not\equiv 0$, the series is equal to

$$\epsilon\sqrt{p}\sum_{\pm} e(-d(\pm c_1 \pm c_2 \ldots \pm c_n)^2/f).$$

If $f = 0$, the sum is zero except for possible combinations of signs giving

$$\pm c_1 \ldots \pm c_n \equiv 0,$$

and then the sum is p for each such possible combination.

Suppose finally that r of the a are zero, say, $a_1 \equiv 0, \ldots, a_r \equiv 0$. The only difference is now that in (14), (15), (16), the h terms start with h_{r+1}, and in (17), the c terms start with c_{r+1}.

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