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THE MAXIMUM SIZE OF (k, l)-SUM-FREE SETS IN CYCLIC GROUPS

BÉLA BAJNOK✉ and RYAN MATZKE

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Abstract

A subset A of a finite abelian group G is called (k, l)-sum-free if the sum of k (not necessarily distinct) elements of A never equals the sum of l (not necessarily distinct) elements of A. We find an explicit formula for the maximum size of a (k, l)-sum-free subset in G for all k and l in the case when G is cyclic by proving that it suffices to consider (k, l)-sum-free intervals in subgroups of G. This simplifies and extends earlier results by Hamidoune and Plagne ['A new critical pair theorem applied to sum-free sets in abelian groups', *Comment. Math. Helv.* **79**(1) (2004), 183–207] and Bajnok ['On the maximum size of a (k, l)-sum-free subset of an abelian group', *Int. J. Number Theory* **5**(6) (2009), 953–971].

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1. Introduction

Let G be an additively written abelian group of finite order n and exponent $e(G)$. When G is cyclic, we identify it with $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$; we consider $0, 1, \ldots, n-1$ interchangeably as integers and as elements of $\mathbb{Z}_n$.

For subsets A and B of G, we use the standard notation of $A + B$ and $A - B$ to denote the sets of two-term sums and differences, respectively, with one term chosen from A and the other from B. If, say, A consists of a single element a, we simply write $a + B$ and $a - B$ instead of $A + B$ and $A - B$. For a subset A of G and a positive integer h, hA denotes the h-fold *sumset* of A, that is, the collection of h-term sums with (not necessarily distinct) elements from A. Note that the h-fold sumset of A is (usually) different from its h-fold *dilation* $h \cdot A = \{ha \mid a \in A\}$.

For positive integers k and l, with $k > l$, we call a subset A of G (k, l)-*sum-free* if kA and lA are disjoint or, equivalently, if

$$0 \notin kA - lA.$$

For example, $A = \{1, 2\}$ is a $(5, 2)$-sum-free set in $\mathbb{Z}_9$ because $5A = \{5, 6, 7, 8, 0, 1\}$ and $2A = \{2, 3, 4\}$. (In this example, kA and lA are not only disjoint, but also partition the group; such (k,l)-sum-free sets are called *complete*.) We denote the maximum size of (k,l)-sum-free subsets in G by $\mu(G,\{k,l\})$. As our main result in this paper, we determine $\mu(\mathbb{Z}_n,\{k,l\})$ for all n, k and l.

Before we state our results, it may be interesting to briefly review the history of this problem. A $(2,1)$-sum-free set is simply called a *sum-free* set. Sum-free sets in abelian groups were first introduced by Erdős in [7] and then studied systematically by Wallis *et al.* [15].

We can construct sum-free sets in G by selecting a subgroup H in G for which G/H is cyclic and then taking the 'middle one-third' of the cosets of H. More precisely, with d denoting the index of H in G,

$$A = \bigcup_{i=\lceil (d-1)/3\rceil}^{2\lceil (d-1)/3\rceil -1} (i+H)$$

is sum-free in G and thus

$$\mu(G,\{2,1\}) \geq \max_{d\mid e(G)} \left\{ \left\lceil \frac{d-1}{3} \right\rceil \cdot \frac{n}{d} \right\}.$$

Using a version of Kneser's theorem, Diananda and Yap proved that we cannot do better in cyclic groups.

Theorem 1.1 (Diananda and Yap, 1969; see [6, 15]). *For all positive integers n,*

$$\mu(\mathbb{Z}_n,\{2,1\}) = \max_{d\mid n} \left\{ \left\lceil \frac{d-1}{3} \right\rceil \cdot \frac{n}{d} \right\}.$$

The fact that the lower bound is also exact in the case of noncyclic groups was established first for some cases by Diananda and Yap; the general question was finally resolved by Green and Ruzsa via complicated methods that, in part, also relied on a computer.

Theorem 1.2 (Green and Ruzsa, 2005; see [8]). *For any abelian group G of order n and exponent $e(G)$,*

$$\mu(G,\{2,1\}) = \max_{d\mid e(G)} \left\{ \left\lceil \frac{d-1}{3} \right\rceil \cdot \frac{n}{d} \right\}.$$

The first result for general k and l was given by Bier and Chin.

Theorem 1.3 (Bier and Chin, 2001; see [4]). *Let p be a positive prime. If $k-l$ is divisible by p, then $\mu(\mathbb{Z}_p,\{k,l\})=0$; otherwise,*

$$\mu(\mathbb{Z}_p,\{k,l\}) = \left\lceil \frac{p-1}{k+l} \right\rceil.$$

This was generalised by Hamidoune and Plagne.

Theorem 1.4 (Hamidoune and Plagne, 2004; see [9]). *If $k-l$ is relatively prime to n, then*

$$\mu(\mathbb{Z}_n,\{k,l\}) = \max_{d\mid n} \left\{ \left\lceil \frac{d-1}{k+l} \right\rceil \cdot \frac{n}{d} \right\}.$$

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The case when n and $k-l$ are not relatively prime is considerably more complicated. We have the following bounds of the first author.

THEOREM 1.5 (Bajnok, 2009; see [1]). *For all positive integers n, k and l with k > l,*

$$\max_{d|n}\left\{\left\lceil\frac{d-\delta}{k+l}\right\rceil\cdot\frac{n}{d}\right\}\le\mu(\mathbb{Z}_n,\{k,l\})\le\max_{d|n}\left\{\left\lceil\frac{d-1}{k+l}\right\rceil\cdot\frac{n}{d}\right\},$$

where $\delta=\gcd(d,k-l)$.

Until now, not even a conjecture was known for the actual value of $\mu(\mathbb{Z}_n,\{k,l\})$. Here we prove the following result.

THEOREM 1.6. *For all positive integers n, k and l with k > l,*

$$\mu(\mathbb{Z}_n,\{k,l\})=\max_{d|n}\left\{\left\lceil\frac{d-(\delta-r)}{k+l}\right\rceil\cdot\frac{n}{d}\right\},$$

where $\delta=\gcd(d,k-l)$ *and r is the remainder of* $l\lceil(d-\delta)/(k+l)\rceil \pmod{\delta}$.

We may observe that $\delta-r$ is between 1 and δ, inclusive, so Theorem 1.5 follows from Theorem 1.6; in particular, we get Theorem 1.4 when n and $k-l$ are relatively prime.

Let us now turn to the discussion of our approach. The main role in our development will be played by *arithmetic progressions*, that is, sets of the form

$$A=\{a+i\cdot b\mid i=0,1,\ldots,m-1\}$$

for some positive integer m and elements a and b of $\mathbb{Z}_n$. (We will assume that $m\le n/\gcd(n,b)$ and thus A has size $|A|=m$. Note also that a and b are not uniquely determined by A; the only time when this will make a difference for us is when $|A|=1$, in which case we set $b=1$.) In [9], Hamidoune and Plagne proved that, if n and $k-l$ are relatively prime, then $\mu(\mathbb{Z}_n,\{k,l\})$ equals

$$\max_{d|n}\left\{\alpha(\mathbb{Z}_d,\{k,l\})\cdot\frac{n}{d}\right\},$$

where $\alpha(\mathbb{Z}_d,\{k,l\})$ is the maximum size of a (k,l)-sum-free arithmetic progression in $\mathbb{Z}_d$. Hamidoune and Plagne only treated the case when n and $k-l$ are relatively prime, as they wrote 'in the absence of this assumption, degenerate behaviours may appear'. Nevertheless, as the first author proved, the identity remains valid in the general case.

THEOREM 1.7 (Bajnok, 2009; see [1]). *For all positive integers n, k and l with k > l,*

$$\mu(\mathbb{Z}_n,\{k,l\})=\max_{d|n}\left\{\alpha(\mathbb{Z}_d,\{k,l\})\cdot\frac{n}{d}\right\}.$$

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When attempting to evaluate $\alpha(\mathbb{Z}_d, \{k, l\})$, one naturally considers two types of arithmetic progressions: those with a common difference b that is not relatively prime to d (in which case the set is contained in a coset of a proper subgroup) and those where b is relatively prime to d (in which case the set, unless of size 1, is not contained in a coset of a proper subgroup). Accordingly, Hamidoune and Plagne [9] defined $\beta(\mathbb{Z}_d, \{k, l\})$ as the maximum size of a (k, l)-sum-free arithmetic progression with $\gcd(b, d) > 1$, and $\gamma(\mathbb{Z}_d, \{k, l\})$ as the maximum size of a (k, l)-sum-free arithmetic progression with $\gcd(b, d) = 1$. Clearly,

$$\alpha(\mathbb{Z}_d, \{k, l\}) = \max\{\beta(\mathbb{Z}_d, \{k, l\}), \gamma(\mathbb{Z}_d, \{k, l\})\}.$$

The authors of [9] evaluated both $\beta(\mathbb{Z}_d, \{k, l\})$ and $\gamma(\mathbb{Z}_d, \{k, l\})$ under the assumption that d and $k - l$ are relatively prime. We are able to find $\gamma(\mathbb{Z}_d, \{k, l\})$ without this assumption.

Theorem 1.8. *For all positive integers d, k and l with $k > l$,*

$$\gamma(\mathbb{Z}_d, \{k, l\}) = \left\lceil \frac{d - (\delta - r)}{k + l} \right\rceil,$$

where $\delta = \gcd(d, k - l)$ and r is the remainder of $l\lceil (d - \delta)/(k + l)\rceil \pmod{\delta}$.

However, evaluating $\beta(\mathbb{Z}_d, \{k, l\})$ in general does not seem feasible. Luckily, as we prove here, this is not necessary, since we have the following result.

Theorem 1.9. *For all positive integers n, k and l with $k > l$,*

$$\max_{d|n} \left\{\alpha(\mathbb{Z}_d, \{k, l\}) \cdot \frac{n}{d}\right\} = \max_{d|n} \left\{\gamma(\mathbb{Z}_d, \{k, l\}) \cdot \frac{n}{d}\right\}.$$

Therefore, Theorem 1.6 follows readily from Theorems 1.7–1.9. In Sections 2 and 3 below we prove Theorems 1.8 and 1.9, respectively. In Section 4 we discuss some further related questions.

2. The maximum size of (k, l)-sum-free intervals

Recall that $\gamma(\mathbb{Z}_d, \{k, l\})$ denotes the maximum size of a (k, l)-sum-free arithmetic progression in $\mathbb{Z}_d$ whose common difference is relatively prime to d. In this section we evaluate $\gamma(\mathbb{Z}_d, \{k, l\})$ and thus prove Theorem 1.8. Note that if

$$A = \{a + i \cdot b \mid i = 0, 1, \ldots, m - 1\},$$

with b relatively prime to d, then $b \cdot c = 1$ for some $c \in \mathbb{Z}_d$ and thus the c-fold dilation

$$c \cdot A = \{c \cdot a + i \mid i = 0, 1, \ldots, m - 1\}$$

of A is the interval $[ca, ca + m - 1]$; furthermore, A is (k, l)-sum-free in $\mathbb{Z}_d$ if and only if $c \cdot A$ is. Therefore, we may restrict our attention to intervals.

First, we prove a lemma.

Lemma 2.1. *Suppose that k, l and d are positive integers and that $k > l$; let $\delta = \gcd(d, k - l)$. Then $\mathbb{Z}_d$ contains a (k, l)-sum-free interval of size m if and only if*

$$k(m - 1) + \lceil (l(m - 1) + 1)/\delta \rceil \cdot \delta < d.$$

Proof. Let $A = [a, a + m - 1]$ with $a \in \mathbb{Z}_d$ and $|A| = m$. (As customary, our notation stands for the interval $\{a, a + 1, \ldots, a + m - 1\}$.) Note that A is (k, l)-sum-free if and only if

$$0 \notin kA - lA.$$

Observe that $kA - lA$ is also an interval, namely

$$kA - lA = [(k - l)a - l(m - 1), (k - l)a + k(m - 1)].$$

Therefore, A is (k, l)-sum-free if and only if there is a positive integer b for which

$$(k - l)a - l(m - 1) \geq bd + 1$$

and

$$(k - l)a + k(m - 1) \leq (b + 1)d - 1.$$

The set of these two inequalities is equivalent to

$$l(m - 1) + 1 \leq (k - l)a - bd \leq d - k(m - 1) - 1$$

or

$$\frac{l(m - 1) + 1}{\delta} \leq \frac{(k - l)}{\delta} \cdot a - \frac{d}{\delta} \cdot b \leq \frac{d - k(m - 1) - 1}{\delta}.$$

Here $(k - l)/\delta$ and d/δ are relatively prime, so every integer can be written in the form

$$\frac{(k - l)}{\delta} \cdot a - \frac{d}{\delta} \cdot b$$

for some a and b; we may also assume that $0 \leq a \leq d/\delta - 1$ and hence $0 \leq a \leq d - 1$. Therefore, $\mathbb{Z}_d$ contains a (k, l)-sum-free interval of size m if and only if there is an integer C with

$$\frac{l(m - 1) + 1}{\delta} \leq C \leq \frac{d - k(m - 1) - 1}{\delta}$$

or, equivalently,

$$\left\lceil \frac{l(m - 1) + 1}{\delta} \right\rceil \leq \frac{d - k(m - 1) - 1}{\delta},$$

which is further equivalent to

$$k(m - 1) + \lceil (l(m - 1) + 1)/\delta \rceil \cdot \delta < d,$$

as claimed. □

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Proof of Theorem 1.8. Let $\gamma_d = \gamma(\mathbb{Z}_d, \{k, l\})$,

$$f = \left\lceil \frac{d-\delta}{k+l} \right\rceil$$

and

$$m_0 = \left\lceil \frac{d-(\delta-r)}{k+l} \right\rceil.$$

We then clearly have

$$f \le m_0 \le f+1.$$

Claim 1. $\gamma_d \ge f$.

Proof of Claim 1. Since $\lceil s/t \rceil \cdot t \le s+t-1$ for positive integers s and t,

$$\lceil (l(f-1)+1)/\delta \rceil \cdot \delta \le l(f-1)+\delta$$

and

$$(k+l)f \le d-\delta+(k+l)-1.$$

Therefore,

$$k(f-1)+\lceil (l(f-1)+1)/\delta \rceil \cdot \delta \le (k+l)(f-1)+\delta \le d-1,$$

from which our claim follows by Lemma 2.1. □

Claim 2. $\gamma_d \le f+1$.

Proof of Claim 2. We can easily see that

$$k(f+1)+\lceil (l(f+1)+1)/\delta \rceil \cdot \delta > (k+l)(f+1) \ge d-\delta+k+l > d,$$

which implies our claim by Lemma 2.1. □

Claim 3. $\gamma_d \ge f+1$ if and only if $m_0 \ge f+1$.

Proof of Claim 3. First note that, since r is the remainder of $lf \pmod{\delta}$,

$$\lceil (lf+1)/\delta \rceil \cdot \delta = lf+\delta-r.$$

Therefore, $\gamma_d \ge f+1$ if and only if

$$kf+lf+\delta-r < d,$$

which is equivalent to

$$f < \frac{d-(\delta-r)}{k+l};$$

since f is an integer, this is further equivalent to $f < m_0$, that is, to $f+1 \le m_0$, as claimed. □

Our result that $\gamma_d = m_0$ now follows, since, if $f = m_0$, then $\gamma_d \ge f$ by Claim 1 and $\gamma_d \le f$ by Claim 3 and, if $f+1 = m_0$, then $\gamma_d \ge f+1$ by Claim 3 and $\gamma_d \le f+1$ by Claim 2. □

As a consequence of Theorem 1.8, we find the following lower bound.

Corollary 2.2. *For all positive integers k, l and d with $k > l$,*

$$\gamma(\mathbb{Z}_d, \{k, l\}) \ge \left\lfloor \frac{d}{k+l} \right\rfloor.$$

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3. Intervals suffice

In this section we prove Theorem 1.9, that is,

$$\max_{d|n} \left\{\alpha(\mathbb{Z}_d, \{k, l\}) \cdot \frac{n}{d}\right\} = \max_{d|n} \left\{\gamma(\mathbb{Z}_d, \{k, l\}) \cdot \frac{n}{d}\right\}.$$

We only need to establish that the left-hand side is less than or equal to the right-hand side, since, obviously,

$$\alpha(\mathbb{Z}_d, \{k, l\}) \geq \gamma(\mathbb{Z}_d, \{k, l\}).$$

Our result will thus follow from the following theorem.

Theorem 3.1. *For all positive integers d, k and l with $k > l$, there exists a divisor c of d for which*

$$\alpha(\mathbb{Z}_d, \{k, l\}) \leq \gamma(\mathbb{Z}_c, \{k, l\}) \cdot \frac{d}{c}.$$

Proof. Since $\alpha(\mathbb{Z}_d, \{k, l\})$ is the larger of $\beta(\mathbb{Z}_d, \{k, l\})$ and $\gamma(\mathbb{Z}_d, \{k, l\})$, we may assume that it equals $\beta(\mathbb{Z}_d, \{k, l\})$. We let β_d denote $\beta(\mathbb{Z}_d, \{k, l\})$.

Let A be a (k, l)-sum-free arithmetic progression in $\mathbb{Z}_d$ of size β_d and suppose that

$$A = \{a + i \cdot b \mid i = 0, 1, \ldots, \beta_d - 1\}$$

for some elements a and b of $\mathbb{Z}_d$; we may assume that $\beta_d \geq 2$ (a one-element subset would be an interval) and that $g = \gcd(b, d) \geq 2$. (We interchangeably consider $0, 1, \ldots, d - 1$ as integers and as elements of $\mathbb{Z}_d$.)

Let H denote the subgroup of index g in $\mathbb{Z}_d$. We then have a unique element $e \in \{0, 1, \ldots, g - 1\}$ for which A is a subset of the coset $e + H$ of H. We consider two cases.

When $k \not\equiv l \pmod g$, then $\gamma_g = \gamma(\mathbb{Z}_g, \{k, l\}) \geq 1$, since (for example) $\{1\}$ is a (k, l)-sum-free set in $\mathbb{Z}_g$. Therefore,

$$\beta_d = |A| \leq |H| = d/g \leq \gamma_g \cdot d/g.$$

We thus see that $c = g$ satisfies our claim.

Assume now that $k \equiv l \pmod g$. In this case $ke + H = le + H$ and thus kA and lA are both subsets of the same coset of H. Since the sets are nonempty and disjoint, we must have $|kA| < |H|$, $|lA| < |H|$ and

$$|kA| + |lA| \leq |H|.$$

Now

$$kA = \{ka + i \cdot b \mid i = 0, 1, \ldots, k \cdot \beta_d - k\},$$

so

$$|kA| = \min\{|H|,\ k \cdot \beta_d - k + 1\} = k \cdot \beta_d - k + 1$$

and similarly

$$|lA| = l \cdot \beta_d - l + 1.$$

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Therefore,

$$(k\cdot\beta_d - k + 1) + (l\cdot\beta_d - l + 1) \le |H| = d/g,$$

from which

$$\beta_d \le \left\lfloor \frac{d/g - 2}{k+l} \right\rfloor + 1.$$

Note that $\beta_d \ge 2$ implies that

$$d/g - 2 \ge k + l;$$

since $g \ge 2$, this then further implies that

$$d - d/g - 2 \ge k + l.$$

Therefore,

$$\beta_d \le \left\lfloor \frac{d/g - 2}{k+l} \right\rfloor + 1 \le \left\lfloor \frac{d-4}{k+l} \right\rfloor \le \left\lfloor \frac{d}{k+l} \right\rfloor.$$

By Corollary 2.2, we thus have $\beta_d \le \gamma_d$, which proves our claim. □

4. Further questions

Having found the maximum size of (k,l)-sum-free sets in cyclic groups, we may turn to some other related questions. Here we only discuss three of them; other intriguing problems, including:

- the number of (k,l)-sum-free sets;
- maximal (k,l)-sum-free sets (with respect to inclusion);
- complete (k,l)-sum-free sets (that is, those where $kA \cup lA = G$);
- maximum-size (k,l)-sum-free sets in subsets;

are discussed in detail in the first author's book [2, Ch. G.1.1].

4.1. Noncyclic groups. Clearly, if A is a (k,l)-sum-free set in G_1, then $A \times G_2$ is (k,l)-sum-free in $G_1 \times G_2$ and thus for any abelian group of order n and exponent $e(G)$,

$$\mu(G,\{k,l\}) \ge \mu(\mathbb{Z}_{e(G)},\{k,l\}) \cdot \frac{n}{e(G)}.$$

Therefore, by Theorem 1.6,

$$\mu(G,\{k,l\}) \ge \max_{d|e(G)} \left\{ \left\lceil \frac{d-(\delta-r)}{k+l} \right\rceil \cdot \frac{n}{d} \right\},$$

where $\delta = \gcd(d, k-l)$ and r is the remainder of $l\lceil (d-\delta)/(k+l)\rceil \pmod{\delta}$. We believe that equality holds. As we mentioned in the Introduction, Green and Ruzsa proved this conjecture for the case $(k,l) = (2,1)$; see Theorem 1.2 above. As their methods were complicated and relied, in part, on a computer, we expect the general case to be challenging.

We have the following partial result.

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THEOREM 4.1 (Bajnok, 2009; cf. [1]). *We have*

$$\mu(G, \{k, l\}) = \mu(\mathbb{Z}_{e(G)}, \{k, l\}) \cdot \frac{n}{e(G)}$$

whenever $e(G)$ *has at least one divisor* d *that is not congruent to any integer between* 1 *and* $\gcd(d, k-l)$ *(inclusive)* $\pmod{k+l}$.

In particular, for elementary abelian p-groups, we have the following result.

THEOREM 4.2. *Let* p *be a positive prime and* $r \in \mathbb{N}$. *If* $k-l$ *is divisible by* p, *then* $\mu(\mathbb{Z}_p^r, \{k, l\}) = 0$. *If* $k-l$ *is not divisible by* p *and* $p-1$ *is not divisible by* $k+l$, *then*

$$\mu(\mathbb{Z}_p^r, \{k, l\}) = \left\lceil \frac{p-1}{k+l} \right\rceil \cdot p^{r-1}.$$

Other cases remain open.

4.2. Classification of maximum-size (k, l)-sum-free sets. The question that we have here is: what can one say about a (k, l)-sum-free subset A of G of maximum size $|A| = \mu(G, \{k, l\})$?

The sum-free case (that is, when $(k, l) = (2, 1)$) has been investigated thoroughly and is now known. It turns out that, when the order n of the group has at least one divisor that is not congruent to 1 (mod 3), then sum-free sets of maximum size are unions of cosets that form arithmetic progressions; see the results of Diananda and Yap in [6] and Street in [13, 14] and also [15, Theorems 7.8 and 7.9]. The situation is considerably less apparent, however, when all divisors of n are congruent to 1 (mod 3). The classification was completed by Balasubramanian *et al.* in 2016; see [3]. The general result is too complicated to present here. We just mention the example that the set

$$\{(n-1)/3\} \cup [(n+5)/3, (2n-5)/3] \cup \{(2n+1)/3\},$$

which is two elements short of an arithmetic progression, is sum-free in $\mathbb{Z}_n$ and has maximum size $\mu(\mathbb{Z}_n, \{2, 1\}) = (n-1)/3$. (The classification of this case for cyclic groups was completed by Yap; see [16].)

The case when $k > 2$ is not known in general, but we have the following result of Plagne.

THEOREM 4.3 (Plagne, 2002; see [10]). *Let* p *be a positive prime and let* k *and* l *be positive integers with* $k > l$ *and* $k \geq 3$. *Suppose also that* $k-l$ *is not divisible by* p. *If* A *is a* (k, l)*-sum-free set in* $\mathbb{Z}_p$ *of maximum size* $\lceil (p-1)/(k+l) \rceil$, *then* A *is an arithmetic progression.*

We are not aware of further results on the classification of (k, l)-sum-free sets of maximum size.

4.3. Additive k-tuples. Given a subset A of G and a positive integer k, we may ask for the cardinality $P(G, k, A)$ of the set

$$\{(a_1, \ldots, a_k) \in A^k \mid a_1 + \cdots + a_k \in A\}.$$

We can then set $P(G, k, m)$ as the minimum value of $P(G, k, A)$ among all m-subsets A of G (with $m \in \mathbb{N}$). By definition, $P(G, k, m) = 0$ whenever $m \leq \mu(G, \{k, 1\})$, but $P(G, k, m) \geq 1$ for $\mu(G, \{k, 1\}) + 1 \leq m \leq n$.

Let us consider the case of $k = 2$ and the cyclic group $\mathbb{Z}_p$ of prime order p. As we observed, the 'middle third' of the elements forms a sum-free set in $\mathbb{Z}_p$ of maximum size $\mu(\mathbb{Z}_p, \{2, 1\}) = \lceil (p-1)/3 \rceil$. For $\lceil (p-1)/3 \rceil + 1 \leq m \leq p$, we may enlarge the set to

$$A(p, m) = \{\lceil (p-m)/2 \rceil + i \mid i = 0, 1, \ldots, m-1\}.$$

Then $A(p, m)$ is the 'middle' m elements of $\mathbb{Z}_p$ and a short calculation yields

$$P(\mathbb{Z}_p, 2, A(p, m)) = \lfloor (3m - p)^2/4 \rfloor.$$

Recently, Samotij and Sudakov proved that we cannot do better and that, in fact, $A(p, m)$ is essentially the only set achieving the minimum value.

Theorem 4.4 (Samotij and Sudakov, 2016; see [11, 12]). *For every positive prime p and integer m with $\lceil (p-1)/3 \rceil + 1 \leq m \leq p$,*

$$P(\mathbb{Z}_p, 2, m) = \lfloor (3m - p)^2/4 \rfloor.$$

Furthermore, if $P(\mathbb{Z}_p, 2, m) = P(\mathbb{Z}_p, 2, A)$ for some $A \subseteq G$, then there is an element b of $\mathbb{Z}_p$ for which $A = b \cdot A(p, m)$.

Soon after, Chervak *et al.* generalised Theorem 4.4 for other values of k, while still remaining in cyclic groups of prime order p. As they showed in [5], the answer turns out to be more complicated, but at least in the case when $k - 1$ is not divisible by p, the value $P(\mathbb{Z}_p, k, m)$ is still given by intervals (though there are other sets A that yield the same value). The general problem of finding $P(G, k, m)$ is largely unsolved.

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BÉLA BAJNOK, Department of Mathematics,
Gettysburg College,
Gettysburg, PA, USA
e-mail: bbajnok@gettysburg.edu

RYAN MATZKE, School of Mathematics,
University of Minnesota,
Minneapolis, MN, USA
e-mail: matzk053@umn.edu

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