AN INEQUALITY FOR SEQUENCE TRANSFORMATIONS

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1.1. Let $A=(a_{\mu\nu})$ be a *normal triangular matrix*, *i.e.*, one for which $a_{\mu\mu}\neq 0$ $(\mu\geqslant 0)$, $a_{\mu\nu}=0$ $(\nu>\mu)$.

Inequalities of the following form have entered naturally into analysis:

$$\left|\frac{1}{R_n}\sum_{\nu=0}^{m} a_{n\nu}s_\nu\right| \leqslant K \max_{0\leqslant\mu\leqslant m}\left|\frac{1}{R_\mu}\sum_{\nu=0}^{\mu} a_{\mu\nu}s_\nu\right|, \tag{1}$$

where (i) $0\leqslant m<n$, (ii) $R_\mu>0$ $(\mu\geqslant 0)$, (iii) K is a constant, depending on the matrix A and the sequence $\{R_\mu\}$, but independent of m, n and the finite sequence $\{s_\nu\}$.

The factor $1/R_\mu$ is convenient for classification, but we may omit it, by replacing $a_{\mu\nu}/R_\mu$, by $c_{\mu\nu}$, so that the inequality becomes

$$\left|\sum_{\nu=0}^{m} c_{n\nu}s_\nu\right| \leqslant K \max_{0\leqslant\mu\leqslant m}\left|\sum_{\nu=0}^{\mu} c_{\mu\nu}s_\nu\right|. \tag{2}$$

The inequalities (1) and (2) only hold for restricted classes of matrices. For example, if $|c_{n\nu}|\to\infty$ as $n\to\infty$, (2) breaks down for large n. Again, if $c_{n\nu}\to 0$, (2) gives no information for large n.

But these cases remain significant if we consider instead inequalities with the constant K replaced by a factor $G=G(m,n)$, independent of $\{s_\nu\}$, but depending on m and n. In the present paper I begin by obtaining an inequality of this kind, for an arbitrary normal triangular matrix with complex elements. The factor G is best-possible, in the sense that equality is attained with a suitable $\{s_\nu\}$, depending on m and n. In §3 I give necessary and sufficient conditions for equality to be attained with a sequence $\{s_\nu\}$ which is *independent* of m and n.

For an important class of matrices with positive elements, the same inequality has been obtained by Wilansky and Zeller [**19**], and discussed further by Zeller [**22**]. An account of earlier results is given in §2, in conjunction with examples. Zeller's later results are discussed in §4.

1.2. If

$$t_\mu=\sum_{\nu=0}^{\mu} a_{\mu\nu}s_\nu \quad (\mu\geqslant 0), \tag{3}$$

then

$$s_\nu=\sum_{\mu=0}^{\nu} b_{\nu\mu}t_\mu \quad (\nu\geqslant 0), \tag{4}$$

where $B=(b_{\mu\nu})$ is the inverse of the normal triangular matrix $A=(a_{\mu\nu})$. The matrix B is also triangular and normal. From the identities

$AB = BA = I$, we have

$$\sum_{\lambda=\nu}^{\mu} a_{\mu\lambda} b_{\lambda\nu} = \begin{cases} 1 & (\nu=\mu) \\ 0 & (\nu<\mu) \end{cases} \tag{5}$$

and

$$\sum_{\lambda=\nu}^{\mu} b_{\mu\lambda} a_{\lambda\nu} = \begin{cases} 1 & (\nu=\mu) \\ 0 & (\nu<\mu). \end{cases} \tag{6}$$

We shall make frequent use of (3), (4) and (5).

THEOREM 1. *If $A=(a_{\mu\nu})$ is a normal triangular matrix, with complex elements, $B=(b_{\mu\nu})$ is its inverse, $R_\mu > 0$ and $0 \leqslant m < n$, then*

$$\left| \frac{1}{R_n} \sum_{\nu=0}^{m} a_{n\nu} s_\nu \right| \leqslant G \max_{0\leqslant\mu\leqslant m} \left| \frac{1}{R_\mu} \sum_{\nu=0}^{\mu} a_{\mu\nu} s_\nu \right| \tag{7}$$

for all finite sequences $\{s_\nu\}$ $(\nu=0, 1, \ldots, m)$, where

$$G = G(m, n) = \frac{1}{R_n} \sum_{\mu=0}^{m} |h_\mu| R_\mu, \tag{8}$$

$$h_\mu = h_\mu(m, n) = \sum_{\nu=\mu}^{m} a_{n\nu} b_{\nu\mu}. \tag{9}$$

If

$$h_\mu = e^{-i\rho_\mu} |h_\mu| \neq 0 \quad (0 \leqslant \mu \leqslant m), \tag{10}$$

where $\rho_\mu = \rho_\mu(m, n)$, then there is equality in (7) if and only if

$$s_\nu = C k_\nu \quad (0 \leqslant \nu \leqslant m) \tag{11}$$

where C is an arbitrary constant, and $k_\nu = k_\nu(m, n)$ satisfies the equation†

$$\sum_{\nu=0}^{\mu} a_{\mu\nu} k_\nu = e^{i\rho_\mu} R_\mu \quad (0 \leqslant \mu \leqslant m), \tag{12}$$

i.e.

$$k_\nu = \sum_{\mu=0}^{\nu} b_{\nu\mu} e^{i\rho_\mu} R_\mu \quad (0 \leqslant \nu \leqslant m). \tag{13}$$

If $h_\mu = 0$ for some μ (and the corresponding ρ_μ is chosen arbitrarily), the conditions (11)–(13) are sufficient for equality.

In all cases, the factor G may be expressed in the form

$$G = \frac{1}{R_n} \left| \sum_{\nu=0}^{m} a_{n\nu} k_\nu \right|. \tag{14}$$

Proof. Let m, n be given integers such that $0 \leqslant m < n$. Write‡

$$t_{n,m} = \sum_{\nu=0}^{m} a_{n\nu} s_\nu = \sum_{\nu=0}^{m} a_{n\nu} \sum_{\mu=0}^{\nu} b_{\nu\mu} t_\mu = \sum_{\mu=0}^{m} t_\mu \sum_{\nu=\mu}^{m} a_{n\nu} b_{\nu\mu} = \sum_{\mu=0}^{m} h_\mu t_\mu,$$

† If $C = 0$, k_ν is arbitrary.

‡ *Cf.* Wilansky and Zeller [19] or Zeller [21; p. 43].

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where h_μ is given by (9). Then

$$\frac{1}{R_n}|t_{n,m}| \leqslant \frac{1}{R_n}\sum_{\mu=0}^{m}|h_\mu t_\mu| \qquad (*)$$

$$= \frac{1}{R_n}\sum_{\mu=0}^{m}|h_\mu| R_\mu . |t_\mu|/R_\mu \leqslant \frac{M}{R_n}\sum_{\mu=0}^{m}|h_\mu| R_\mu \qquad (**)$$

$$= GM,$$

where

$$M = \max_{0\leqslant\mu\leqslant m}|t_\mu|/R_\mu \tag{15}$$

and $G = G(m, n)$ is given by (8).

This establishes the inequality (7).

We next show that $G(m, n)$ is best-possible, in the sense that there is a finite sequence $\{s_\nu'\} = \{s_\nu'(m, n)\}$, such that

$$\frac{1}{R_n}\left|\sum_{\nu=0}^{m} a_{n\nu} s_\nu'\right| = G \max_{0\leqslant\mu\leqslant m}\left|\frac{1}{R_\mu}\sum_{\nu=0}^{\mu} a_{\mu\nu} s_\nu'\right|. \tag{16}$$

Now equality is attained in (7) if and only if nothing is thrown away at the steps (*) and (**). Thus there is equality in (7) if and only if

(i) $$h_\mu t_\mu = e^{i\theta}|h_\mu t_\mu| \qquad (0\leqslant\mu\leqslant m),$$

where θ is real and independent of μ,

(ii) $$|h_\mu t_\mu| = M|h_\mu| R_\mu \qquad (0\leqslant\mu\leqslant m),$$

where M is non-negative and independent of μ.

First suppose that $h_\mu \neq 0$ $(0\leqslant\mu\leqslant m)$. Then the conditions become

(i)′ $t_\mu = e^{i\theta}.e^{i\rho_\mu}|t_\mu|$

(ii)′ $|t_\mu| = MR_\mu$ $\quad (0\leqslant\mu\leqslant m)$,

for some real θ and some $M \geqslant 0$, where ρ_μ satisfies (10).

Conditions (i)′ and (ii)′ hold if and only if

$$t_\mu = Me^{i\theta}.e^{i\rho_\mu}R_\mu$$

i.e., $$t_\mu = Ce^{i\rho_\mu}R_\mu \qquad (0\leqslant\mu\leqslant m) \tag{17}$$

for some constant C.

Thus there is equality in (7) if and only if

$$\sum_{\nu=0}^{\mu} a_{\mu\nu} s_\nu = Ce^{i\rho_\mu}R_\mu \quad (0\leqslant\mu\leqslant m). \tag{18}$$

Now let k_ν be defined as so to satisfy (12)–(13). Then the conditions for equality are satisfied if and only if (11) holds.

Next suppose that $h_\mu = 0$ *for one or more* μ. *If* $h_{\mu'} = 0$, where $0\leqslant\mu'\leqslant m$, conditions (i) and (ii) are automatically satisfied for $\mu = \mu'$, and impose no

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restriction on $t_{\mu'}$, except that $|t_{\mu'}| \leqslant MR_{\mu'}$ by (15). Hence the conditions for equality are satisfied, in particular, when s_ν satisfies (11).

Finally, if we substitute $s_\nu' = k_\nu$ into (16) ,we obtain

$$\frac{1}{R_n}\left|\sum_{\nu=0}^{m} a_{n\nu} k_\nu\right| = G \max_{0\leqslant\mu\leqslant m}\left|\frac{1}{R_\mu} e^{i\rho_\mu} R_\mu\right| = G.$$

Thus G is always expressible in the form (14).

This completes the proof of the theorem.

Remarks. (1) More precisely, there is equality in (7) if and only if

$$\left.\begin{array}{ll}\text{(I)}\ t_\mu = Ce^{i\rho_\mu} R_\mu & (h_\mu \neq 0)\\ \text{(II)}\ |t_\mu| \leqslant |C| R_\mu & (h_\mu = 0)\end{array}\right\} \quad (0 \leqslant \mu \leqslant m). \tag{19}$$

and

In particular, if $h_\mu = 0$ for all μ $(0 \leqslant \mu \leqslant m)$, then C is indeterminate, and there is always equality in (7), both sides being zero. In this case $a_{n\nu} = 0$ for $0 \leqslant \nu \leqslant m$.

(2) If $h_{\mu'} = 0$, and $t_{\mu'} = T$, then the contribution of $t_{\mu'}$ to $s_\nu = \sum_{\mu=0}^{\nu} b_{\nu\mu} t_\mu$ is $b_{\nu\mu'} T$, if $\nu \geqslant \mu'$, and zero if $\nu < \mu'$. Hence its contribution to $t_{n,m} = \sum_{\nu=0}^{m} a_{n\nu} s_\nu$ is $\sum_{\nu=\mu'}^{m} a_{n\nu} b_{\nu\mu'} T = h_{\mu'} T = 0$.

Thus the arbitrary $t_{\mu'}$ contributes nothing to the left-hand side of the inequality. And it contributes nothing to the right-hand side provided $|T| \leqslant M' R_{\mu'}$, where $M' = \max_{0\leqslant\mu\leqslant m,\ \mu\neq\mu'} |t_\mu|/R_\mu$.

2.1. The inequality

$$\left|\sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1} s_\nu\right| \leqslant \max_{0\leqslant\mu\leqslant m}\left|\sum_{\nu=0}^{\mu} A_{\mu-\nu}^{\delta-1} s_\nu\right|, \tag{20}$$

where $0 < \delta < 1$, $0 \leqslant m < n$ and

$$A_n{}^\sigma = \binom{n+\sigma}{n} \quad (\sigma > -1), \tag{21}$$

was first stated by Jacob [**8**], and later found independently by myself [**2**], [**3**]. The proof in [**2**] is valid for complex s_ν.

Jacob said that the inequality was " already known ", and referred to Lemma 7 of Hardy and Riesz [**7**], which is Riesz's inequality (or mean value theorem) for typical means:

$$\left|\int_0^\xi (x-u)^{\delta-1} A(u)\, du\right| \leqslant \max_{0\leqslant v\leqslant\xi}\left|\int_0^v (v-u)^{\delta-1} A(u)\, du\right|, \tag{22}$$

where $0 < \delta < 1$, $0 < \xi < x$ and

$$A(u) = \sum_{\lambda_n < u} a_n \qquad (\lambda_0 < \lambda_1 < \ldots \text{ and } \lambda_n \to \infty). \tag{23}$$

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Riesz's inequality is not the same as Jacob's, even when $\lambda_n = n$, but an examination of the proof in [7] shows that (20) may be established by a similar argument, with Abel's lemma playing the role of the second mean value theorem.

2.2. In a more general form of Riesz's inequality, $A(u)$ is replaced by a function $\phi(u) \in L(0, \xi)$ and the maximum by the essential upper bound.

The proofs of the inequality [**13**], [**7**], [**14**], [**16**], show that a factor

$$G = G(\xi, x) = \frac{1}{\Gamma(\delta)\,\Gamma(1-\delta)} \int_0^{\xi} (x-u)^{\delta-1} u^{-\delta}\, du < 1 \tag{24}$$

may be inserted on the right-hand side. Moreover, this factor is best-possible. For equality is attained if we put $\phi(u) = Cu^{-\delta}$.

I have obtained some extensions of Riesz's inequality in an earlier investigation not yet presented for publication.

2.3. The general inequality (2) was introduced by Jurkat and Peyerimhoff [**10**].

Peyerimhoff [**12**] had shown that, if the normal triangular matrix $C = (c_{\mu\nu})$ defines a regular sequence-to-sequence transformation, then the space C_0, of sequences $s = \{s_\nu\}$ whose transforms $t = \{t_\mu\}$ are null sequences, will have the property† that every sequence $s = \{s_\nu\}$ in C_0 is the weak limit of the sequence‡

$$s^{(k)} = (s_0, s_1, \ldots, s_k, 0, 0, \ldots), \tag{25}$$

if and only if

$$\left| \sum_{\nu=0}^{m} c_{n\nu} s_\nu \right| \leqslant K \sup_{\mu \geqslant 0} \left| \sum_{\nu=0}^{\mu} c_{\mu\nu} s_\nu \right|, \tag{26}$$

whenever $0 \leqslant m < n < \infty$ and $s \in C_0$.

Jurkat and Peyerimhoff showed that (26) holds for all $s \in C_0$ if and only if the inequality (2) holds for all finite sequences $(s_0, s_1, \ldots, s_m)$ and all m, n such that $0 \leqslant m < n < \infty$. Their argument is valid for any normal triangular matrix and with C_0 replaced by any set G containing all the sequences s whose transforms are terminating sequences

$$t^{(m)} = (t_0, t_1, \ldots, t_m, 0, 0, \ldots). \tag{27}$$

A number of other properties have also been shown to be equivalent to (26), and hence also to (2). For an account of these, see Wilansky [**18**] and the references there to [**4**], [**11**], [**17**], [**19**], [**20**], [**21**].

† Schwache Abschnittskonvergenz (SAK); weak sectional convergence.

‡ *i.e.* $F(s) = \lim F(s^{(k)})$ for every linear functional $F(s)$ defined on C_0.

2.4. Jurkat and Peyerimhoff [**10**] found sufficient conditions for the inequality (2) (or (1) with $R_\mu = 1$) to hold. Their conditions were

$$\text{(i) } a_{\mu\nu} > 0 \quad (0 \leqslant \nu \leqslant \mu \leqslant n), \quad \text{(ii) } \frac{a_{\mu_2 \nu_1}}{a_{\mu_1 \nu_1}} \geqslant \frac{a_{\mu_2 \nu_2}}{a_{\mu_1 \nu_2}}, \quad \text{(iii) } \frac{a_{\mu_2 0}}{a_{\mu_1 0}} \leqslant K, \tag{28}$$

for $0 \leqslant \nu_1 \leqslant \nu_2 \leqslant \mu_1 < \mu_2 \leqslant n$.

If condition (iii) is omitted and $a_{\mu\nu}$ is replaced by $c_{\mu\nu} = a_{\mu\nu}/a_{\mu 0}$, then $c_{\mu\nu}$ satisfies the same conditions, with $K = 1$, and hence, as their proof shows,

$$\left| \frac{1}{a_{n0}} \sum_{\nu=0}^{m} a_{n\nu} s_\nu \right| \leqslant \max_{0 \leqslant \mu \leqslant m} \left| \frac{1}{a_{\mu 0}} \sum_{\nu=0}^{\mu} a_{\mu\nu} s_\nu \right|, \tag{29}$$

for $0 \leqslant m \leqslant n$, which is an inequality of the form (1), with $R_\mu = a_{\mu 0}$. Their proof, which holds for real s_ν, is an extension of that in [**3**].

2.5. Wilansky and Zeller [**19**] (see Zeller [**21**, p. 43]) obtained the inequality (1), with $R_\mu = 1$, under the conditions

$$\left.\begin{aligned} &a_{\mu\mu} = 1/b_{\mu\mu} > 0 \ \ (\mu \geqslant 0), \quad a_{\mu\nu} \geqslant 0 \ \ (0 \leqslant \nu < \mu), \\ &b_{\mu\nu} \leqslant 0 \ \ (0 \leqslant \nu < \mu), \quad B_\nu = \sum_{\mu=0}^{\nu} b_{\nu\mu} \geqslant 0 \ \ (\nu \geqslant 0). \end{aligned}\right\} \tag{30}$$

Their proof, which is valid for complex s_ν, shows that the constant $K = 1$ may be replaced by the factor

$$G = \sum_{\mu=0}^{m} h_\mu, \text{ where } h_\mu \geqslant 0, \tag{31}$$

h_μ being given by (9). This implies, since $B_\nu \geqslant 0$, that

$$G = \sum_{\nu=0}^{m} a_{n\nu} B_\nu \leqslant \sum_{\nu=0}^{n} a_{n\nu} B_\nu = \sum_{\mu=0}^{n} \sum_{\nu=\mu}^{n} a_{n\nu} b_{\nu\mu} = 1, \tag{32}$$

and that equality is attained in (7), with $R_\mu = 1$, when $t_\mu = C$, *i.e.*, $s_\nu = CB_\nu$.

They also observed that Jurkat and Peyerimhoff's criterion, with $K = 1$, is included in theirs. This is a generalisation of Kaluza's theorem; *cf.* Hardy [**6**; Theorem 22]. Later Zeller [**22**] remarked that the conditions $b_{\mu\mu} > 0$ $(\mu \geqslant 0)$, $b_{\mu\nu} \leqslant 0$ $(0 \leqslant \nu < \mu)$ themselves imply that $a_{\mu\nu} \geqslant 0$ $(0 \leqslant \nu < \mu)$, so that the hypothesis on $a_{\mu\nu}$ may be omitted. This is a generalisation of a sort of converse of Kaluza's theorem; *cf.* Dienes [**5**].†

These results may be incorporated in the following theorem, which is a corollary of Theorem 1. The hypotheses of Theorem 2 are satisfied, in particular, under conditions (28), with (iii) omitted.

† Dr. Vermes showed me that if B is $n \times n$ and triangular, with diagonal matrix D, then $B^{-1} = \sum_{\nu=0}^{n-1} D^{-\nu-1}(D-B)^\nu$, since $(D-B)^n = 0$. See also Tatchell [**15**]. Professor Peyerimhoff gave me two further references: (1) G. de Rham, *Publ. Inst. Math. Belgrade*, 4 (1952), 133–134, (2) W. B. Jurkat, *Proc. International Congress of Mathematicians Amsterdam*, 2 (1954), 126.

Theorem 2. *If $B=(b_{\mu\nu})$ is the inverse of a normal triangular matrix $A=(a_{\mu\nu})$, and if*

$$b_{\mu\mu}>0 \quad (0\leqslant\mu\leqslant n), \qquad b_{\mu\nu}\leqslant 0 \quad (0\leqslant\nu<\mu\leqslant n) \tag{33}$$

and $R_\mu>0$ $(0\leqslant\mu\leqslant n)$, then $a_{\mu\nu}\geqslant 0$ $(0\leqslant\nu\leqslant\mu\leqslant n)$ and, if $0\leqslant m<n$,

$$h_\mu\geqslant 0 \quad (0\leqslant\mu\leqslant m), \tag{34}$$

where h_μ is given by (9). *Further the inequality* (7) *holds, with*

$$G=\left|\sum_{\nu=0}^{m} a_{n\nu}k_\nu\right| \Big/ \sum_{\nu=0}^{n} a_{n\nu}k_\nu, \tag{35}$$

where k_ν satisfies

$$\sum_{\nu=0}^{\mu} a_{\mu\nu}k_\nu=R_\mu \quad (0\leqslant\mu\leqslant n), \tag{36}$$

i.e.,

$$k_\nu=\sum_{\mu=0}^{\nu} b_{\nu\mu}R_\mu \quad (0\leqslant\nu\leqslant n). \tag{37}$$

Proof. Since conditions (33) imply that $a_{\mu\nu}\geqslant 0$ $(0\leqslant\nu\leqslant\mu\leqslant n)$, we have, as in Wilansky and Zeller's theorem,

$$h_\mu=\sum_{\nu=\mu}^{m} a_{n\nu}b_{\nu\mu}=-\sum_{\nu=m+1}^{n} a_{n\nu}b_{\nu\mu}\geqslant 0 \quad (0\leqslant\mu\leqslant m).$$

Thus

$$h_\mu=e^{-0i}|h_\mu| \quad (0\leqslant\mu\leqslant m),$$

and (13), with $\rho_\mu=0$, becomes (37) for $0\leqslant\nu\leqslant m$. Hence (35) follows from (14), together with (36) for $\mu=n$.

This proves the theorem.

In particular, we have

Corollary 1. *If $k_\nu\geqslant 0$ $(0\leqslant\nu\leqslant n)$, then $G\leqslant 1$.*

If $R_\mu=1$, this is Wilansky and Zeller's theorem.

Corollary 2. *If $k_\nu\geqslant 0$ $(0\leqslant\nu\leqslant n)$ and $a_{n\nu'}k_{\nu'}>0$ for some ν' such that $m<\nu'\leqslant n$, then $G<1$.*

Corollary 3. *If $k_\nu=0$ $(m<\nu\leqslant n)$, then $G=1$.*

Corollary 4. *If $k_\nu\leqslant 0$ $(m<\nu\leqslant n)$ and $a_{n\nu'}k_{\nu'}<0$ for some ν' such that $m<\nu'\leqslant n$, then $G>1$.*

In Corollaries 1–2,

$$0\leqslant\sum_{\nu=0}^{m} a_{n\nu}k_\nu \begin{Bmatrix}\leqslant\\ <\end{Bmatrix} \sum_{\nu=0}^{n} a_{n\nu}k_\nu=R_n\neq 0.$$

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In Corollaries 3–4,

$$\sum_{\nu=0}^{m} a_{n\nu} k_\nu \left\{ \begin{matrix} = \\ > \end{matrix} \right\} \sum_{\nu=0}^{n} a_{n\nu} k_\nu = R_n > 0.$$

Examples. (1) If $0 < \delta < 1$, $\alpha > -1$, then †

$$\left| \frac{1}{A_n{}^\alpha} \sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1} s_\nu \right| \leqslant G \max_{0 \leqslant \mu \leqslant m} \left| \frac{1}{A_n{}^\mu} \sum_{\nu=0}^{\mu} A_{\mu-\nu}^{\delta-1} s_\nu \right| \tag{38}$$

for $0 \leqslant m < n$, where

$$G = \frac{1}{A_n{}^\alpha} \left| \sum_{\nu=0}^{m} A_{n-\nu}^{\delta-1} A_\nu^{\alpha-\delta} \right|, \tag{39}$$

and equality is attained in (38) if and only if

$$s_\nu = C A_\nu{}^{\alpha-\delta} \quad (0 \leqslant \nu \leqslant m). \tag{40}$$

Further,

$$G \left\{ \begin{matrix} < 1 & (\alpha > \delta - 1) \\ = 1 & (\alpha = \delta - 1) \\ > 1 & (-1 < \alpha < \delta - 1). \end{matrix} \right\} \tag{41}$$

Here $b_{\mu\nu} = A_{\mu-\nu}^{-\delta-1}$, $R_\mu = A_\mu{}^\alpha > 0$, $h_k > 0$ and

$$k_\nu = \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{-\delta-1} A_\mu{}^\alpha = A_\nu{}^{\alpha-\delta}. \tag{42}$$

Hence $k_0 = 1$ and, for $\nu \geqslant 1$,

$$k_\nu \left\{ \begin{matrix} > 0 & (\alpha - \delta > -1) \\ = 0 & (\alpha - \delta = -1) \\ < 0 & (-1 - \delta < \alpha - \delta < -1). \end{matrix} \right\} \tag{43}$$

(2) An example of Corollary 3 is the extension of the inequality (29) to complex s_ν.

(3) Let $A = (a_{n\nu})$ be the Hausdorff matrix $(\mathfrak{H}, \mu)$, where $\mu_\nu \neq 0$ $(\nu \geqslant 0)$ *i.e.*,

$$a_{n\nu} = \binom{n}{\nu} \Delta^{n-\nu} \mu_\nu, \quad b_{n\nu} = \binom{n}{\nu} \Delta^{n-\nu}(1/\mu_\nu), \tag{44}$$

where $\Delta u_\lambda = u_\lambda - u_{\lambda+1}$.

Then, if $\mu_0 > 0$ and $\Delta^p(1/\mu_\nu) \leqslant 0$ $(\nu \geqslant 0, p \geqslant 1)$, the matrix A satisfies the hypotheses of Theorem 2. If $R_\mu = 1$, then

$$\{k_\nu\} = (\mathfrak{H}, 1/\mu)\{1\} = \{1/\mu_0\}, \tag{45}$$

and equality is attained in (7) when $s_\nu = C$ $(0 \leqslant \nu \leqslant m)$.

† The case $\alpha \geqslant \delta - 1$, with G replaced by $K = 1$, has been given by Andersen [1].

We may take

(i) $$\mu_\nu = 1\bigg/\binom{\nu+\delta}{\nu} = \delta\int_0^1 t^\nu(1-t)^{\delta-1}\,dt, \tag{46}$$

where $0<\delta<1$. Then A is the Cesàro matrix of order $\delta: a_{\mu\nu} = A^{\delta-1}_{\mu-\nu}/A_\mu{}^\delta$.

(ii) $$\mu_\nu = (\nu+1)^{-\delta} = \frac{1}{\Gamma(\delta)}\int_0^1 t^\nu\{\log(1/t)\}^{\delta-1}\,dt, \tag{47}$$

where $0<\delta<1$. Then A is the Hölder matrix of order δ.

In this case, if $R_\mu = 1$, we obtain

$$\left|\sum_{\nu=0}^{m}\binom{n}{\nu}\Delta^{n-\nu}\{(\nu+1)^{-\delta}\}s_\nu\right| \leqslant G\max_{0\leqslant\mu\leqslant m}\left|\sum_{\nu=0}^{\mu}\binom{\mu}{\nu}\Delta^{\mu-\nu}\{(\nu+1)^{-\delta}\}s_\nu\right|, \tag{48}$$

where
$$G = \sum_{\nu=0}^{m}\binom{n}{\nu}\Delta^{n-\nu}\{(\nu+1)^{-\delta}\}. \tag{49}$$

Here $h_\mu > 0$ $(0\leqslant\mu\leqslant m)$ and $\{k_\nu\} = H_\nu{}^{-\delta}\{1\} = 1$. Hence there is equality in (48) if and only if $s_\nu = C$ $(0\leqslant\nu\leqslant m)$.

Also $G(m,n)\leqslant G(n-1,n) < R_n = 1$.

Zeller [**20**] has stated that the inequality for Hölder means of order δ, $0<\delta<1$, with $K=1$ in place of G, was obtained by Peyerimhoff [**12**]. But the inequality does not occur in Peyerimhoff's paper.

Jacob [**9**] has given an inequality for the integral analogue of Hölder means of order δ, $0<\delta<1$.

2.6. The next simplest case of Theorem 1 is the following result.

THEOREM 3. *If* $A=(a_{\mu\nu})$, $R_\mu>0$ *and*

$$a_{\mu\mu}>0 \quad (0\leqslant\mu\leqslant n), \qquad a_{\mu\nu}\leqslant 0 \quad (0\leqslant\nu<\mu\leqslant n), \tag{50}$$

then $b_{\mu\nu}\geqslant 0$ $(0\leqslant\nu\leqslant\mu\leqslant n)$ *and, if* $0\leqslant m<n$,

$$h_\mu\leqslant 0 \quad (0\leqslant\mu\leqslant m), \tag{51}$$

where h_μ *is given by* (9). *Further, the inequality* (7) *holds, with*

$$G = \sum_{\nu=0}^{m} a_{n\nu}k_\nu\bigg/\left|\sum_{\nu=0}^{n} a_{n\nu}k_\nu\right|, \tag{52}$$

where
$$k_\nu = -\sum_{\mu=0}^{\nu} b_{\nu\mu}R_\mu \quad (0\leqslant\nu\leqslant n), \tag{53}$$

and also†

$$G(m,n)\leqslant G(n-1,n) = \frac{1}{R_n b_{nn}}\sum_{\mu=0}^{n-1} b_{n\mu}R_\mu. \tag{54}$$

† *Cf.* §4, Theorem 7.

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Proof. Since (50) implies that $b_{\mu\nu} \geqslant 0$ $(0 \leqslant \nu \leqslant \mu \leqslant n)$, we have

$$h_\mu = \sum_{\nu=\mu}^{m} a_{n\nu} b_{\nu\mu} \leqslant 0.$$

Hence $h_\mu = e^{-\pi i}|h_\mu|$, so that (13) becomes (53) for $0 \leqslant \nu \leqslant m$, and (52) follows as (35) does in Theorem 2. Finally, (54) follows, since $a_{n\nu} k_\nu \geqslant 0$ $(0 \leqslant \nu < n)$ and

$$h_\mu(n-1, n) = -a_{nn} b_{n\mu} = -b_{n\mu}/b_{nn} \leqslant 0 \quad (0 \leqslant \mu < n).$$

Example. $a_{\mu\nu} = A_{\mu-\nu}^{-\eta-1}$ $(0 < \eta < 1)$. $R_\mu = A_\mu^{\alpha}$ $(\alpha > -1)$. Here $b_{\mu\nu} = A_{\mu-\nu}^{\eta-1}$, $k_\nu = A_\nu^{\alpha+\eta}$ and

$$G(m, n) = \frac{1}{A_n^{\alpha}} \left| \sum_{\nu=0}^{m} A_{n-\nu}^{-\eta-1} A_\nu^{\alpha+\eta} \right|. \tag{55}$$

Clearly $G(m, n)$ is bounded if $m < \theta n$ $(0 < \theta < 1)$, but

$$G(n-1, n) = (A_n^{\alpha+\eta} - A_n^{\alpha})/A_n^{\alpha},$$

which is unbounded.

3.1. In Theorems 2 and 3, and the examples on them, the numbers k_ν were independent of m and n. In other words, if the hypotheses are given for all m, n such that $0 \leqslant m < n < \infty$, then the sequences $\{s_\nu\} = \{Ck_\nu\}$ give equality in (7) for all m and n such that $0 \leqslant m < n < \infty$.

In this section we first obtain necessary and sufficient conditions for a matrix $(a_{\mu\nu})$ to be such that equality can be attained in (7) throughout

$$\Delta: \textit{the set of all pairs } (m, n) \textit{ such that } 0 \leqslant m < n < \infty, \tag{56}$$

with one and the same sequence $\{s_\nu\}$, where $s_0' \neq 0$. If $s_\nu = 0$ $(0 \leqslant \nu \leqslant p)$, $h_\mu(m, n)$ is unrestricted for $0 \leqslant \mu \leqslant p$.

For $\mu \geqslant 0$, we write Δ_μ for the set of pairs (m, n) such that $\mu \leqslant m < n < \infty$.

THEOREM 4. *If $A = (a_{\mu\nu})$ is triangular and normal, then a necessary and sufficient condition for equality to be attainable in* (7) *throughout Δ, defined by* (56), *with a sequence $\{s_\nu\}$ $(s_0 \neq 0)$ independent of m and n, is that there should be real numbers $\sigma_{n,m}$ $(0 \leqslant m < n < \infty)$ and ρ_μ $(\mu \geqslant 0)$ such that*

$$h_\mu(m, n) = e^{i(\sigma_{n,m} - \rho_\mu)}|h_\mu(m, n)| \quad (0 \leqslant \mu \leqslant m) \tag{57}$$

for all (m, n) in Δ, where $\sigma_{n,m}$ is independent of μ and ρ_μ is independent of m and n, $h_\mu(m, n)$ being defined by (9).

Proof. There will be equality in (7) throughout Δ, with a given sequence $\{s_\nu\}$, if and only if nothing is thrown away at the steps corresponding to (*) and (**) in the proof of Theorem 1.

If t_μ' is the transform of s_ν', given by (3), necessary and sufficient conditions are

$$\left.\begin{array}{ll}\text{(i)} & h_\mu t_\mu' = e^{i\sigma_{n,m}}|h_\mu t_\mu'| \\ \text{(ii)} & |h_\mu t_\mu'| = M_m |h_\mu| R_\mu\end{array}\right\} \quad (0 \leqslant \mu \leqslant m) \tag{58}$$

for all (m, n) in Δ, where $\sigma_{n,m}$ is independent of μ, and if $s_0' \neq 0$,

$$M_m = \max_{0 \leqslant \mu \leqslant m} |t_\mu'|/R_\mu > 0.$$

Necessity. Assume that there is a sequence $\{s_\nu')$ $(s_0' \neq 0)$ giving equality in (7) throughout Δ.

First suppose that (a) *for each* μ $(\mu \geqslant 0)$ *there is a pair* $m = m_\mu$, $n = n_\mu$ *in* Δ_μ *such that* $h_\mu(m_\mu, n_\mu) \neq 0$. Then, if we put $m = m_\mu$, $n = n_\mu$ in (ii) and cancel the non-zero factor, we see that (ii) implies

$$\text{(ii)}' \qquad |t_\mu'| = M_{m_\mu} R_\mu \quad (\mu \geqslant 0).$$

Since $m_\mu \geqslant \mu$, (ii)$'$ implies

$$\text{(ii)}'' \qquad |t_\mu'| = M_\mu R_\mu \quad (\mu \geqslant 0).$$

Since t_μ' is independent of m and n, it follows from (ii)$''$ that there are numbers ρ_μ such that

$$t_\mu' = M_\mu e^{i\rho_\mu} R_\mu \quad (\mu \geqslant 0). \tag{59}$$

Substituting from (59) into (i), we obtain

$$h_\mu M_\mu e^{i\rho_\mu} R_\mu = e^{i\sigma_{n,m}}|h_\mu| M_\mu R_\mu \quad (0 \leqslant \mu \leqslant m < n < \infty),$$

which is the equivalent to (57), since $M_\mu > 0$, $R_\mu > 0$.

Thus (57) is *necessary* when (a) holds.

Next suppose that condition (a) *does not hold*, and let G be the set of values of μ for which $h_\mu(m, n) = 0$ throughout Δ_μ. Then for every μ not in G, the same argument shows that there must be numbers $\sigma_{n,m}$ and ρ_μ satisfying (57). But for every μ in G, conditions (i), (ii) and (57) are automatically satisfied. Thus (57) is *necessary* when condition (a) does not hold.

Sufficiency. If (57) holds, then (i) and (ii) are satisfied, with $M_m = M > 0$, when $t_\mu' = Me^{i\lambda} e^{i\rho_\mu} R_\mu$, and hence there is equality in (7) throughout Δ when $s_\nu' = Ck_\nu$ $(C \neq 0)$, with k_ν as in (13).

Thus (57) is *sufficient*, and the theorem is proved.

3.2. The class of sequences $\{s_\nu\}$ giving equality in (7) throughout Δ is most easily determined in the case where $h_\mu(m, n) \neq 0$ $(0 \leqslant \mu \leqslant m)$ for all (m, n) in Δ. In the general case, where some of the $h_\mu(m, n)$ may vanish, we must analyse the distribution of the non-vanishing $h_\mu(m, n)$.

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We shall say that two non-negative integers μ_1, μ_2 are *linked*, if there is a pair (m', n') in $\Delta_{\mu_1} \cap \Delta_{\mu_2}$ such that $h_{\mu_1}(m', n') \neq 0$ and $h_{\mu_2}(m', n') \neq 0$. If a non-negative integer μ is not linked to any other non-negative integer we shall call μ *isolated.*

We shall say that a set E of non-negative integers is *connected,* if for every pair μ_α, μ_β in E there is a finite chain of linked pairs (μ_0, μ_1), (μ_1, μ_2), $\ldots$, (μ_{N-1}, μ_N) such that $\mu_0 = \mu_\alpha$, $\mu_N = \mu_\beta$.

The simplest case from this point of view is that in which the set $\mu \geqslant 0$ is connected. In the next theorem we determine in this case the class of $\{s_\nu\}$, with $s_0 \neq 0$, giving equality in (7) throughout Δ.

THEOREM 5. *If condition* (57) *holds, and the set* $\mu \geqslant 0$ *is connected (in the sense defined), then there is equality in* (7) *throughout* Δ, *with* $s_0 \neq 0$, *if and only if*

$$s_\nu = Ck_\nu \quad (\nu \geqslant 0), \tag{60}$$

where C is a non-zero constant and

$$k_\nu = \sum_{\mu=0}^{\nu} b_{\nu\mu} e^{i\rho_\mu} R_\mu \quad (\nu \geqslant 0), \tag{61}$$

$(b_{\mu\nu})$ *being the inverse of* $(a_{\mu\nu})$.

If (57) *holds and the set* $\mu \geqslant 0$ *is not connected, then conditions* (60)–(61) *are sufficient for equality in* (7) *throughout* Δ.

Proof. Let $\{s_\nu\}$ be an arbitrary sequence, with $s_0 \neq 0$, and write

$$t_\mu = e^{i\theta_\mu} |t_\mu| \quad (\mu \geqslant 0), \tag{62}$$

where t_μ is given by (3) and θ_μ is real. Then $\{s_\nu\}$ will give equality in (7) throughout Δ if and only if

$$\left.\begin{array}{ll} \text{(i)}_a & h_\mu e^{i\theta_\mu} |t_\mu| = e^{i\phi_{n,m}} |h_\mu t_\mu| \\ \text{(ii)}_a & |h_\mu t_\mu| = M_m R_\mu |h_\mu| \end{array}\right\} \quad (0 \leqslant \mu \leqslant m)$$

for all (m, n) in Δ, where $M_m > 0$ and $\phi_{n,m}$ is real and independent of μ.

Necessity. Suppose that there is equality in (7) throughout Δ with t_μ given by (62).

Since (57) holds, it follows from (i)_a and (ii)_a, as in the proof of Theorem 4, that

$$e^{i(\phi_{n,m} - \theta_\mu))} |h_\mu(m, n)| = e^{i(\sigma_{n,m} - \rho_\mu)} |h_\mu(m, n)| \quad (0 \leqslant \mu \leqslant m) \tag{63}$$

for all (m, n) in Δ.

Now let μ_α, μ_β be an arbitrary pair of non-negative integers. Since the set $\mu \geqslant 0$ is connected, there is a finite chain of linked pairs (μ_{j-1}, μ_j) $(j = 1, 2, \ldots, N)$, with $\mu_0 = \mu_\alpha$, $\mu_N = \mu_\beta$, and pairs (m_j, n_j) in $\Delta_{\mu_{j-1}} \cap \Delta_{\mu_j}$ such that $h_{\mu_{j-1}}(m_j, n_j) \neq 0$, $h_{\mu_j}(m_j, n_j) \neq 0$ $(j = 1, 2, \ldots, N)$.

Substituting $m=m_j$, $n=n_j$ into (63), with $\mu=\mu_{j-1}$ and $\mu=\mu_j$, and eliminating the terms involving (m_j, n_j), we find that

$$\exp\{i(\theta_{\mu_{j-1}}-\rho_{\mu_{j-1}})\}=\exp\{i(\theta_{\mu_j}-\rho_{\mu_j})\} \quad (j=1, 2, \ldots, N), \tag{64}$$

and hence that

$$\exp\{i(\theta_{\mu_\alpha}-\rho_{\mu_\alpha})\}=\exp\{i(\theta_{\mu_\beta}-\rho_{\mu_\beta})\}. \tag{65}$$

Since μ_α, μ_β are arbitrary non-negative integers, this proves that

$$e^{i(\theta_\mu-\rho_\mu)}=e^{i\lambda} \quad (\mu\geqslant 0), \tag{66}$$

where λ is real and independent of μ, m and n.

Also, from (ii)$_\alpha$, $M_{\mu_{j-1}}=M_{m_j}=M_{\mu_j}$, $(j=1, 2, \ldots, N)$, so that $M_{\mu_\alpha}=M_{\mu_\beta}$. Thus $M_\mu=M>0$, and

$$t_\mu=Me^{i\lambda}e^{i\rho_\mu}R_\mu=Ce^{i\rho_\mu}R_\mu \quad (\mu\geqslant 0), \tag{67}$$

where C is a non-zero constant, and hence $\{s_\nu\}$ satisfies (60)–(61).

Sufficiency. We have already shown that the condition $t_\mu=Ce^{i\rho_\mu}R_\mu$, which is equivalent to (60)–(61), is sufficient.

This completes the proof.

Remark. In the general case, the set $\mu\geqslant 0$ will be composed of a sequence $\{D_j\}$ of sets, where each D_j is either a connected set or an isolated integer. Also the union of the intervals $\nu\leqslant\mu\leqslant m_\nu$, where $h_\nu(m_\nu, n_\nu)\neq 0$, is a set of distinct intervals which, together with the point-intervals formed by the remaining isolated integers, is a sequence $\{I_k\}$ of intervals $\sigma_k\leqslant\mu<\sigma_{k+1}$. Each D_j is covered by some I_k, and in the case of equality $e^{i\lambda}$ is constant in each D_j, while M_μ is constant in each I_k and M_μ is non-decreasing. Thus there will be equality in (7) throughout Δ, with $s_0\neq 0$, if and only if (I) $t_\mu=M_{\sigma_k}e^{i\lambda_j}e^{i\rho_\mu}R_\mu$ for $\mu\in D_j\subseteq I_k$, whenever D_j is a connected set, and (II) $|t_\mu|\leqslant M_{\sigma_k}R_\mu$ whenever μ is an isolated integer such that $\mu\neq 0$ and $\mu\in I_k$, where $|t_0|=M_0R_0=|s_0|>0$.

4. In his later paper [**22**], Zeller gave two theorems (Satz 1 and Satz 2) which would not normally have been suggested by the results of the present paper.

Zeller's Satz 2 completes the theorem of Wilansky and Zeller, mentioned above in §2.5, in a necessary and sufficient form. It may be modified so as to complete Theorem 2 similarly.

Zeller assumed in Satz 2 that the matrix A is triangular and normal and has non-negative elements, but he need only have chosen the sign of the *diagonal* elements. By omitting the redundant hypothesis, we are led to a similar completion of Theorem 3. The converse results may be combined into the following single theorem, in which *either* the upper signs *or* the lower signs may be taken.

THEOREM 6. *If $A=(a_{\mu\nu})$ is triangular and normal, and $a_{\mu\mu}>0$, $R_\mu>0$ for $0\leqslant\mu\leqslant n\leqslant N$, and if*

$$\gamma(m, n)=\frac{1}{R_n}\sum_{\nu=0}^{m} a_{n\nu} r_\nu \geqslant 0 \quad (0\leqslant m<n\leqslant N), \tag{68}$$

where

$$r_\nu=\pm\sum_{\mu=0}^{\nu} b_{\nu\mu} R_\mu \quad (0\leqslant \nu\leqslant N), \tag{69}$$

$(b_{\mu\nu})$ being the inverse of $(a_{\mu\nu})$, and if the inequality

$$\left|\frac{1}{R_n}\sum_{\nu=0}^{m} a_{n\nu} s_\nu\right| \leqslant \gamma(m, n) \max_{0\leqslant\mu\leqslant m}\left|\frac{1}{R_\mu}\sum_{\nu=0}^{\mu} a_{\mu\nu} s_\nu\right| \tag{70}$$

holds for all m, n such that $0\leqslant m<n\leqslant N$, and for all finite sequences $\{s_\nu\}$ $(0\leqslant\nu<N)$, then

$$a_{\mu\nu}\begin{cases}\geqslant 0\\ \leqslant 0\end{cases}, \quad b_{\mu\nu}\begin{cases}\leqslant 0\\ \geqslant 0\end{cases} \quad (0\leqslant\nu<\mu\leqslant N), \tag{71}$$

and $\gamma(m, n)$ is the best possible factor in (70).

Proof. Since $G(m, n)$, given by (8)–(9), is the best-possible factor in (7), the truth of (70) in the cases stated implies that

$$G(m, n)\leqslant\gamma(m, n) \quad (0\leqslant m<n\leqslant N) \tag{72}$$

and hence, by (8), (9), (68) and (69), that

$$\frac{1}{R_n}\sum_{\mu=0}^{m}|h_\mu(m, n)|\, R_\mu\leqslant \pm\frac{1}{R_n}\sum_{\nu=0}^{m} a_{n\nu}\sum_{\mu=0}^{\nu} b_{\nu\mu} R_\mu=\pm\frac{1}{R_n}\sum_{\mu=0}^{m} h_\mu(m, n)\, R_\mu.$$

This implies that

$$h_\mu(m, n)\begin{cases}\geqslant 0\\ \leqslant 0\end{cases} \quad (0\leqslant\mu\leqslant m<n\leqslant N). \tag{73}$$

With the upper signs, (73) implies that

$$h_\mu(n-1, n)=-a_{nn} b_{n\mu}\geqslant 0 \quad (0\leqslant\mu<n\leqslant N). \tag{74}$$

Since $a_{nn}>0$, (74) implies that $b_{n\mu}\leqslant 0$ for $0\leqslant\mu<n\leqslant N$, and hence also that $a_{n\mu}\geqslant 0$ for $0\leqslant\mu<n<N$.

With the lower signs, (73) implies that

$$h_m(m, n)=a_{nm} b_{mm}\leqslant 0 \quad (0\leqslant m<n\leqslant N). \tag{75}$$

Since $b_{mm}>0$, (75) implies that $a_{nm}\leqslant 0$ for $0\leqslant m\leqslant n\leqslant N$, and hence also $b_{nm}\geqslant 0$ for $0\leqslant m<n\leqslant N$.

Thus the relations (71) hold, with $a_{\mu\mu}=1/b_{\mu\mu}>0$, and hence it follows from Theorems 2 and 3, (35) and (52), that

$$\gamma(m, n)=G(m, n). \tag{76}$$

This completes the proof.

Zeller's Satz 1 may be restated in the form

Theorem 7. *If the matrix $A=(a_{\mu\nu})$ is triangular and normal, then a necessary and sufficient condition for the inequality* (1) *to hold, with $K=1$, for all m, n such that $0 \leqslant m < n \leqslant N$, and all finite sequences $\{s_\nu\}$ $(0 \leqslant \nu < N)$, is that*

$$\sum_{\mu=0}^{n-1} |b_{n\mu}| R_\mu \leqslant |b_{nn}| R_n \quad (0<n \leqslant N), \tag{77}$$

where $B=(b_{\mu\nu})$ is the inverse of A.

Proof. The condition is *necessary*, since

$$h_\mu(n-1, n)=-a_{nn} b_{n\mu}=-b_{n\mu}/b_{nn},$$

and hence (77) is equivalent to

$$G(n-1, n)=\frac{1}{R_n} \sum_{\mu=0}^{n-1} |h_\mu(n-1, n)| R_\mu \leqslant 1 \quad (0<n \leqslant N).$$

The *sufficiency* will now follow from †

Lemma. *If $G(n-1, n) \leqslant 1$ for $1 \leqslant n \leqslant N$, then*

$$G(0, n) \leqslant G(1, n) \leqslant \ldots \leqslant G(n-1, n)$$

for $2 \leqslant n \leqslant N$.

Let m have any value such that $0 \leqslant m < n-1 \leqslant N-1$. Then

$$\begin{aligned} R_n G(m, n) &= \sum_{\mu=0}^{m} R_\mu \left| \sum_{\nu=\mu}^{m} a_{n\nu} b_{\nu\mu} \right| \\ &\leqslant \sum_{\mu=0}^{m} R_\mu \left| \sum_{\nu=\mu}^{m+1} a_{n\nu} b_{\nu\mu} \right| + |a_{n,m+1}| \sum_{\mu=0}^{m} |b_{m+1,\mu}| R_\mu \\ &\leqslant \sum_{\mu=0}^{m} R_\mu \left| \sum_{\nu=\mu}^{m+1} a_{n\nu} b_{\nu\mu} \right| + |a_{n,m+1} b_{m+1,m+1}| R_{m+1} \\ &= \sum_{\mu=0}^{m+1} R_\mu \left| \sum_{\nu=\mu}^{m+1} a_{n\nu} b_{\nu\mu} \right| = R_n G(m+1, n). \end{aligned}$$

This proves the lemma, and hence the theorem.

† Zeller's argument proceeded from the identity (6), which we have not used.

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